What do financial markets say about the exchange rate?*

Mikhail Chernov,† Valentin Haddad,‡ and Oleg Itskhoki§

First draft: April, 2022
This draft: September 13, 2023

Preliminary and incomplete, comments welcome!

Abstract

Financial markets play two roles with implications for the exchange rate: they accommodate risk-sharing and act as a source of shocks. In prevailing theories, these roles are seen as mutually exclusive and individually face challenges in explaining exchange rate dynamics. However, we demonstrate that this is not necessarily the case. We develop an analytical framework that characterizes the link between exchange rates and finance across all conceivable market structures. Our findings indicate that full market segmentation is not necessary for financial shocks to explain exchange rates. Moreover, risk-sharing can have a significant role without leading to the traditional puzzles associated with the macro disconnect. We identify plausible market structures where both roles coexist, addressing challenges faced when examined separately.

JEL classification codes: E44, F31, G15.

Keywords: exchange rates, FX disconnect, market completeness, spanning

*We are grateful to Mark Aguiar, Andy Atkeson, William Cassidy, Hanno Lustig, Ian Martin, Dmitry Mukhin, Tyler Muir, Stefan Nagel, Irina Zviadadze for comments on earlier drafts and participants in the seminars and conferences sponsored by Boston University, the 2023 NBER Summer Institute, the 2023 Paul Woolley Conference at LSE, the 2023 SED meeting, UCLA, University of Luxembourg, the 2023 Yiran Fan Memorial Conference at University of Chicago, the 2023 Vienna FX Symposium. We thank Kibeom Lee for excellent research assistance. The latest version is available here.

†UCLA, NBER, and CEPR; mikhail.chernov@anderson.ucla.edu.
‡UCLA and NBER; valentin.haddad@anderson.ucla.edu.
§UCLA, NBER, and CEPR; itskhoki@econ.ucla.edu.
Introduction

A broad set of evidence suggests that exchange movements rate have little to no relation with macroeconomic aggregates (Meese and Rogoff, 1983; Obstfeld and Rogoff, 2001). This observation has prompted researchers to search for the origin of these fluctuations somewhere else. Financial markets seem like a natural place to look. This approach has led to many valuable insights but to challenges as well. In this paper, we provide a general analysis of how the finance block of an equilibrium model interacts with the exchange rate. This perspective clarifies the root cause of the main challenges to existing theories of this interaction, and allows us to identify frameworks that overcome these challenges.

We focus on the duality between two roles of financial markets in the determination of the exchange rate:

- Financial markets are where sharing of macroeconomic risks across countries takes place. Households use financial claims to line up their marginal rates of substitution, and this determines the exchange rate that smooths out certain macro shocks.\(^1\)

- Financial markets are also a source of shocks to the exchange rate. For example, shocks to financial institutions intermediating international trading affect the exchange rate.

The literature usually adopts market structures in which only one of these roles is emphasized. Each of them runs into significant challenges. On the one hand, models of risk-sharing typically assume integrated markets, complete markets, or both.

\(^1\)By households we generally refer to the local representative agents.
Complete means that every Arrow-Debreu claim is available, integrated means that everyone can trade with each other. This rich market structure leads to a tight connection of the exchange rate with the macroeconomy, at odds with the classic evidence of disconnect, or, alternatively, imposing strong constraints on difficult-to-measure aspects of macroeconomic dynamics. On the other hand, models of financial shocks typically focus on limited and segmented market structures. These assumptions are at odds with the existence of widely accessed local markets and of many global multi-market intermediaries.

While the literature has explored specific market structures beyond these two extremes, general results have been elusive, as each case seemingly requires a separate analysis. The first contribution of this paper is to provide an analytical framework which characterizes the link between the exchange rate and financial markets across all possible market structures. We fully map out when and how the two roles of finance shape the properties of the exchange rate.

One possible reading of the literature is that the two roles of financial markets are mutually exclusive. Moreover, it may appear that the two challenges, which we have highlighted, are intrinsic to each role of the financial markets. Our second contribution is to show that, while there is a tension between the two roles, both conjectures are incorrect. In particular, we highlight plausible market structures in which risk-sharing and financial shocks jointly determine the exchange rate, and addressing both challenges.

Specifically, we demonstrate that extreme market segmentation is not a prerequisite for shocks in the financial sector to have a substantial impact on the exchange rate. Even if households trade assets in their respective countries and a few risky assets in
common, all with global intermediaries, there is still a lot of flexibility for financial sector shocks to determine the exchange rate.

Next, we show that the disconnect puzzle persists when markets are incomplete but integrated, or intermediated but complete. However, this conclusion is not intrinsic to all structures with a considerable extent of macroeconomic risk-sharing. Specifically, this challenge can disappear if markets are both intermediated and incomplete, even when many risks are still shared in such structures.

Finally, our framework leads to an empirical method to quantify how the properties of asset returns discipline the exchange rate in any given market structure. We implement this approach using data on stocks and bonds. Their lack of strong correlation across countries and with the exchange rate identifies a scope for solving both challenges simultaneously. We conclude that a market structure in which households in each country trade their local stocks and bonds with a global intermediary features both roles without their challenges, and hence is particularly promising for building models of the exchange rate.

A useful starting point for our analytical results is the case of complete and integrated markets. In this setting, risk-sharing between local households completely pins down the exchange rate, which must equal the difference between their (log) intertemporal marginal rates of substitution $m$ and $m^*$:

$$\Delta s_{t+1} = m^*_{t+1} - m_{t+1},$$  \hspace{1cm} (1)

where $\Delta s$ is the log home currency depreciation rate (see, e.g., Backus, Foresi, and Telmer, 2001).
This expression highlights why complete and integrated markets struggle with the macro disconnect: many models connect the discount factors $m$ and $m^*$, and hence the exchange rate, to macroeconomic aggregates. This issue implies a mismatch between several moments of the model-based and the empirical exchange rate: the volatility (Brandt, Cochrane, and Santa-Clara, 2006), cyclicality (Backus and Smith, 1993), and risk premium (Fama, 1984) puzzles. Another implication of the relation in equation (1) is that the associated risk-sharing leaves no room for the second role of financial markets: even if financial frictions are modeled, they have no impact on the depreciation rate above and beyond what can be learned from households’ marginal utility.

Our framework generalizes this baseline case to all possible market structures including deviations from market completeness, from market integration, or both simultaneously. Households in each country trade a potentially distinct set of assets in their local currency. That is, Euler equations hold with respect to each country’s stochastic discount factor (SDF) for these assets. We summarize the remainder of what happens in international financial markets by the assumption of no arbitrage. This corresponds to assuming that interactions between all players in international markets result in an “international” SDF which may or may not coincide with one of the two local SDFs. In the latter case, it could be the discount factor of a global intermediary.

In this setting, we characterize all restrictions imposed on the exchange rate based on the households’ discount factors, thereby producing a general version of the relation in equation (1). These restrictions take a simple form that can be summarized in two relations. First, innovations to the depreciation rate coincide with innovations in the relative discount factor across countries when projected on risks that both home
and foreign households can trade and, therefore, share:

\[ \text{proj}(\Delta s_{t+1} | \epsilon^g_{t+1}) = \text{proj}(\tilde{m}_{t+1} - \overline{m}_{t+1} | \epsilon^g_{t+1}), \]  

(2)

with \( \epsilon^g \) the set of globally traded risks. Second, we show that the expected depreciation rate is similar to the one under complete markets when traded asset returns span the exchange rate; otherwise, it is unconstrained by local discount factors. These two results exhaust all possible restrictions imposed by local SDFs on the exchange rate, i.e. they are necessary and sufficient for precluding international arbitrage opportunities. In general, these constraints do not pin down the exchange rate completely. That leaves space for the second role of financial markets, as a source of financial shocks, to determine the remainder of the exchange rate.

To address the conjectures about the interaction between the two financial roles and the challenges associated with them, we conduct the following exercise. We fix households’ intertemporal marginal rates of substitution (IMRS) \( m \) and \( m^* \).\(^{2}\) We assume that these discount factors are such that applying equation (1) leads to the exchange rate puzzles. We use equation (2) to study the equilibrium behavior of the exchange rate implied by various choices of market structure. This path is complementary to the large literature which fixes financial markets to be complete and integrated, while varying assumptions about preferences (CRRA, habits, Epstein-Zin, ...) and aggregate dynamics (random walk, long-run risks, disasters, ...).

We ask whether the currency puzzles remain in market structures other than complete and integrated. Examining equation (2) reveals when risk-sharing puts as tight a restriction on the exchange rate as equation (1). Specifically, if households can use

\(^{2}\)For example, with CRRA preferences, \( m_{t+1} = -\rho - \gamma \Delta c_{t+1} \). So, given knowledge of the preference parameters and consumption data, \( m_{t+1} \) is a fixed observable quantity.
assets to trade exposure to the depreciation rate and to innovations in their relative
discount factor, these two variables become globally traded risks. Then, the pro-
jections in equation (2) disappear and this relation coincides with equation (1); the
puzzles arise.

We show that even if only one of the two variables is spanned by globally traded
risks, the puzzles remain. This more general situation arises when markets are in-
complete and integrated. Then, the exchange rate is globally traded as soon as both
households can trade risk-free bonds in both countries (the case considered in Lustig
and Verdelhan, 2019). This situation also arises when markets allow to trade all
macro risks and are intermediated. In this case, innovations in relative marginal
utility are globally traded. Taken together, these results show that relaxing market
incompleteness alone or market integration alone does not solve the puzzles.

At the other extreme, we ask which market structures are not constrained by house-
hold risk-sharing at all. This lack of constraints is a central ingredient in typical
models focused on financial shocks. These models start from extreme segmentation:
households can only trade their respective risk-free bonds, and intermediaries engage
in the carry trade only. They then have the freedom to choose sources of financial
shocks — frictions to intermediaries, noise traders, etc. — to pin down realistic
exchange rate dynamics. Equation (2) reveals the precise source of this freedom.
Because households cannot trade any risks in common, \( \epsilon^{g}_{t+1} \) is empty, and the pro-
jections in equation (2) are degenerate. As a result, there are no restrictions on the
exchange rate.

The lack of global shocks and the flexibility of financial shocks that comes with it
survive in a larger set of market structures. First, intermediaries can be sophisticated
and trade an arbitrarily large set of assets. Second, households can trade a risky asset in common — a contrast to the case of trading risk-free bonds. Finally, households in the two countries can each trade many local assets, as long as the risks in the two countries are not related. Such market structures are more compelling than the extreme versions with little to no trading considered in the literature.

Of course, there are many local assets in each country and at least some of their returns are correlated across countries. This suggests that there might be some global sources of risk that can be traded by all investors. We show that such situations do not necessarily take us back to the puzzles. Risk-sharing can play a substantial role without removing all the flexibility of financial shocks to determine the exchange rate. This occurs when there are some global shocks in equation (2), but they do not explain 100% of the variation in either the depreciation rate or the relative discount factor. We show how to quantify the magnitude of global risks empirically.

We consider a market structure in which investors in each country can trade a wide set of stock indices (the market, value-growth, and industry portfolios) and sovereign bonds with maturities ranging from 2 to 10 years. We focus on G-10 countries, U.S. vs foreign on a bilateral basis, from 1988 to 2022 at a monthly frequency.

First, we evaluate whether the depreciation rate can be spanned by these asset returns. The answer is no: the largest spanning regression $R^2$ is 45% for Canada (vs the U.S.), the lowest is 25% for Switzerland. Thus, unspanned shocks play an important role in the variation of the exchange rate, enriching the results of Chernov and Creal (2023) for the case of the international yield curves.

Next, we quantify global shocks using two methods. First, we use canonical correlation analysis to find maximally correlated portfolios in a pair of countries. Second,
we use shocks that are commonly used as global in the literature: the Volatility Index (VIX), the Global Financial Cycle (GFC, Miranda-Agrippino and Rey, 2020), and the Excess Bond Premium (EBP, Gilchrist and Zakrajsek, 2012). Regardless of the method, global shocks contribute mildly to the variation in exchange rates: most countries have no more than 10% of FX variation explained by global shocks. Thus, the evidence supports the conclusion that, in this market structure, the combination of risk-sharing and financial shocks, with a more prominent role for the latter, can explain exchange rate movements.

Naturally, this empirical analysis leaves some questions unanswered. Our empirical conclusions depend on the choice of spanning assets. While we find the intermediated market structure with local stock and bond trading appealing, one can follow our approach and revisit the quantification of spanning and global shocks in any structure they prefer. Second, while we show that financial shocks naturally play a major role in determining the exchange rate, our analysis focused on asset prices alone as a source of discipline. More direct evidence on the trading of these financial players is necessary to establish this conclusion tightly.

**Related literature.** We derive general restrictions on the exchange rate given financial market structure in each of the two countries. We apply these restrictions to the famous facts about exchange rates such as their relatively low volatility and weak relation to business cycles. The resulting implication that exchange rates features large unspanned and local shocks is very much in the spirit of Hansen and Jagannathan (1991) agnostic characterization of asset-pricing models. Departures from complete markets in the context of currency puzzles is explored by Lustig and Verdelhan (2019). They consider a special case where the exchange rate is spanned
because each country’s investor can trade the other country’s risk-free bond.\footnote{In this sense, such an assumption leads to the same conclusions as the full market integration, which is explicitly assumed by Maurer and Tran (2021) and Sandulescu, Trojani, and Vedolin (2021). These authors do not investigate the puzzles associated with the macro disconnect.} That makes it difficult to capture volatility and cyclicality puzzles jointly. Jiang, Krishnamurthy, Lustig, and Sun (2022) consider a similar incomplete-market setting with international access to trading in risk-free bonds but complemented by safe asset demand for dollar bonds. This feature leads to wedges in the Euler equations, which are ruled out in our setting. One implication of these wedges is that the exchange rate is affected by the convenience yield in addition to risks spanned by the SDFs. Chernov and Creal (2023) emphasize inability of bonds to span exchange rates and propose an affine term structure model with martingale shocks to the SDF, which affect the exchange rate but not bond prices. Real business cycle (RBC) models of exchange rates with complete and integrated markets are represented by Verdelhan (2010) (habits), Colacito and Croce (2011) (long-run risk), and Farhi and Gabaix (2016) (disasters), among many others. Exchange rate models with intermediation include Gabaix and Maggiori (2015), Gourinchas, Ray, and Vayanos (2022), Greenwood, Hanson, Stein, and Sunderam (2022), and Itskhoki and Mukhin (2021).

1 Framework

We introduce our framework to represent and analyze a variety of financial market structures.
1.1 Market structure

We consider settings with two representative households, $h$ for home, and $f$ for foreign. Each household can trade a set of assets, $H$ and $F$, respectively. Those sets can contain subsets of local assets and foreign assets converted to local currency. Figure 1 demonstrates some examples. For instance, in autarky $H$ contains domestic stocks and bonds, while $F$ contains the foreign ones. When markets are integrated, $H$ and $F$ contain identical assets but expressed in respective currencies, e.g., $H$ may include a domestic sovereign bond and a foreign equity index converted to domestic currency, while $F$ contains domestic bond converted to foreign currency and foreign equity index. If markets are complete, $H$ and $F$ contain the full set of Arrow-Debreu securities expressed in respective currencies. Market completeness is a particular case of full market integration where securities span all possible risks.

Further, we consider a set $I$ of assets traded in international markets. Assets can
be included in this set for two reasons. First, it could be that home and foreign households trade some assets in common, as in the partially integrated cases above. Then, either $h$ or $f$ can be considered as an international arbitrageur, with $I = H$ or $I = F$, respectively. Second, it could be that financial intermediaries trade across borders even if households do not, as in the examples in panel C of Figure 1. In this case, $I$ are the assets from $H$ and $F$ that intermediaries can trade. We require that $H$ and $F$ each contain a risk-free bond in the respective currency, and $I$ contains both risk-free bonds.

Our main result is that, in this large family of market structures, restrictions on the exchange rate coming from risk-sharing between households are determined by the properties of returns in $H \cap I$ expressed in domestic currency and returns in $F \cap I$ expressed in foreign currency. To continue our examples, if markets are partially integrated and $I = H$, then $H \cap I = H$ are the assets traded by the domestic household, $F \cap I = F \cap H$ are the assets traded by both households. In intermediated markets, $H \cap I$ is the set of assets traded both by the domestic household and the intermediaries; ditto for $F \cap I$.

The base assets in the set $H \cap I$ have log returns $r_{t+1} = (r_{1,t+1}, \ldots, r_{N,t+1})$. We assume this collection includes a risk-free asset with return $r_{ft}$ in home currency known at time $t$. The corresponding set of all feasible portfolio returns is $r_{p,t+1} = \{r_{p,t+1} | \exists w_t \in \mathbb{R}^N : w_t' \mu = 1, r_{p,t+1} = \log (w_t' \exp (r_{t+1}))\}$. Furthermore, we assume that asset returns are log-normal, that is $r_{t+1}$ are multivariate normal, $MVN(\mu_t, \Sigma_t)$. Similarly, the returns of base assets in $F \cap I$ are $r_{t+1}^*$ in foreign currency, log-normal of size $N^*$, and contain a foreign-currency risk-free rate of $r_{ft}^*$. The corresponding set of portfolio returns is $r_{p,t+1}^*$. Throughout the paper, we use the Campbell and Viceira (2002) approximation for log portfolio excess returns in the relevant derivations as
described in the Appendix.

1.2 Pricing Assumptions

We introduce two sets of assumptions to ask how risk-sharing between households constrains the behavior of the exchange rate.

**Local Euler equations.** We specify valuation mechanisms by each representative household with a given SDF \( m \) at home and \( m^* \) abroad. These SDFs value assets as follows.

**Assumption 1.** The domestic (log) stochastic discount factor \( m_{t+1} \) prices all assets in \( H \) in domestic currency. In particular, it satisfies the Euler equation:

\[
\forall r_{t+1} \in r_{p,t+1} : \quad E_t \left[ \exp(m_{t+1} + r_{t+1}) \right] = 1.
\]

Similarly, the foreign log SDF \( m^*_{t+1} \) prices all assets in \( F \) in foreign currency, and

\[
\forall r^*_{t+1} \in r^*_{p,t+1} : \quad E_t \left[ \exp(m^*_{t+1} + r^*_{t+1}) \right] = 1.
\]

Recall that \( r_{p,t+1} \) (\( r^*_{p,t+1} \)) is the set of feasible portfolio returns constructed from assets in \( H \cap I \) (\( F \cap I \)). Thus (3) and (4) require only pricing of assets in sets \( H \cap I \) and \( F \cap I \), respectively. These Euler equations are all that is needed for our formal results. Nevertheless, in many economic environments it is reasonable to assume that the same home and foreign SDFs price all assets in \( H \) and \( F \), respectively.
Assumption 1 can be viewed as the definition of local financial market equilibrium that we use in our analysis.\footnote{Note that equilibrium in the financial market may involve borrowing or short-sale constraints, infrequent portfolio adjustment, or convenience yield on certain assets. In all such cases, some Euler equations do not always hold with equality, and in our analysis this simply requires redefining sets $H$ and $F$ to exclude such assets (for a given time period $t$). In this case, conditions (3) and (4) can be thought of as definitions of sets $H$ and $F$ rather than an assumption.}

We focus on situations with log-normal SDFs. The Euler equations imply that expected excess returns are proportional to the covariance with the stochastic discount factors. In our log-normal setting, this corresponds to:

\[ \forall r_{t+1} \in \mathbf{r}_{p,t+1} : \quad E_t(r_{t+1}) + \frac{1}{2} \text{var}_t(r_{t+1}) = r_{ft} - \text{cov}_t(m_{t+1}, r_{t+1}), \quad (5) \]

\[ \forall r^*_{t+1} \in \mathbf{r}^*_{p,t+1} : \quad E_t(r^*_{t+1}) + \frac{1}{2} \text{var}_t(r^*_{t+1}) = r^*_{ft} - \text{cov}_t(m^*_{t+1}, r^*_{t+1}). \quad (6) \]

These Euler equations are the point of contact of the economy with financial markets, and hold irrespective of the remainder of the economic environment. For example, with CRRA utility, $m_{t+1} = -\gamma \Delta c_{t+1}$ where $\gamma$ is the coefficient of risk aversion and $c_t$ is log aggregate domestic consumption, which could be exogenous as in an endowment economy, or allowed to change endogenously.

In Section XX, we show how this framework can be used in model-free applications with a different interpretation. Instead of coming from intertemporal marginal rates of substitutions, the discount factors could simply be a representation of the risk-return relation among assets traded in a country. For example, the SDF could be constructed from asset returns as $m_{t+1} = \lambda_t' \mathbf{r}_{t+1}$, with $\lambda_t \in \mathbb{R}^N$, in the spirit of Hansen and Jagannathan (1991).\footnote{More precisely, this expression could be viewed as a log-approximation to SDFs which are}
**International arbitrage.** So far none of our assumptions involve the exchange rate. In order to analyze how it interacts with local Euler equations, one has to take a stand on how international markets operate. To isolate the risk-sharing role of these markets, we make a minimal assumption about them: there are no arbitrage opportunities for assets in \( I \).

Specifically, the set of returns in \( I \) combines the domestic and foreign set of international returns converted to the domestic currency.\(^6\) Following our notations, international portfolios are generated by the base assets \( r_{t+1}' = (r_{t+1}, r_{t+1}' + \Delta s_{t+1}) \), where \( \Delta s_{t+1} \) is the log home currency depreciation rate. We denote the set of international portfolios generated by these base assets by \( r_{p,t+1}' \).

**Assumption 2.** There are no arbitrage opportunities in the set of international returns \( r_{p,t+1}' \), that is:

\[
\forall r_{p,t+1} \in r_{p,t+1}' : \text{var}_t(r_{p,t+1}) = 0 \Rightarrow E_t(r_{p,t+1}) = r_f.
\]  

(7)

In words, any portfolio that has no risk must earn the risk-free rate of return.\(^7\) This is equivalent to the existence of an international SDF \( m_t' \). For example, this would be the discount factor of an international arbitrageur. However, unlike for households, at this stage we do not assume any knowledge of this SDF beyond its existence. This is not to say that the specificities of international arbitrageurs do not matter, simply that we consider them as part of the second role of financial markets, as a source of

\(^6\)Our conclusions are unchanged if we focus on international arbitrage in foreign currency.

\(^7\)In our log-normal setting, condition (7) is equivalent to the absence of arbitrage opportunities. In more general settings, it is a necessary condition for no arbitrage.
shocks.\(^8\)

### 1.3 Global, local and unspanned shocks

Intuitively, returns are affected by a collection of shocks, some of which are local to each economy, \(\epsilon_{t+1}\) or \(\epsilon^*_{t+1}\), while others are common to both, i.e. global shocks \(\epsilon^g_{t+1}\). We denote with tilde the innovation (or shock) for any variable \(x\), that is \(\tilde{x}_{t+1} \equiv x_{t+1} - E_t x_{t+1}\).\(^9\) With this, we define the set of globally-traded shocks, or global shocks for short.

**Definition 1.** The set of global shocks is \(\epsilon^g_{t+1} = \{\epsilon^g_{t+1} | \exists \lambda \in \mathbb{R}^N, \lambda^* \in \mathbb{R}^{N^*} : \epsilon^g_{t+1} = \lambda \tilde{r}_{t+1} = \lambda^* \tilde{r}^*_{t+1}\}\).

Global shocks can be traded by local investors in their local currency in both countries. Formally, this means the coincidence of two properties. First, such a shock must affect returns in the two countries. Second, investors must have access to a trading strategy in each country that isolates the shock from other sources of risk. Appendix A shows how to construct a basis of this space from the covariance matrix of \(r_{t+1}\) and \(r^*_{t+1}\). Local shocks \(\epsilon_{t+1}\) and \(\epsilon^*_{t+1}\) are the residuals of return innovations, \(\tilde{r}_{t+1}\) and \(\tilde{r}^*_{t+1}\), after controlling for global shocks.

Global shocks can arise because of common underlying economic shocks (e.g., productivity) that determine returns in both countries as long as such shocks can be

\(^{8}\)It might be tempting to replace both representative households by the arbitrageur in Assumption 1, but such an approach is mostly vacuous as it effectively considers twice the same investor. Mechanically the conversion of an intermediary’s SDF from domestic to foreign currency is \(m^t = m^t + \Delta s\), irrespective of market structure — an accounting relation, not an equilibrium relation.

\(^{9}\)Note that \(\text{var}_t(\tilde{x}_{t+1}) = \text{var}_t(x_{t+1})\) and we use this notation interchangeably.
replicated by investors in the two countries. Alternatively, global shocks can emerge without common fundamental shocks as a result of asset trading across countries — either directly by households or via an intermediary.

As an example, consider partially integrated markets such as the ones in Figure 1B. Imagine that $r_{t+1} = (r_{ft}, r_{1,t+1}, r_{2,t+1}, r_{ft}^*, r_{1,t+1}^*, r_{1,t+1} + \Delta s_{t+1})$ and $r_{t+1}^* = (r_{ft}^*, r_{1,t+1}^*, r_{2,t+1}^*, r_{ft} - \Delta s_{t+1}, r_{1,t+1} - \Delta s_{t+1})$. In such a setting, the domestic investor can construct a portfolio with a return $r_{1,t+1}^* - r_{ft}^*$ by buying the foreign risky asset 1 and by selling the foreign risk-free asset, both converted into domestic currency. Similarly, the foreign investor can construct a portfolio with a return $r_{1,t+1} - r_{ft}$. As a result, both $r_{1,t+1}^*$ and $r_{1,t+1}$ are in the set of global shocks $\epsilon_{g,t+1}$. Furthermore, here the FX risk $\Delta s_{t+1}$ is also a global shock: it can be traded by both households through their respective carry trades.

Finally, we refer to any other sources of variation orthogonal to asset returns $(\hat{r}_{t+1}, \hat{r}_{t+1}^*)$, or equivalently orthogonal to local and global shocks $(\epsilon_{t+1}, \epsilon_{t+1}^*)$, as unspanned shocks.

**Exchange rate depreciation.** We can use this taxonomy of shocks to uniquely decompose the innovation to the depreciation rate as follows:

$$\tilde{\Delta} s_{t+1} = g_{t+1} + \ell_{t+1} + u_{t+1}, \quad (8)$$

where $g_{t+1}$ is a linear combination of global shocks $\epsilon_{t+1}^g$, $\ell_{t+1}$ is a linear combination of both types of local shocks, $\epsilon_{t+1}$ and $\epsilon_{t+1}^*$, and $u_{t+1}$ is unspanned.

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10 A practical example of such global shocks arises in the context of commodity (e.g., oil) futures denominated in different currencies, or stocks of the same company traded in jurisdictions with different currencies (e.g., Royal Dutch Shell).
Thus, there are four components to the exchange rate depreciation, $\Delta s_{t+1}$. The first is the conditional expectation $E_t \Delta s_{t+1}$. Then, there are two types of shocks spanned by assets, a global component $g_{t+1}$ and a local component $\ell_{t+1}$. Finally, there can be unspanned shocks $u_{t+1}$. This decomposition will play a central role in our characterization of restrictions on the behavior of the exchange rate.

Relatedly, one can construct the spanned components directly from asset returns:

$$\tilde{\Delta}s_{t+1} = \tilde{r}_{p,t+1} - \tilde{r}^*_{p,t+1} + u_{t+1},$$  \hspace{1cm} (9)$$

with $r_{p,t+1} \in r_{p,t+1}$ and $r^*_{p,t+1} \in r_{p,t+1}$ two portfolios with best $R^2$ for explaining the exchange rate.\(^{11}\) Mechanically, the residual coincides with the unspanned component $u_{t+1}$ in equation (8). If this unspanned component is equal to 0, the depreciation rate is spanned by asset returns, and the difference between the shocks to returns on the two portfolios replicates the exchange rate shock exactly.

\section{The general risk-sharing view of exchange rates}

In this section, we characterize the restrictions on the behavior of the exchange rate imposed by the absence of international arbitrage and given the properties of returns on traded assets, $r$ and $r^*$, and local SDFs $m$ and $m^*$ that price them. We show that Assumptions 1 and 2 impose two sets of necessary restrictions on the depreciation rate — one on the shocks to the depreciation rate $\tilde{\Delta}s_{t+1}$ and another on the expected depreciation rate $E_t \Delta s_{t+1}$.

\(^{11}\)Formally, the portfolios maximize $R^2 = 1 - \frac{\text{var}_t(\Delta s_{t+1} - (\tilde{r}_{p,t+1} - \tilde{r}^*_{p,t+1}))}{\text{var}_t(\Delta s_{t+1})}$.

This pair of portfolios is not unique when global shocks are present. All of our results hold for any such pair.
We demonstrate that in a complete market setting these two sets of restrictions lead to the well-known asset market view of exchange rates and the puzzles that come with it. Subsequent analysis spells out the implications of these restrictions for a much larger set of market structures and revisits the puzzles in light of these results. All the proofs are in Appendix B. Appendix C proves the sufficiency of our key results: if the two sets of restrictions hold, Assumption 2 about the absence of international arbitrage opportunities is valid. Appendix D derives exact non-linear versions of the results.

2.1 Exchange rate shocks

We show that the component of the depreciation rate that loads on global shocks, \( g_{t+1} \), must coincide with the component of the difference of SDFs that loads on global shocks.

**Proposition 1.** Under Assumptions 1 and 2,

\[
proj(\tilde{m}^*_{t+1} - \tilde{m}_{t+1} | \epsilon_{t+1}^g) = proj(\Delta s_{t+1} | \epsilon_{t+1}^g) = g_{t+1}.
\] (10)

Said differently, start from any pair of admissible local SDFs and regress them on all global shocks. The predicted value of this regression is equal to the global component of the exchange rate, \( g_{t+1} \):

\[
m^*_{t+1} - m_{t+1} = g_{t+1} + v_{t+1} \quad \text{with} \quad v_{t+1} \perp \epsilon_{t+1}^g.
\] (11)

What is missing in Proposition 1 is just as important as what is there. Local financial
markets do not impose any restrictions on the component of the depreciation rate loading on either local shocks \((\epsilon_{t+1},\epsilon^*_t)\) or its unspanned component \(u_{t+1}\). Thus, in general, financial markets impose less restrictions on the exchange rate as compared with complete markets.

How does the absence of arbitrage lead to this result? In complete markets, local and foreign investors must agree on the price of all payoffs after conversion to a common currency: 
\[
\text{cov}_t(m_{t+1}, r_{t+1}) = \text{cov}_t(m^*_{t+1} - \Delta s_{t+1}, r_{t+1}) \text{ for every } r_{t+1}.
\]
Proposition 1 comes from a generalization of this result. To preclude arbitrage opportunities, local and foreign investors must only agree on the price of risks that they both trade — the global shocks.

Without a change of currency, the argument is standard: the international arbitrageur can buy the global shock \(\epsilon^g_{t+1}\) in the home market and sell it in the foreign market.\(^{12}\) Because this portfolio is riskless, the two risk premia must coincide, 
\[
\text{cov}_t(m_{t+1}, \epsilon^g_{t+1}) = \text{cov}_t(m^*_{t+1}, \epsilon^g_{t+1}).
\]
In Appendix B.2, we show that this logic extends to the case with currency conversion, and no arbitrage requires an adjustment to expected returns of 
\[
\text{cov}_t(\Delta s_{t+1}, \epsilon^g_{t+1}),
\]
the so-called quanto adjustment. This implies that the comovement of the depreciation rate with global shocks must be the same as that of the relative SDFs, 
\[
\text{cov}_t(m^*_{t+1} - m_{t+1}, \epsilon^g_{t+1}) = \text{cov}_t(\Delta s_{t+1}, \epsilon^g_{t+1}).
\]
Conversely, for shocks that are not traded by both investors, it is impossible to construct candidate arbitrage portfolios that relate pricing in the two markets (see Appendix C).

\(^{12}\)See, for example, Chen and Knez (1995).
2.2 Expected depreciation rate

We turn to restrictions on the behavior of the expected depreciation rate. These restrictions depend on the relation of the exchange rate with asset returns. Start from the projection of the exchange rate on asset returns, represented by two portfolio \( r_{p,t+1} \) and \( r^*_{p,t+1} \) as in equation (9). Recall that when \( r_{p,t+1} \) and \( r^*_{p,t+1} \) span the exchange rate, the unspanned component \( u_{t+1} \) is equal to 0. We define \( \delta_t \) as the difference of the two portfolios’ expected returns:

\[
\delta_t \equiv r_{ft} - \text{cov}_t(m_{t+1}, r_{p,t+1}) - \frac{1}{2}\text{var}_t(r_{p,t+1}) - r^*_{ft} - \text{cov}_t(m^*_{t+1}, r^*_{p,t+1}) - \frac{1}{2}\text{var}_t(r^*_{p,t+1}).
\]  

(12)

The following proposition relates the behavior of the expected depreciation rate to spanning of the exchange rate and this quantity, which only depends on asset returns and local SDFs.

**Proposition 2.** The expected depreciation rate is pinned down if and only if the exchange rate is spanned by asset returns, that is when \( u_{t+1} = 0 \). In this case, it is:

\[
E_t \Delta s_{t+1} = \delta_t = r_{ft} - r^*_{ft} - \text{cov}_t(m_{t+1}, \Delta s_{t+1}) - \frac{1}{2}\text{var}_t(\Delta s_{t+1}) + \theta_t,
\]  

(13)

where \( \theta_t = \text{cov}_t(m^*_{t+1} - m_{t+1} - \Delta s_{t+1}, r^*_{p,t+1}) \). This quantity collapses to \( \theta_t = 0 \) when the exchange rate is spanned by global shocks.

The most important implication of Proposition 2 is that it delineates two cases: either local market pricing determines expected depreciation exactly, or it says nothing
about it. The expected depreciation rate is closely related to the risk premium for exchange rate risk. Exposure to this risk can be obtained by engaging in the carry trade. This risk premium is pinned down by pricing in local financial markets only if the international arbitrageur can use locally traded assets to perfectly offset this risk. Therefore, the absence of arbitrage has no bearing on this quantity if the exchange rate is not spanned by asset returns, that is, $u_{t+1} \neq 0$.

**Spanned exchange rate.** When the exchange rate is spanned, the international arbitrageur uses the two local markets to price the exchange rate risk. Hence, the two local SDFs play a role in the expected depreciation rate. This insight explains the presence of the novel adjustment term $\theta_t$ in equation (13) relative to the standard complete market formula (with $\theta_t = 0$). It also leads to a symmetric expression to equation (13) which emphasizes the foreign SDF $m^*_{t+1}$:

$$\delta_t = r_{ft} - r^*_{ft} - \text{cov}_t(m^*_{t+1}, \Delta s_{t+1}) + \frac{1}{2} \text{var}_t(\Delta s_{t+1}) + \theta^*_t,$$

with $\theta^*_t = \text{cov}_t(m^*_{t+1} - m_{t+1} - \Delta s_{t+1}, r_{p,t+1})$.

It is only when the local investors are able to replicate the exchange rate on their own that their individual Euler equations are enough to obtain the expected depreciation. If the home (foreign) investor can trade both spanning portfolios, then $\theta_t = 0$ ($\theta^*_t = 0$), and the standard complete market formula for the home (foreign) investor holds. For example, this situation occurs in settings in which the home investor acts as an international arbitrageur. For both home and foreign investors to price the exchange rate risk, they must be able to trade it, that is, the exchange rate is a global shock.
Unspanned exchange rate. When the exchange rate is not spanned by traded assets, its expectation can deviate from this formula by an arbitrary wedge,

$$E_t \Delta s_{t+1} = \delta_t + \psi_t. \quad (15)$$

This complete flexibility might lead to implausibly large trading profits for the international investor. One can be more informative about these deviations $\psi_t$ by imposing a condition that is stronger than the absence of arbitrage (Assumption 2).

**Assumption 3.** *(No quasi-arbitrage)* There is an upper bound $B$ on Sharpe ratios in international markets:

$$\forall r^I_{p,t+1} \in \mathcal{R}^I_{p,t+1} : \quad \left| E_t (r^I_{p,t+1}) + \frac{1}{2} \text{var}_t (r^I_{p,t+1}) - r_{ft} \right| \leq B \sqrt{\text{var}_t (r^I_{p,t+1})}. \quad (16)$$

This assumption restricts the Sharpe ratio of trades in international markets. Such bounds have a long tradition in finance, going back to Cochrane and Saa-Requejo (2000), Kozak, Nagel, and Santosh (2020), and Ross (1976). Intuitively, it can be motivated by the view that if trades that are too profitable emerged in equilibrium, new financial institutions would step in to take advantage of them. Under this view we obtain the following condition.

**Proposition 3.** Under Assumption 3, the wedge $\psi_t$ in the expected depreciation rate must satisfy:

$$\left| \psi_t + \frac{1}{2} \text{var}_t (u_{t+1}) \right| \leq B \sqrt{\text{var}_t (u_{t+1})} \equiv B \sqrt{(1 - R^2) \text{var}_t (\Delta s_{t+1})}, \quad (17)$$
where $R^2$ is the R-squared in the regression of $\Delta s_{t+1}$ on $r_{t+1}$ and $r_{t+1}^*$. 

This proposition limits possible expected depreciations in the case of an unspanned exchange rate. It indicates that deviations from the risk premium in the spanned case are bounded by the volatility of unspanned shocks.

3 Implications for the currency puzzles

In this section we discuss how the different market structures are capable of speaking to the key currency puzzles: cyclicality, volatility, and forward premium. Specifically, we take the households’ IMRS $m$ and $m^*$ as given. Further, we assume that these IMRS are such that three puzzles arise when the markets are complete and integrated. For example, that would be the case for IMRS that depend on directly observable macroeconomic variables. We then relax various assumptions associated with complete markets and evaluate the associated implications for the exchange rate.

While the literature typically focuses on the three puzzles simultaneously, our Propositions suggest that forward premium puzzle, i.e., an empirical measure of the expected depreciation rate, is affected by different features of the financial markets as compared to the cyclicality and volatility puzzles, which are driven by the properties of the exchange rate innovations. We follow the structure of the Propositions and start the discussion with the implications for innovations and their interaction with the volatility and cyclicality puzzles. Next, we proceed with the discussion of the exchange rate expectations and its relation to the currency risk premium.
3.1 Complete and integrated markets

The complete and integrated markets case is the relevant benchmark for our discussion as it forms the backbone of many models attempting to explain the currency puzzles. Financial markets are complete when investors have access to the full set of Arrow-Debreu securities in both markets. The financial markets are integrated when both households can trade all available securities. In this setting, \( e_{t+1}^g \) spans all possible risks. Then Proposition 1 implies

\[
\tilde{m}_{t+1}^* - \tilde{m}_{t+1} = \tilde{\Delta} s_{t+1}. \tag{18}
\]

Innovations to the depreciation rate must equal innovations to the difference of stochastic discount factors, completely pinning down exchange rate shocks.

This result leads to two puzzles about the behavior of the exchange rate. First, consider the variance of the depreciation rate:

\[
\text{var}_t(\Delta s_{t+1}) = \text{var}_t(m_{t+1}^* - m_{t+1}) \\
= \text{var}_t(m_{t+1}^*) + \text{var}_t(m_{t+1}) - 2 \text{cov}_t(m_{t+1}, m_{t+1}^*). \tag{19}
\]

Brandt, Cochrane, and Santa-Clara (2006) argue that this equation creates a volatility puzzle, with the exchange rate being not volatile enough. Typically observed Sharpe ratios on domestic assets imply highly volatile SDFs, much more so than exchange rate depreciation. The mild correlation of macro quantities across countries suggests that the IMRS are not correlated enough for the last term of equation (19) to offset this high variance and obtain realistic exchange rate risk.
Figure 2: Proposition 1 in complete and integrated markets

\[
\text{Variance } \text{var}_t(\Delta s_{t+1})
\]

\[
\text{Cyclicality } \text{cov}_t(m_{t+1}^* - m_{t+1}, \Delta s_{t+1})
\]

The figure illustrates implications of the complete market setting, labeled as CM, for the properties of depreciation rates. The point labeled Data is a stylized representation of the evidence regarding the depreciation rates. The grey area represents the infeasible combinations of volatility and criticality of depreciation rates due to the Cauchy-Schwarz inequality.

Further, the result (18) implies

\[
\text{var}_t(\Delta s_{t+1}) = \text{cov}_t(\Delta s_{t+1}, m_{t+1}^* - m_{t+1}),
\]

(20)

and \( \text{corr}_t(\Delta s_{t+1}, m_{t+1}^* - m_{t+1}) = 1 \). Changes in exchange rates must be perfectly correlated with relative marginal utilities of the domestic and foreign households, that is, the home currency depreciates in relatively good times for home investors. As pointed out by Backus and Smith (1993), this implication is counterfactual for various measures of good times leading to the cyclical puzzle.

We introduce a visualization of these puzzles which we will revisit for other market structures. Figure 2 demonstrates the tension in capturing volatility, on the vertical axis, and cyclicality, on the horizontal axis, at once. The point labeled ‘Data’ is a
stylized representation of our assumption that the selected $m$ and $m^*$ lead to the puzzles when the markets are complete and integrated. Equation (20) implies that the complete markets case should be on the $45^\circ$ line. We select a point on the vertical axis that is equal to $\text{var}_t(m^*_{t+1} - m_{t+1})$ and, according to the volatility puzzle, is higher than $\text{var}_t(\Delta s_{t+1})$ that we see in the data. The point labeled ‘CM’ shows what the complete market setting implies. The distance between Data and CM is the essence of the volatility and cyclicality puzzles in complete markets.

3.2 Risk-sharing

This section studies how the risk-sharing role of financial markets interacts with various market structures to resolve the cyclicality and volatility puzzles. We gradually depart from the complete and integrated market setting, which implies equation (18), by assuming that the global shocks $\epsilon_{t+1}$ are such that only the projection on one of the sides of equation (10) of Proposition 1 does not fully recover the variable being projected. This corresponds to departures from either market completeness or market integration at a time.

We start with the case when assets in each country span both IMRS. This would happen if households in each country can trade each other’s shocks to marginal utility. Such a situation does not necessarily require integrated markets, it can arise when $H \cap F = \emptyset$ and the markets are intermediated. In this case, equation (10)

\[ \text{cov}_t^2(\Delta s_{t+1}, m^*_{t+1} - m_{t+1}) \leq \text{var}_t(\Delta s_{t+1}) \cdot \text{var}_t(m^*_{t+1} - m_{t+1}). \]

We fix $\text{var}_t(m^*_{t+1} - m_{t+1})$ at the value indicated on the vertical axis. This leads us to a space of mathematically feasible combinations of volatility and cyclicality. The grey area on the chart indicates the infeasible combinations.

\[ \text{The Cauchy-Schwarz inequality implies that} \]

\[ \text{cov}_t^2(\Delta s_{t+1}, m^*_{t+1} - m_{t+1}) \leq \text{var}_t(\Delta s_{t+1}) \cdot \text{var}_t(m^*_{t+1} - m_{t+1}). \]
simplifies to
\[ \tilde{m}_{t+1}^* - \tilde{m}_{t+1} = \text{proj}(\tilde{m}_{t+1}^* - \tilde{m}_{t+1}|e_{t+1}^\theta) = \text{proj}(\tilde{\Delta}s_{t+1}|e_{t+1}^\theta) = g_{t+1}. \] (21)

The projection of the exchange rate innovations on global shocks, that is, its global component, is equal to the difference in shocks in the IMRS. That is, a regression of the exchange rate depreciation on the difference of log IMRS yields a coefficient of 1.

The unspanned component \( u_{t+1} \) is unbounded, and Equation (21) implies:
\[ \text{var}_t(\Delta s_{t+1}) = \text{var}_t(m_{t+1}^* - m_{t+1}) + \text{var}_t(\ell_{t+1} + u_{t+1}) \geq \text{var}_t(m_{t+1}^* - m_{t+1}). \]

This result deepens the volatility puzzle. If economies are entirely driven by global shocks, exchange rate volatility can only be larger than in the complete market case. The cyclicality is not affected because Equation (21) implies:
\[ \text{cov}_t(m_{t+1}^* - m_{t+1}, \Delta s_{t+1}) = \text{cov}_t(g_{t+1}, g_{t+1} + \ell_{t+1} + u_{t+1}) = \text{var}_t(g_{t+1}) = \text{var}_t(m_{t+1}^* - m_{t+1}). \]

We depict this situation in Figure 3 via a vertical line emanating from CM (complete markets case).

Spanned IMRS frequently arises in models with small number of common macro risks are traded in both countries. RBC models would fall into such category. One could consider various settings popular in finance, such as habits, or long-run risk as long as the setting features imperfect integration or intermediation.
The figure illustrates the trade-offs in matching volatility and cyclicality of the exchange rate. The point labeled Data is a stylized representation of the evidence regarding the depreciation rates. The point labeled CM represents the complete market setting where \( \text{var}(\Delta s_t) = \text{var}(m^*_{t+1} - m_{t+1}) = \text{cov}(m^*_{t+1} - m_{t+1}, \Delta s_{t+1}) \). The grey area represents the infeasible combinations of volatility and cyclicality of depreciation rates due to the Cauchy-Schwarz inequality. We consider a scenario when financial assets span the IMRS of households. The red line shows the feasible variance-cyclicality combinations in such a scenario. We also consider a scenario when the exchange rate is spanned by global shocks. The blue 45° line shows the feasible variance-cyclicality combinations in that case.

The next case that we consider is when the exchange rate can be traded directly by households, or when it can be spanned by the traded shocks. In this case, equation (10) simplifies to

\[
\text{proj}(\tilde{m}_{t+1} - \tilde{m}_{t+1} | \epsilon^g_{t+1}) = \text{proj}(\tilde{\Delta}s_{t+1} | \epsilon^g_{t+1}) = \tilde{\Delta}s_{t+1}. \tag{22}
\]

The projection of the difference in shocks in the IMRS on global shocks is equal to the innovations in the depreciation rate. That is, a regression of the difference of log IMRS on the exchange rate depreciation yields a coefficient of 1.

As a result, one can make only limited progress in addressing the exchange rate
puzzles. Specifically,

$$\text{var}_t(\Delta s_{t+1}) = \text{var}_t(\text{proj}(\tilde{m}_{t+1}^* - \tilde{m}_{t+1}|\epsilon_{t+1}^g) \leq \text{var}_t(m_{t+1}^* - m_{t+1}),$$

which potentially alleviates the volatility puzzle. As regards the cyclicality puzzle, the covariance of the depreciation rate with the SDF differential must equal the variance of the exchange rate,

$$\text{cov}_t(\Delta s_{t+1}, m_{t+1}^* - m_{t+1}) = \text{cov}_t(\Delta s_{t+1}, \text{proj}(m_{t+1}^* - m_{t+1}|\epsilon_{t+1}^g)) = \text{var}_t(\Delta s_{t+1}).$$

Therefore, just like in the complete markets case, there is a cyclicality puzzle. Figure 3 summarizes these constraints on the cyclicality and volatility of the depreciation rate: the exchange must be on the 45-degree line segment between the origin and the complete markets point.

Such a situation does not require market completeness or even full market integration as long as households in both countries can trade risk-free assets of both countries, that is, partial market integration. Lustig and Verdelhan (2019) consider this specific case (their Assumption 2) and our conclusions concur with theirs. We note that allowing the intermediary associated with the SDF $m^I$ from Assumption 2 to trade both risk-free bonds instead of households leads to different conclusions as the exchange rate is no longer spanned by the assets traded by the households, and proj($\Delta s_{t+1}|\epsilon_{t+1}^g) \neq \Delta s_{t+1}$.\(^{14}\)

\(^{14}\)Having said that, the correlation between relative discount factors in the domestic and foreign economies and depreciation rate is less than perfect:

$$\text{corr}_t(m_{t+1}^* - m_{t+1}, \Delta s_{t+1}) = \frac{\text{cov}_t(m_{t+1}^* - m_{t+1}, \Delta s_{t+1})}{\sqrt{\text{var}_t(m_{t+1}^* - m_{t+1}) \cdot \text{var}_t(\Delta s_{t+1})}} \leq \frac{\text{cov}_t(m_{t+1}^* - m_{t+1}, \Delta s_{t+1})}{\text{var}_t(\Delta s_{t+1})} = 1.$$
The results of this section imply that the exchange rate is constrained by the properties of the households’ IMRS as long as there is a single departure from the complete and integrated markets. If one modifies who can trade assets by allowing for imperfect market integration or intermediation as long as available set of assets is rich, then we end up in the spanned IMRS scenario. If one modifies which assets can be traded by allowing a limited set of assets as long as markets are integrated along these assets, then we find ourselves in the spanned exchange rate scenario. As Figure 3 summarizes, these cases maintain the joint volatility-cyclicality puzzles.

Thus, a natural question is whether relaxing both market completeness and integration at the same time could be helpful in resolving the tension between volatility and cyclicality of the exchange rate. We turn to this issue next and consider the most extreme form of relaxing completeness and integration: no role for risk sharing.

Workhorse models featuring financial shocks feature extreme segmentation by assumption. Households can trade the global risk-free asset, while the intermediary bears currency risk only. As a result, there are no global shocks and equation (10) of Proposition 1 becomes degenerate. There are no restrictions on the exchange rate innovations that are associated with the household. In this case, any point within the white cone in Figure 3 is feasible potentially resolving the puzzles.

This extreme form of market segmentation is not appealing as households can trade more than one asset and intermediaries participate in more than one market. One can, thus, contemplate markets structures that still relax both completeness and integration and not as restrictive as fully segmented markets as long as there no global shocks. For instance, households can trade the respective local assets as long as their returns are not related. Intermediaries can trade the assets in all local
markets combined with various derivative contracts. The question that we address in the next section is whether the two roles of finance: risk-sharing and financial shocks can fruitfully coexist in one framework so that risk-sharing would still impose helpful restrictions on the exchange rate while financial shocks offer the flexibility of capturing the puzzles.

### 3.3 Combining the two roles of financial markets

Proposition 1 and the previous section suggest that in order to resolve the cyclicality and volatility puzzles, the share of global shocks in the economy must be less than 100%. In order to assess how small this contribution should be, we revisit the graphic representation of the two puzzles in Figure 2.

The following relation quantifies the trade-off between cyclicality and volatility

\[
\text{volatility} \quad \text{cyclicality} \\
\var_t(\Delta s_{t+1}) \geq \text{var}(g_{t+1}) + \frac{\text{cov}_t(m^*_{t+1} - m_{t+1}, \Delta s_{t+1}) - \text{var}(g_{t+1})}{\text{var}_t(m^*_{t+1} - m_{t+1}) - \text{var}(g_{t+1})},
\]

which is visualized by the red cones on Figure 4.\(^{15}\) According to Proposition 1, the minimum variance of the exchange rate is \(\text{var}(g_{t+1})\); it is attained when the cyclicality has the same value, \(\text{cov}_t(m^*_{t+1} - m_{t+1}, \Delta s_{t+1}) = \text{var}(g_{t+1})\). This point corresponds to the vertex of the parabola, which lies on the segment of the 45-degree line between the origin and the complete markets point.

To reduce cyclicality of the exchange rate, one has to increase the variance of the exchange rate by adding non-global components, which can have arbitrary correlation

\(^{15}\)This relation is a consequence of the Cauchy-Schwartz inequality applied to the non-global components of the exchange rate and SDF differential, \((l_{t+1} + u_{t+1})\) and \((m^*_{t+1} - m_{t+1} - g_{t+1})\).
Figure 4: The impact of global shocks on variance and cyclicality of exchange rates

Note: The figure illustrates the trade-offs in matching volatility and cyclicality of the exchange rate (see text). The point labeled Data is a stylized representation of the evidence regarding the depreciation rates; the point labeled CM represents the complete market setting. The grey area represents infeasible combinations of volatility and cyclicality due to the Cauchy-Schwarz inequality.

with the SDFs. The lighter cone in the Figure reflects such an increase in non-global components. This is exactly what is captured by the inequality in equation (23), which corresponds to the space inside of the parabola in the figure.

Across models, the smaller is the role of global shocks, the larger is the space of combination of volatility and cyclicality. This leads us to a high-level conclusion about the two roles of financial markets. Extreme forms of market segmentation are not required to capture various features of the exchange rate realistically. Risk-sharing may also be present and usefully restrict the properties of the exchange rate. Thus, the two roles of finance can have a joint impact on the exchange rate and avoid both challenges associated with these role when considered individually: tight connection of the exchange rate to the macroeconomy and complete segmentation of the market participation by households and intermediaries.
3.4 The currency risk premium

Moving onto the forward premium puzzle, consider the application of Proposition 2 when markets are complete and integrated. In this case, $\tilde{r}_{t+1}$ and $\tilde{r}^*_{t+1}$ span $\Delta s_{t+1}$, and $\psi_t = 0$. The resulting risk premium for currency does generate deviations from uncovered interest parity (UIP). Standard international models struggle with generating the empirically observed magnitude of currency risk premium simultaneously with addressing the cyclicality and volatility puzzles.

Using the household Euler equations, we can express $r_{ft} - r^*_{ft}$ in equation (13) in terms of SDFs. As a result,

$$E_t \Delta s_{t+1} = E_t (m_{t+1} - m^*_{t+1}).$$

The mean depreciation rate must equal the mean of the difference of IMRS. Combining this equation with equation (18) we obtain the classic “asset market view” result for exchange rates

$$m^*_{t+1} - m_{t+1} = \Delta s_{t+1}, \quad (24)$$

which completely pins down the depreciation rate.

Interestingly, our derivation highlights that this result does not hinge on the classic notion of market completeness. It requires neither integration of markets nor spanning of all states of the world. Indeed, it is enough to be able to span $\tilde{m}^*_{t+1} - \tilde{m}_{t+1}$ and $\Delta s_{t+1}$ in each country in order to apply Propositions 1 and 2 and obtain the complete-market result of equation (24).
When the exchange rate is spanned, \( u_{t+1} = 0 \) mechanically because the asset spanning the depreciation rate is the depreciation rate itself, or, more precisely, the carry return on the strategy based on risk-free assets. Therefore, the implication of Proposition 2 coincides with the complete markets case. Equation (13) with \( \theta_t = 0 \) and equation (22) are also equivalent to the ones in Proposition 1 of Lustig and Verdelhan (2019).

When the IMRS are spanned the risk premium depends on whether the unspanned component of the currency depreciation rate is zero or not. If it is not, the risk premium is not restricted by the IMRS properties. If it is zero, equation (13) applies with \( \theta_t \neq 0 \). As we decrease the role of global shocks, this implication of Proposition 2 does not change.

The Proposition offers a simple path towards resolving the forward puzzle. As long as the underlying economic structure is such that the traded assets cannot span the depreciation rate, there are no constraints whatsoever on what its expectation should be. That opens a window to generating a theoretical currency premium that would be consistent with the observed one.

4 Empirical Analysis

In this section we investigate empirically the tightness of constraints on the exchange rate associated with risk-sharing. We consider a specific market structure by assuming that markets are intermediated and that households in each country can trade a broad collection of local assets. We limit the asset set in each country to sovereign bonds and various stock portfolios of that country.
We first demonstrate that exchange rates appear to have a large component $u_{t+1}$ unrelated to the returns of other traded assets. Then, we provide methods to characterize global shocks. Both of these exercises lead to the conclusion that, for the data we consider, the effects of risk-sharing are present, but play a modest role.

### 4.1 Data

We consider countries corresponding to G10 currencies between 2/1988 and 12/2022. We consider Germany as the representative country for the euro. Prior to the introduction of the euro, we use the German Deutschemark and splice these series together beginning in 1999. Our analysis focuses at the monthly frequency. We obtain exchange rates from WM/Reuters. Government bond yields are from each country’s central bank websites. Monthly bond returns are computed from bond yields using a second-order Taylor approximation. We obtain equity indices from MSCI. For each country, 10 different industry indices and 3 different style equity indices (Large + Mid Cap, Value, Growth) are sourced. Risk-free rates are calculated by dividing the 1-year yield by 12.

### 4.2 Is the exchange rate spanned?

Motivated by Proposition 2, we ask whether the depreciation rate is spanned by combination of domestic and foreign asset returns. We implement regressions of the form:

\[
\Delta s_{t+1} = \alpha + \beta' r_{t+1} + \beta'' r^*_{t+1} + u_{t+1}. \tag{25}
\]
Here the residual $u_{t+1}$ is a direct estimate of the unspanned component of the depreciation rate in equation (8).

We report the adjusted $R^2$ of these regressions. Exact spanning corresponds to an $R^2$ of 1. Furthermore, Proposition 3 highlights that $R^2$ is an appropriate measure of economic distance to the case of perfect spanning.

Table 1 reports the results. We always report the results for the combination of assets in the United States and another country. Each column in the table corresponds to that other country. Each row reflects a particular combination of assets used in the regression. Broadly speaking, we consider bonds and equities separately and in combination. Within each asset class, we zoom in on various individual contributions.

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<tr>
<td>Stocks</td>
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<tr>
<td>Mkt + Value/Growth</td>
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<td>6.75</td>
<td>5.06</td>
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<tr>
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<td>41.61</td>
<td>18.55</td>
<td>22.78</td>
<td>29.41</td>
<td>24.53</td>
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<td>19.61</td>
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<td>406</td>
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</tr>
</tbody>
</table>

Notes: The table reports the adjusted $R^2$ of a regression of the depreciation rate on various subsets of asset returns, as in equation (25). Domestic asset returns are in domestic currency; foreign asset returns are in foreign currency. Each column is a different country’s currency relative to the U.S. dollar. The first row uses only 10-year bonds, while the second entertains maturities between 2 and 10 years, obtained from various central banks. The next three row successively add various stock portfolios: the market (a combination of large and mid-cap stocks), value and growth portfolios, and 10 industry portfolios, all from MSCI. The final row considers all assets simultaneously.
Major asset classes do not span exchange rates. When looking at all assets together, the $R^2$s range from 25% for Switzerland to 45% for Canada (in each case combined with the U.S.). Most of the explanatory power comes from the equity side. For example in the case of Canada, the combination of market, value, growth and industry returns explain 42% of variation in the depreciation rate. While the market alone gets to some substantial amount of variation — 27% for Canada —, the addition of industry returns is particularly informative. Consistent with the evidence in Chernov and Creal (2023), bond returns only explain a modest amount of variation in exchange rates: between 0.2% and 7% for the 10-year bond alone, and between 7.2% and 14% for the combination of bonds at all maturities.

We refer to the observation that asset returns do not span changes in exchange rates as the financial exchange rate disconnect. While the $R^2$s we obtain from regressions on asset returns are meaningfully larger than their counterpart with real quantities, these magnitudes are much too small for leading to meaningful theoretical implications. Taking the strictest definition of absence of arbitrage, only a value of 1 leads to the relevance of Proposition 2. According to Proposition 3, even the largest numbers we measure imply a bound for the expected depreciation that is only $\sqrt{1 - 0.45} = 67\%$ of the bound with an $R^2$ of 0, not much tighter. Thus, observing the properties of returns on other assets is not informative about the expected currency depreciation rates.

The flipside of this conclusion is that the unspanned component of the depreciation rates, $u_{t+1}$, is large. In the context of models of intermediated markets, this result offers more flexibility in capturing realistic currency risk premium. As we discussed in section ??, partially integrated markets still imply tight restrictions on the currency premium because Proposition 2 holds.
4.3 Identifying global shocks

In this section we quantify the importance of global shocks $\epsilon^g_{t+1}$, which play the key role in Proposition 1. We do so using two empirical approaches. The undirected approach uses canonical correlation analysis (CCA) to identify these shocks from the asset return data. The directed approach starts from candidates for global shocks such as global macro and financial variables proposed in the literature.

4.3.1 Undirected approach

The CCA procedure finds a US and a foreign portfolio of asset returns consisting of $r_{t+1}$ and $r^*_t$, respectively, such that they have the highest correlation possible in sample. Next, conditional on finding this pair, the procedure looks for the next maximally correlated pair of portfolios that are orthogonal to their first pair. And so on.

According to Definition 1, global shocks would manifest themselves as innovations to portfolios with perfect correlation. In that case, Proposition 1 implies that projections of the depreciation rate and the difference in the SDFs on the global shocks coincide. In the data, even the largest correlation could be less than 1. So, in practice we would have to use an ad-hoc cut-off to decide which portfolios are sufficiently close to each other to constitute a measure of a global shock.

Table 2 reports the results. Each column represents a foreign country. For a given country, each row reports the canonical correlation between the assets of that country and the US assets, reported in order of importance, starting from the largest.
Table 2: Maximally correlated shocks across asset markets

<table>
<thead>
<tr>
<th>Rank</th>
<th>AU</th>
<th>CA</th>
<th>DE</th>
<th>JP</th>
<th>NO</th>
<th>NZ</th>
<th>SE</th>
<th>CH</th>
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<td>75.01</td>
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<td>60.41</td>
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<td>14.58</td>
<td>33.82</td>
<td>35.18</td>
<td>51.39</td>
</tr>
</tbody>
</table>

N 419 395 419 419 406 419 414 419 419

Notes: The table reports the correlation in % between the maximally correlated portfolios of asset returns between the U.S. and each country. The successive pairs of portfolio are orthogonal to each other, and obtained by canonical correlation analysis. Domestic asset returns are in domestic currency; foreign asset returns are in foreign currency. Each column is for a different country’s assets relative to the U.S. assets. The assets include government bonds of maturities between 2 and 10 years (obtained from various central banks) and various stock portfolios: the market (a combination of large and mid-cap stocks), value and growth portfolios, and 10 industry portfolios (from MSCI).
The values of the largest correlations range from 64% for New Zealand to 90% for Canada. In some cases lower ranked correlations are similar to the largest one, like for Canada or the UK. In other cases, the magnitude of correlation drops off quickly, e.g., for New Zealand or Norway. Strictly speaking, the evidence suggests that there are no global shocks amongst the assets that we consider.

As alluded to earlier, we can be more generous with interpreting the evidence in Table 2 and assign a value of 1 to each estimated correlation that is above a certain threshold. We consider the value of 60% as such a threshold. We denote the matrix of foreign portfolio weights by $w^*$; if there is only one global shock, this is a vector. We ask how much variation in the depreciation rate is explained by global shocks. We implement regressions of the form:

$$\Delta s_{t+1} = \alpha + \beta^g (w^* r^*_t) + \varepsilon_{t+1}. \quad (26)$$

The $R^2$ of such a regression is the fraction variance in exchange rate explained by global shocks. Because we assume that the corresponding domestic portfolio is perfectly correlated with its foreign counterpart, we do not include it in the regression. The regression residual is a direct estimate of the contribution of local and unspanned shocks to the depreciation rate, $\varepsilon_{t+1} = \ell_{t+1} + u_{t+1}$.

Combining with the results of regression (25), we can decompose variation in the depreciation rate into the contribution of global, local, and unspanned shocks. Specifically, we have $\text{var}(\beta^g (w^* r^*_t))$ for global shocks, and $\text{var}(\varepsilon_{t+1}) - \text{var}(u_{t+1})$ for local shocks. Figure 5 reports these quantities as fraction of the variation in depreciation rate; the contributions mechanically add up to 1.

For all currencies, at least half of the variation in exchange rates is unspanned by asset
Figure 5: Decomposition of exchange rate innovations, undirected

Notes: The figure reports the fraction of variance in exchange rates explained by globally traded shocks, local shocks, and shocks that are not spanned by asset returns. Each bar is a different country’s currency relative to the U.S. dollar. Global shocks are measured via returns of the assets that we use in our analysis using CCA.

returns — the financial disconnect we have already noted. Global shocks contribute very little to variation in the depreciation rates. The contribution is of the order of a few percentage points, with the exception of Australia and Canada with contributions around 25%. These estimates should be seen as an upper bound on the role of global shocks; remember that estimated global shocks include any pair of portfolios with correlation above 60%, far from the strict Definition 1.
4.3.2 Directed approach

Instead of being agnostic about the nature of global shocks we rely on macroeconomic research and assume that they are known. Specifically, we take VIX, GFC (Miranda-Agrippino and Rey, 2020), and EBP (Gilchrist and Zakrajsek, 2012) as such shocks. This approach requires a strong assumption that portfolios of traded assets in each economy can span these shocks.

For each country, we regress its depreciation rate vs USD on the global shocks. The $R^2$ from such a regression produce the fraction of the exchange rate variation due to global shocks. Next, we implement the regression in Equation (25) where the set of returns is complemented by the three global shocks to obtain the unspanned component. Naturally, it is going to be smaller than that in the previous section. The knowledge of the variation due to global and unspanned shocks delivers the variation due to local shocks.

Figure 6 reports the resulting decomposition of the variation in the exchange rate into the three types of shocks. The directed approach delivers somewhat larger contribution of global shocks, but qualitatively the conclusions are unchanged. The unspanned shocks represent the largest share of shocks. Contribution of the global shocks is modest with Australia and Canada, who approach 50%, being the exception.

Just like the financial disconnect leads to weak restrictions about the expected depreciation rate, the small role of global shocks implies weak restrictions about exchange rate risks. The flipside of this conclusion is that the settings of Section ??m which have sizable exposure to local shocks relative to global shocks are capable of resolving the cyclicality and volatility puzzles jointly. Given that partially integrated markets
Notes: The figure reports the fraction of variance in exchange rates explained by globally traded shocks, local shocks, and shocks that are not spanned by asset returns. Each bar is a different country’s currency relative to the U.S. dollar. Global shocks are measured via VIX, GFC, and EBP.

still impose tight restrictions on the currency risk premium, the intermediated market structure appears to be the most promising avenue for describing the equilibrium behavior of the exchange rate.

4.4 Relation to returns-based SDFs

Because of our discussion of currency puzzles, we have focused on the IMRS in the
earlier sections. Our Propositions apply to any SDFs, even the one inferred from no-arbitrage models, which are estimated using asset prices, or directly from the returns data in a model-free fashion. In such context, our results allow to address the following question. Suppose a researcher is given a set of domestic and foreign asset returns without the knowledge of market structure that generated them and without labels, i.e., a domestic asset could be a foreign asset converted to home currency. What can SDFs constructed from these returns say about the exchange rate?

This is a substantial generalization of the extant returns-based analysis where market integration is assumed (e.g., Maurer and Tran, 2021, Sandulescu, Trojani, and Vedolin, 2021), or the international SDF $m'$ is estimated (e.g., Chernov and Creal, 2023). In all of these cases, the exchange rate is a traded asset, and, thus, the estimated SDFs do not impose restrictions on the exchange rate. That is contrast to IMRS-based conclusions when the exchange rate is spanned. Equation (22) imposes equilibrium restrictions on the joint behaviour of the IMRS difference and the currency depreciation rate.

5 Conclusion

In this paper, we propose a general framework for understanding how much financial markets determine the behavior of exchange rates. Our theory accommodates many settings: complete or incomplete markets, arbitrary forms of market integration, or situations in which international financial trade happens through intermediaries. We characterize restrictions on the behavior of exchange rates due to the absence of
international arbitrage. These restrictions can be summarized by two conditions that share the simplicity of the complete market result while having richer implications.

We use these results to study many different market structures. We find that in theoretical settings where financial markets are informative about the exchange rate, they lead to the same counterfactual implications as in complete markets. In contrast, some structures, such as intermediated markets, do not impose much restrictions on exchange rates. This lack of structure is consistent with two properties of the data. First, there is a financial exchange rate disconnect: depreciation rates are not that correlated to asset returns. Second, few shocks are globally traded, and they explain even less of the variation in exchange rates. Thus, we conclude that intermediated market structures are the most promising avenues for modeling the equilibrium exchange rate.
References


Appendix

A Global shocks

A.1 Identification and construction

We show how to identify a basis for the set of global shocks $e_{t+1}^g$. We drop time indices for parsimony.

First, recall what canonical correlation analysis does.

**Definition 2.** Canonical correlation analysis identifies pairs $(\lambda_i, \lambda_i^*)$ for $i = 1, \ldots, K$ for some $K$ such that:

1. $\forall i \var(\lambda_i'r) \neq 0$
2. $\forall i \lambda_i'r = \lambda_i^*r^*$
3. $\forall i \neq j \lambda_i'r \perp \lambda_j'r$
4. $\forall r \in \text{span}(r), r^* \in \text{span}(r^*)$ if $\forall i r \perp \lambda_i'r$ and $r^* \perp \lambda_i'r$ then $r \neq r^*$

Then we show that this procedure identifies a basis of $e^g$.

**Lemma 1.** The collection $(\lambda_1'r, \ldots \lambda_K'r)$ identified by canonical correlation analysis is a basis of $e^g$.

**Proof.** By point 2 of Definition 2, all the $\lambda_i'r$ are in $e^g$. Thus, $\text{span}(\lambda_1'r, \ldots \lambda_K'r) \subset e^g$.

Let us show the other direction. Assume that there exists $r \in e^g$ such that $r \notin \text{span}(\lambda_1'r, \ldots \lambda_K'r)$. We can orthogonalize $r$ to all the $\lambda_i'r$ and obtain $\hat{r}$. Because $\hat{r}$ is a linear combination of $r$ and $\lambda_i'r$ which are all in $e^g$, it is also in $e^g$, and therefore in $\text{span}(r)$ and $\text{span}(r^*)$. By substituting $\hat{r}$ for both $r$ and $r^*$ in point 4 of Definition 2, we immediately obtain a contradiction. Therefore $\text{span}(\lambda_1'r, \ldots \lambda_K'r) \supset e^g$; the two sets are equal. By point 3 of the CCA definition, $\text{dim}(\text{span}(\lambda_1'r, \ldots \lambda_K'r)) = K$, so $(\lambda_1'r, \ldots \lambda_K'r)$ is indeed a basis of $e^g$. ■

Furthermore, we relate the dimension of $e^g$ to the rank of covariance matrices of $r$, $r^*$, and the two combined.
Lemma 2. The dimension of \( \epsilon^g \) is:

\[
\dim(\epsilon^g) = \text{rank}(\text{var}(r)) + \text{rank}(\text{var}(r^*)) - \text{rank}(\text{var}(r, r^*)).
\]

Proof. Observe that, by construction,

\[
\dim(\text{span}(r, r^*)) = [\dim(\text{span}(\epsilon^g)) + \dim(\text{span}(\epsilon))] + \{\dim(\text{span}(\epsilon^*))\}
\]

\[
= [\dim(\text{span}(r))] + \{\dim(\text{span}(r^*) - \dim(\epsilon^g))\}.
\]

Therefore,

\[
\dim(\epsilon^g) = \dim(\text{span}(r)) + \dim(\text{span}(r^*)) - \dim(\text{span}(r, r^*)),
\]

which yields the result.

A.2 Examples and counter-examples

It may be intuitively appealing to think about sources of common variation in domestic and foreign assets as global shocks. There is a critical difference between such intuition and the formal definition of global shocks, which requires replication of the exposure to such shock solely using assets of either country.

As an example, consider economies with \( N \) risky assets each, with all of these assets having exposure to a shock \( \epsilon_{t+1} : \tilde{r}_{i,t+1} = \alpha_i \epsilon_{t+1} + \beta_i \epsilon_{i,t+1}, \) and \( \tilde{r}^*_{i,t+1} = \alpha^*_i \epsilon_{t+1}, \) and all the shocks are orthogonal to each other. If \( \beta_i = 0 \) for at least one domestic asset \( i, \) then \( \epsilon_{t+1} \) is a global shock. If none of the \( \beta_i \) is equal to zero, then \( \epsilon_{t+1} \) is a global shock if \( N \to \infty \) which allows a portfolio of \( \tilde{r}_{i,t+1} \) to isolate \( \epsilon_{t+1} \) via diversification. If neither condition holds then \( \epsilon_{t+1} \) is not a global shock despite affecting common variation in domestic and foreign assets.
B Derivation of the main results

B.1 Portfolio approximation

To maintain tractability, we follow Campbell and Viceira (2002) and approximate the log portfolio excess returns relative to a risk-free rate $r_{ft}$:

$$r_{p,t+1} - r_{ft} = \log \left( \mathbf{w}' e^{r_{t+1} - r_{ft}} \right)$$

$$\approx \mathbf{w}' (r_{t+1} - r_{ft}) + \frac{1}{2} \mathbf{w}' \text{diag}(\Sigma_t) - \frac{1}{2} \mathbf{w}' \Sigma_t \mathbf{w},$$

(27)

where $\Sigma_t$ is the variance-covariance matrix of log returns. This approximation allows us to represent portfolios returns as linear combination of log returns. Importantly, it is stable by recombination, leading to the same result when applied in two steps or all at once for a portfolio of portfolios. The approximation becomes exact as time becomes continuous and the underlying data-generating process for returns converges to a purely diffusive stochastic process.

B.2 Two international portfolios

Two international portfolios are useful for the derivation of our main results.

**Carry trade.** One zero-cost portfolio, often referred to as carry, entails taking long and short positions in related assets:

$$R_{\text{carry},t+1} = R_{t+1} - R^*_{t+1} \cdot S_{t+1} / S_t.$$  

(28)

Traditionally, the traded assets are taken to be domestic and foreign risk-free (one-period) bonds. But carry does not have to be limited by that. For instance, Lustig, Stathopolous, and Verdelhan (2019) consider long-term bonds. More generally, one could use any pair of assets that are close to each other, e.g., $\text{corr}_t(r_{t+1}, r^*_{t+1}) \approx 1$.

The key characteristic of the carry trade is that it exposes the arbitrageur to currency risk.

**Lemma 3.** The conversion from foreign to home returns in the carry portfolio introduces exposure to currency risk, $\tilde{r}_{\text{carry},t+1} = \tilde{r}_{t+1} - \tilde{r}^*_{t+1} + \Delta S_{t+1}.$
Proof. We map the zero-cost portfolio (28) into the log approximation of a funded portfolio in equation (27) by adding a position in the risk-free asset:

\[ R_{p,t+1} \equiv R_{\text{carry},t+1} + R_{f,t} = R_{t+1} - R_{t+1}^* \cdot S_{t+1}/S_t + R_{f,t}. \]

The portfolio \( R_{p,t+1} \) corresponds to the weights \( w_1 = 1 \) in the domestic risky asset \( R_{t+1} \), \( w_2 = -1 \) in the foreign risky asset converted to USD, \( R_{t+1}^* \cdot S_{t+1}/S_t \), and \( w_3 = 1 \) in the domestic risk-free asset with \( \mathbf{w}_t = (w_1, w_2, w_3)' \). These weights lead to an expression for the log gross return relative to the risk-free rate \( R_{p,t+1}/R_{f,t} \):

\[ r_{\text{carry},t+1} \equiv r_{p,t+1} - r_{ft} = r_{t+1} - r_{t+1}^* - \Delta s_{t+1} + \text{cov}_t(r_{t+1} - r_{t+1}^* - \Delta s_{t+1}, r_{t+1}^* + \Delta s_{t+1}). \] (29)

Thus, the shocks to the exchange rate have an impact on the portfolio performance.

Differential carry. That carry is exposed to currency risk prompts us to consider another zero-cost portfolio, labeled as differential carry, which is long one unit of the domestic asset, and short one unit of the foreign asset, financed at the respective risk-free rates:

\[ R_{\text{diff},t+1} = (R_{t+1} - R_{ft}) - (R_{t+1}^* - R_{ft}^*) \cdot S_{t+1}/S_t. \] (30)

Intuitively, this portfolio does not introduce additional currency exposure because, in contrast to carry, only the foreign excess return is converted to USD. We demonstrate this formally in the following lemma.

Lemma 4. The conversion from foreign to US returns in the diff portfolio does not introduce additional exposure to currency risk, \( \tilde{r}_{\text{diff},t+1} = \tilde{r}_{t+1} - \tilde{r}_{t+1}^* \).

Proof. We map the zero-cost portfolio (30) into a funded portfolio to use the approximation of equation (27):

\[ R_{p,t+1} \equiv R_{\text{diff},t+1} + R_{f,t} = R_{t+1} - (R_{t+1}^* - R_{ft}^*) \cdot S_{t+1}/S_t. \]

The portfolio \( R_{p,t+1} \) corresponds to the weights \( w_1 = 1 \) in the domestic risky asset \( R_{t+1} \), \( w_2 = -1 \) in the foreign risky asset converted to USD, \( R_{t+1}^* \cdot S_{t+1}/S_t \), and \( w_3 = 1 \) in the foreign risk-free asset converted to USD, \( R_{ft}^* \cdot S_{t+1}/S_t \), with \( \mathbf{w}_z = (w_1, w_2, w_3)' \).
These weights lead to an expression for the relative log return:

\[ \text{r}_{\text{diff}+1} = \text{r}_{p,t+1} - \text{r}_{ft} = \text{r}_{t+1} - \text{r}_{ft} - (\text{r}_{t+1}^{*} - \text{r}_{ft}^{*}) - \text{cov}_{t}(\text{r}_{t+1}^{*}, \Delta s_{t+1}) + \text{cov}_{t}(\text{r}_{t+1}^{*}, \text{r}_{t+1} -\text{r}_{t+1}^{*}). \] (31)

Thus, only the covariance of the foreign return with the exchange rate has a material impact on portfolio performance, not the shocks to the exchange rate. ■

The disappearance of exchange rate risk for the diff returns is in part due to our portfolio approximation. In Appendix Section E, we confirm that this approximation is very tight empirically. We compare the excess returns on various stock portfolios and sovereign bonds in their origin currency and in converted currency. The correlation between the two monthly series is always around 99.9%.

**B.3 Proof of Proposition 1**

Consider one of the global shocks, \( \epsilon_{g,t+1}^{q} \). By definition 1, there exist two portfolios \( r_{p,t+1} \in r_{p,t+1} \) and \( r_{p,t+1}^{*} \in r_{p,t+1}^{*} \) such that \( \epsilon_{g,t+1}^{q} = r_{p,t+1} = r_{p,t+1}^{*} \).

The differential carry portfolio of Lemma 4 is in \( r_{p,t+1}^{I} \). In this case, the portfolio has no risk because \( r_{p,t+1} = r_{p,t+1}^{*} \). The shocks to foreign and domestic return perfectly offset each other. By assumption 2, the portfolio must have expected returns equal to the risk-free rate. That is:

\[ 0 = \text{E}_{t}[r_{p,t+1} - \text{r}_{ft}] - \text{E}_{t}[r_{p,t+1}^{*} - \text{r}_{ft}^{*}] - \text{cov}_{t}(r_{p,t+1}, \Delta s_{t+1}) + \text{cov}_{t}(r_{p,t+1}^{*}, r_{p,t+1} - r_{p,t+1}^{*}). \]

The last term is equal to 0 because \( r_{p,t+1} - r_{p,t+1}^{*} \) has no risk. We can replace the first two terms by covariances with the SDFs using the domestic and foreign Euler equations (5) and (6),

\[
0 = -\text{cov}_{t}(m_{t+1}, r_{p,t+1}) - \frac{1}{2}\text{var}_{t}(r_{p,t+1}) + \text{cov}_{t}(m_{t+1}^{*}, r_{p,t+1}^{*}) + \frac{1}{2}\text{var}_{t}(r_{p,t+1}^{*}) - \text{cov}_{t}(r_{p,t+1}^{*}, \Delta s_{t+1}).
\]

Remembering that both portfolio shocks are equal to \( \epsilon_{t+1}^{q} \), this expression simplifies to:

\[
\text{cov}_{t}(m_{t+1}^{*} - m_{t+1} - \Delta s_{t+1}, \epsilon_{t+1}^{g}) = 0.
\]
This equation is equivalent to

\[ \text{cov}(\tilde{m}^*_{t+1} - \tilde{m}_{t+1} - \tilde{\Delta}s_{t+1}, \epsilon^g_{t+1}) = 0, \]

which under log-normality implies equation (10).

Because this result holds for any global shock, it must also hold in terms of multivariate projections on all global shocks \( \epsilon^g_{t+1} \).

**B.4 Proof of Proposition 2**

Consider the carry portfolio of Lemma 3 constructed with a pair of portfolios \( r_{p,t+1} \in r_{p,t+1} \) and \( r^*_{p,t+1} \in r^*_{p,t+1} \) which span the exchange rate (equation (9)). In this case, the portfolio has no risk because \( \tilde{r}_{p,t+1} - \tilde{r}^*_{p,t+1} = \tilde{\Delta}s_{t+1} \). The shocks to foreign and domestic return perfectly offset exchange rate risk. By assumption 2, the portfolio must have expected returns equal to the risk-free rate. This corresponds to

\[ 0 = E_t[r_{p,t+1} - r^*_{p,t+1} - \Delta s_{t+1}] + \text{cov}_t(r_{p,t+1} - r^*_{p,t+1} - \Delta s_{t+1}, r^*_{p,t+1} + \Delta s_{t+1}). \]

The covariance term is equal to 0, because \( r_{p,t+1} = r^*_{p,t+1} - \Delta s_{t+1} \) has no risk. We can replace expected returns using the domestic and foreign Euler equations (5) and (6):

\[ E_t[\Delta s_{t+1}] = r_{ft} - \text{cov}_t(m_{t+1}, r_{p,t+1}) - \frac{1}{2} \text{var}_t(r_{p,t+1}) - r^*_{ft} + \text{cov}_t(m^*_{t+1}, r^*_{p,t+1}) + \frac{1}{2} \text{var}_t(r^*_{p,t+1}) = \delta_t \]

We replace \( \tilde{r}_{p,t+1} = \tilde{r}^*_{p,t+1} + \tilde{\Delta}s_{t+1} \):

\[ E_t[\Delta s_{t+1}] = r_{ft} - r^*_{ft} - \text{cov}_t(m_{t+1}, \Delta s_{t+1}) + \text{cov}_t(m^*_{t+1} - m_{t+1}, r^*_{p,t+1}) + \frac{1}{2} \text{var}_t(r^*_{p,t+1}) - \frac{1}{2} \text{var}_t(\Delta s_{t+1}) - \frac{1}{2} \text{var}_t(r^*_{p,t+1}) - \text{cov}_t(\Delta s_{t+1}, r^*_{p,t+1}) \]

\[ = r_{ft} - r^*_{ft} - \text{cov}_t(m_{t+1}, \Delta s_{t+1}) + \frac{1}{2} \text{var}_t(\Delta s_{t+1}) - \frac{1}{2} \text{var}_t(\Delta s_{t+1}) + \text{cov}_t(m^*_{t+1} - m_{t+1} - \Delta s_{t+1}, r^*_{p,t+1}). \]

This proves part b) of Proposition 2. If markets are fully integrated, all asset returns
are global shocks, and proposition 1 implies that the last term in the equation above is equal to 0, part a) of the proposition. If the exchange rate is not spanned by asset returns, it is impossible to construct a trade with expected returns involving the expected depreciation rate that is risk-free. Therefore, no arbitrage imposes no restriction on the expected depreciation rate.

B.5 Proof of Proposition 3

Recall our decomposition of the depreciation rate into a spanned and unspanned components, \(
\Delta s_{t+1} = E_t(\Delta s_{t+1}) + g_{t+1} + \ell_{t+1} + u_{t+1}
\). Because \(g_{t+1} + \ell_{t+1}\) is spanned by asset returns, there exists \(r_{p,t+1} \in \mathbf{r}_{p,t+1}\) and \(r^*_{p,t+1} \in \mathbf{r^*}_{p,t+1}\) such that \(\tilde{r}_{p,t+1} = g_{t+1} + \ell_{t+1}\). Using Lemma 3, we see that the risk of this portfolio is equal to \(\text{var}\(u_{t+1}\)\).

We apply Assumption 3 to relate this risk to the expected return of the carry trade.

\[
\left| E_t[r_{p,t+1} - r^*_{p,t+1} - \Delta s_{t+1}] + \text{cov}(r_{p,t+1} - r^*_{p,t+1} - \Delta s_{t+1}, r^*_{p,t+1} + \Delta s_{t+1}) + \frac{1}{2}\text{var}(u_{t+1}) \right| 
\leq B \sqrt{\text{var}(u_{t+1})}
\]

Examining the terms in the left-hand-side, we have:

\[
E_t[r_{p,t+1} - r^*_{p,t+1} - \Delta s_{t+1}] = \delta_t - E_t[\Delta s_{t+1}] = -\psi_t
\]

\[
\text{cov}(r_{p,t+1} - r^*_{p,t+1} - \Delta s_{t+1}, r^*_{p,t+1} + \Delta s_{t+1}) = \text{cov}(-u_{t+1}, r_{p,t+1} + u_{t+1})
= -\text{var}(u_{t+1})
\]

Plugging back into the inequality, we obtain:

\[
|\psi_t + \frac{1}{2}\text{var}(u_{t+1})| \leq B \sqrt{\text{var}(u_{t+1})}.
\]

C Propositions 1 and 2 are sufficient

We show that the results of Propositions 1 and 2 are not only necessary for the absence of international arbitrage — Assumption 2 — but also sufficient. Specifically we show the following.
Proposition 4. If:

1. Assumption 1 holds,
2. \( E(\tilde{m}_{t+1}^* - \tilde{m}_{t+1}|\epsilon_{t+1}^g) = E(\tilde{\Delta}_{s_{t+1}}|\epsilon_{t+1}^g) \),
3. (a) Either \( \exists r_{p,t+1}^s \in r_{p,t+1}^s, r_{p,t+1}^{s*} \in r_{p,t+1}^{s*} \) such that \( \tilde{\Delta}_{s_{t+1}} = r_{p,t+1}^s - r_{p,t+1}^{s*} \) and
   \[
   E_t(\Delta_{s_{t+1}}) = r_{f,t} - r_{f,t}^* - \text{cov}_t(m_{t+1}^s, \Delta_{s_{t+1}}) + \frac{1}{2} \text{var}_t(\Delta_{s_{t+1}})
   + \text{cov}_t(m_{t+1}^s - m_{t+1} - \Delta_{s_{t+1}}, r_{p,t+1}),
   \]
   (b) Or \( \forall r_{p,t+1}^s \in r_{p,t+1}^s, r_{p,t+1}^{s*} \in r_{p,t+1}^{s*}, \tilde{\Delta}_{s_{t+1}} \neq r_{p,t+1}^s - r_{p,t+1}^{s*} \)

then there are no arbitrage opportunities in international markets, Assumption 2 holds.

Proof. We proceed by contradiction. Assume that there exists an international arbitrage:

\[ \exists r_{p,t+1}^I \in r_{p,t+1}^I, \text{var}_t(r_{p,t+1}^I) = 0 \text{ and } E_t(r_{p,t+1}^I) \neq r_{f,t}, \]

and denote \( w \) and \( w^* \) the set of weights of such a portfolio on \( r_{t+1} \) and \( r_{t+1}^* \). Remember that \( 1'w + 1'w^* = 1 \). We consider the cases of 3a and 3b in turn.

Assume condition 3a holds. As a preliminary, note that this condition is equivalent to saying that a carry portfolio constructed with \( r_{p,t+1}^s \) and \( r_{p,t+1}^{s*} \) has no risk and no average return in excess of the risk-free rate. Consider the following portfolio: long \( w^t r_{t+1} \), long \( (1'w^*) r_{p,t+1}^s \), long \( w^{s*} (r_{t+1}^s + \Delta s_{t+1}) \), short \( (1'w^*) r_{p,t+1}^{s*} \). Because we have added and subtracted the same total weights, the new weights still add up to 1, so this is still a portfolio. Because this portfolio combines two risk-free portfolio, our assumed arbitrage and the risk-free carry trade, its expected return is the sum of the two expected returns, \( E_t(r_{p,t+1}^I) \). The total weight on foreign in the portfolio are \( 1'w^* - 1'w^* = 0 \). Therefore, this trade is a differential carry portfolio. Because it has no risk, its home and foreign leg offset each other. They form a global shock. Applying condition 1 in the proposition and Lemma 4 leads immediately to the result that the portfolio return must equal the risk-free rate. This contradicts the assumption that \( E_t(r_{p,t+1}^I) \neq r_{f,t} \).
Now assume that condition 3b holds. If $\mathbf{1}^\prime \mathbf{w}^* \neq 0$, then the arbitrage portfolio has a non-zero loading on $\Delta s_{t+1}$ in addition to the home and foreign returns. Because the portfolio is riskless this implies that we can find a pair of home and foreign returns that spans the depreciation rate, a contradiction of condition 3b. If $\mathbf{1}^\prime \mathbf{w}^* = 0$, then the two legs of the portfolio in their home currency perfectly offset each other. Their innovations constitute a global shock and applying condition 1 in the proposition jointly with Lemma 4 implies that the arbitrage portfolio has 0 expected return, a contradiction as well.

D Exact non-linear version of the propositions

Our proofs heavily rely on log-linearization of portfolio returns as described in Appendix B.1. In this section we address a question of how the propositions would change if the derivations are exact.

D.1 A version of Proposition 1

Consider two portfolios, domestic with returns $R_{p,t+1}$ and foreign with returns $R_{p,t+1}^*$ such that their innovations coincide with one of the global shocks, that is, they can be represented as $R_{p,t+1} = \alpha_t + R_{p,t+1}^*$. The local Euler equations imply:

$$E_t(M_{t+1} R_{p,t+1}) = 1,$$
$$E_t(M_{t+1}^* R_{p,t+1}^*) = 1.$$

The local Euler equations can be re-written as

$$E_t(R_{p,t+1}) = R_{ft} - cov_t \left( \frac{M_{t+1}}{E_t(M_{t+1})}, R_{p,t+1} \right) \quad (32)$$
$$E_t(R_{p,t+1}^*) = R_{ft}^* - cov_t \left( \frac{M_{t+1}^*}{E_t(M_{t+1}^*)}, R_{p,t+1}^* \right). \quad (33)$$

Now consider an intermediary whose SDF expressed in the units of domestic currency,
\( M_{t+1}^I \) satisfies the following Euler equations:

\[
E_t(M_{t+1}^I R_{ft}) = 1, \quad (34)
E_t(M_{t+1}^I R^*_{ft} S_{t+1}/S_t) = 1, \quad (35)
E_t(M_{t+1}^I (R_{p,t+1} - R_{ft})) = 0, \quad (36)
E_t(M_{t+1}^I (R^*_{p,t+1} - R^*_{ft}) S_{t+1}/S_t) = 0. \quad (37)
\]

The intermediary trades the zero-cost differential carry portfolio:

\[
0 = E_t \left( M_{t+1}^I \left[ (R_{p,t+1} - R_{ft}) - (R^*_{p,t+1} - R^*_{ft}) \cdot S_{t+1}/S_t \right] \right)
= E_t \left( M_{t+1}^I \left[ (R_{p,t+1} - R_{ft}) - (R_{p,t+1} - \alpha_t - R^*_{ft}) \cdot S_{t+1}/S_t \right] \right)
= E_t \left( M_{t+1}^I \left[ (R_{p,t+1} - R_{ft})(1 - S_{t+1}/S_t) + (\alpha_t + R^*_{ft} - R_{ft}) \cdot S_{t+1}/S_t \right] \right). \quad (38)
\]

Replace the risk-free rates by the expressions from the local Euler equations (32) and (33), divide the equation by \( E_t(M_{t+1}^I) \), and define

\[
cov^I_t \left( \frac{S_{t+1}}{S_t}, R^*_{p,t+1} \right) \equiv E_t \left( \frac{M_{t+1}^I}{E_t(M_{t+1}^I)} \left( R^*_{p,t+1} - R^*_{ft} \right) \frac{S_{t+1}}{S_t} \right) - E_t \left( \frac{M_{t+1}^I}{E_t(M_{t+1}^I)} \left( R^*_{p,t+1} - R^*_{ft} \right) \right) \cdot E_t \left( \frac{M_{t+1}^I}{E_t(M_{t+1}^I)} \frac{S_{t+1}}{S_t} \right),
\]

\[
E_t^I \left( \frac{S_{t+1}}{S_t} \right) \equiv E_t \left( \frac{M_{t+1}^I}{E_t(M_{t+1}^I)} \frac{S_{t+1}}{S_t} \right) = \frac{R_{ft}}{R^*_{ft}}.
\]

Then

\[
0 = -cov^I_t \left( \frac{S_{t+1}}{S_t}, R^*_{p,t+1} \right) + cov_t \left( \frac{M_{t+1}^*}{E_t(M_{t+1}^*)} - \frac{M_{t+1}^I}{E_t(M_{t+1}^I)}, R^*_{p,t+1} \right).
\]

(We replace \( R_{p,t+1} \) with \( R^*_{p,t+1} \) in the \( cov^I_t \) term because of our assumption about \( R_{p,t+1} \) and \( R^*_{p,t+1} \).)
This expression implies

\[
\text{cov}_t \left( \frac{M_{t+1}^*}{E_t(M_{t+1})} - \frac{M_{t+1}}{E_t(M_{t+1})}, \frac{S_{t+1}/S_t}{E_t(S_{t+1}/S_t)}: R_{p,t+1}^* \right) = \text{cov}_t \left( \frac{S_{t+1}/S_t}{E_t(S_{t+1}/S_t)}: R_{p,t+1}^* \right) \]

(38)

As we noted in section B.1, the log approximation that we use in Proposition 1 becomes exact if time is continuous and the data-generating process converges to a pure diffusion. Under such scenario, the covariance in the equation above is observable, and, thus, has the same value with and without risk adjustment (via \(M_{t+1}^I\)). As a result, \(W = 0\). Also, each Arrow-Debreu claim makes the corresponding state global. For such global risk \(W = 0\).

Further, the projection result depends on the knowledge of intermediary’s SDF, \(M_{t+1}^I\) via the term with \(\text{cov}_t^I\). The log approximation relies only on the existence of such SDF, due to Assumption 2, and allows us to be agnostic about its actual values.

The first term in the second line is equal to \(R_{ft}^* \cdot QRP_t\), where \(QRP_t\) is the quanto-implied risk premium of Kremens and Martin (2019). Its role in our paper is different from that of these authors. They use it to approximate the currency risk premium assigned by the intermediary, \(R_{ft}^* E_t(S_{t+1}/S_t) - R_{ft} = -R_{ft} \text{cov}_t(M_{t+1}^I, R_{ft}^* \cdot S_{t+1}/S_t)\). Here it measures the gap in projections of the relative discount factor and the depreciation rate on global risks.

**D.2 A version of Proposition 2**

Consider two portfolios, domestic with returns \(R_{p,t+1}\) and foreign with returns \(R_{p,t+1}^*\) such that their innovations span the exchange rate, that is, they can be represented as \(R_{p,t+1} = \alpha_t + R_{p,t+1}^* S_{t+1}/S_t\). The local Euler equations (32) and (33) hold for these portfolios. Also, we consider a (domestically funded) intermediary whose SDF, \(M_{t+1}^I\), satisfies the following Euler equations:

\[
E_t(M_{t+1}^I R_{p,t+1}) = 1, \\
E_t(M_{t+1}^I R_{p,t+1}^* S_{t+1}/S_t) = 1.
\]

59
First, we show that $\alpha_t = 0$. The intermediary can form a zero-cost carry portfolio:

$$0 = E_t \left( M^l_{t+1} [R_{p,t+1} - R^*_{p,t+1} S_{t+1}/S_t] \right) = \alpha_t E_t \left( M^l_{t+1} \right).$$

Therefore, expected return on the carry portfolio is equal to zero:

$$0 = E_t \left( R_{p,t+1} - R^*_{p,t+1} S_{t+1}/S_t \right)$$

$$= R_{ft} - \text{cov}_t \left( \frac{M_{t+1}}{E_t(M_{t+1})}, R_{p,t+1} \right) - E_t(R^*_{p,t+1}) E_t(S_{t+1}/S_t) - \text{cov}_t(R^*_{p,t+1}, S_{t+1}/S_t)$$

$$= R_{ft} - \text{cov}_t \left( \frac{M_{t+1}}{E_t(M_{t+1})}, R_{p,t+1} \right) - \text{cov}_t(R^*_{p,t+1}, S_{t+1}/S_t)$$

$$- \left[ R^*_{ft} - \text{cov}_t \left( \frac{M^*_t}{E_t(M^*_t)}, R^*_{p,t+1} \right) \right] E_t(S_{t+1}/S_t),$$

where we substituted the local Euler equations (32) and (33) in lines 2 and 5, respectively. This equation implies the currency risk premium:

$$R^*_{ft} E_t \left( \frac{S_{t+1}}{S_t} \right) - R_{ft} = -\text{cov}_t \left( \frac{M_{t+1}}{E_t(M_{t+1})}, R^*_{p,t+1} \frac{S_{t+1}}{S_t} \right)$$

$$+ \text{cov}_t \left( \frac{M_{t+1}}{E_t(M^*_t)}, R^*_{p,t+1} \right) E_t \left( \frac{S_{t+1}}{S_t} \right) - \text{cov}_t \left( R^*_{p,t+1}, \frac{S_{t+1}}{S_t} \right)$$

$$- R^*_{ft} \text{cov}_t \left( \frac{M_{t+1}}{E_t(M_{t+1})}, \frac{S_{t+1}}{S_t} \right) - \text{cov}_t \left( \frac{M_{t+1}}{E_t(M_{t+1})}, [R^*_{p,t+1} - R_{ft}] \frac{S_{t+1}}{S_t} \right)$$

$$+ \text{cov}_t \left( \frac{M_{t+1}}{E_t(M^*_t)} E_t \left( \frac{S_{t+1}}{S_t} \right) - \frac{S_{t+1}}{S_t}, R^*_{p,t+1} \right)$$

$$= -R^*_{ft} \text{cov}_t \left( \frac{M_{t+1}}{E_t(M_{t+1})}, \frac{S_{t+1}}{S_t} \right)$$

$$+ \text{cov}_t \left( \frac{M_{t+1}}{E_t(M_{t+1})} - \frac{M_{t+1}}{E_t(M^*_t)} - \frac{S_{t+1}/S_t}{E_t(S_{t+1}/S_t)}, R^*_{p,t+1} \right) E_t \left( \frac{S_{t+1}}{S_t} \right)$$

$$- \text{cov}_t \left( \frac{M_{t+1}}{E_t(M_{t+1})}, [R^*_{p,t+1} - R_{ft}] \left[ \frac{S_{t+1}}{S_t} - E_t \left( \frac{S_{t+1}}{S_t} \right) \right] \right),$$
where in the first line we take advantage of spanning and replace $R_{p,t+1}$ with $R_{p,t+1}^* S_{t+1}/S_t$; the third line is obtained from the first by adding and subtracting the leading term in line 3; the fourth line is obtained by combining the two terms in the second line; the 6th and 7th lines are obtained by adding and subtracting $\text{cov}_t(M_{t+1}/E_t(M_{t+1}), R_{p,t+1}^*)$.

The term $B$ in the seventh line is the domestic household’s risk premium for quanto exposure and disappears in the log-normal approximation. Also, $B = 0$ if $R_{p,t+1}^*$ happens to be $R_{ft}^*$, that is, domestic household can trade foreign risk-free bond. The term $A$ in the sixth line is equal to zero in this case as well.

Next, if financial markets are integrated then the innovation to $R_{p,t+1}^*$ is global shock. Then, equation (38) from the non-linear version of Proposition 1 implies that

$$A = W \cdot E_t(S_{t+1}/S_t).$$

As is the case for Proposition 1, the log approximation treats this term as close to zero.

It might appear that departure from log-normality in the case of integrated markets leads to two extra terms, $A$ and $B$. In fact, when markets are integrated $A - B$ can be simplified to a term with a single source of departures from zero. Indeed, we obtain

$$A - B = \text{cov}_t \left( \frac{S_{t+1}/S_t}{E_t(S_{t+1}/S_t)}, R_{p,t+1}^* \right) - \text{cov}_t \left( \frac{S_{t+1}/S_t}{E_t(S_{t+1}/S_t)}, R_{p,t+1}^* \right)$$

$$- \text{cov}_t \left( \frac{M_{t+1}}{E_t(M_{t+1})} - 1, [R_{p,t+1}^* - R_{ft}^*] \left[ \frac{S_{t+1}/S_t}{E_t(S_{t+1}/S_t)} - 1 \right] \right)$$

$$= E_t \left( \frac{M_{t+1}}{E_t(M_{t+1})} [R_{p,t+1}^* - R_{ft}^*] \left[ \frac{S_{t+1}/S_t}{E_t(S_{t+1}/S_t)} - 1 \right] \right)$$

Thus, $A - B$ is close to zero when the intermediary pricing the global (quanto) risk the same way as the domestic household.

If there is no spanning, $R_{p,t+1} \neq \alpha_t + R_{p,t+1}^* S_{t+1}/S_t$, then it is impossible to find a risk-free strategy and derive restrictions on the currency risk premium.
Table 3: Correlation between excess returns converted in different currencies: foreign stocks

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<tr>
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<tr>
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Notes: The table reports the correlation (in %) between the excess return on various stock indices expressed in their home currency and converted to U.S. dollar. The portfolios include the market (a combination of large and mid-cap stocks), value and growth portfolios, and 10 industry portfolios, all from MSCI. Each column corresponds to a different country.

E Evaluating the portfolio approximation

We report the correlation (in %) between the excess return on various stock portfolios —Table 3— and bonds of different maturities —Table 5— in their origin currency and converted to U.S. dollars. Tables 4 and 6 start from the U.S. version of these portfolios and converts them to foreign currency. These correlations are pervasively extremely high, almost all over 99.9%.
Table 4: Correlation between excess returns converted in different currencies: U.S. stocks

<table>
<thead>
<tr>
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<td>99.92</td>
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</table>

Notes: The table reports the correlation (in %) between the excess return on various stock indices expressed in the U.S. dollars and converted to foreign currency. The portfolios include the market (a combination of large and mid-cap stocks), value and growth portfolios, and 10 industry portfolios, all from MSCI. Each column corresponds to a different country.
Table 5: Correlation between excess returns converted in different currencies: foreign bonds

<table>
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<tr>
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<td>99.96</td>
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</table>

Notes: The table reports the correlation (in %) between the excess return on government bonds of different maturity expressed in their home currency and converted to U.S. dollars. Bond returns are constructed from yields obtained from each country’s central bank. Each column corresponds to a different country.
Table 6: Correlation between excess returns converted in different currencies: U.S. bonds

<table>
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<tr>
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</thead>
</table>

Notes: The table reports the correlation (in %) between the excess return on U.S. government bonds of different maturity expressed in U.S. dollars and converted to foreign currency. Bond returns are constructed from yields obtained from the Federal Reserve. Each column corresponds to a different country.