University of California, Davis
Department of Economics
Microeconomics

Date: August 12, 2021
Time: 3 hours
Reading Time: 20 minutes

PRELIMINARY EXAMINATION FOR THE Ph.D. DEGREE

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Please answer **two** of the following three questions.
If you answer more than two questions please indicate which two you want to be graded.

QUESTION 1

It is common knowledge between a Seller (he) and a Buyer (she) that the Seller owns an object that is worth \$0 to him and is worth \$$v$ to the Buyer. We consider three versions of the game. In all games the payoffs are as follows: if the sale does not take place then both players get a payoff of 0; if the sale takes place at price p then the Seller's payoff is p and the Buyer's payoff is $v-p$.

Version 1 (simultaneous game). Let $\mathbb{N}$ denote the set of non-negative integers. The two players simultaneously announce prices $p_S \in \mathbb{N}$ (Seller's price) and $p_B \in \mathbb{N}$ (Buyer's price). If $p_B \geq p_S$, the Buyer gets the object and pays p_S to the seller. If $p_S > p_B$ then the game ends with no sale taking place. Assume that the value of the object to the Buyer is an integer $v \geq 2$.

- **(a)** Does the Seller have any strictly dominated strategies? If Yes, find one and mention the strategy that dominates it; if No explain why not.
- **(b)** Does the Buyer have any strictly dominated strategies? If Yes, find one and mention the strategy that dominates it; if No explain why not.
- **(c)** Does the Seller have any weakly dominated strategies? If Yes, find one and mention the strategy that dominates it; if No explain why not.
- **(d)** Does the Buyer have any weakly dominated strategies? If Yes, find one and mention the strategy that dominates it; if No explain why not.
- **(e)** Find all the pure-strategy Nash equilibria and prove that they are the only Nash equilibria.

Version 2 (two-stage game). In stage 1 the Seller announces a price $p_1 \in \mathbb{N}$, which the Buyer accepts or rejects. If the Buyer accepts then she gets the object and pays p_1 to the Seller. If the Buyer rejects, then in stage 2 the Seller makes a second offer $p_2 \in \mathbb{N}$, which the Buyer accepts or rejects. If the Buyer accepts then she gets the object and pays p_2 to the Seller while if the Buyer rejects the game ends with no sale. Assume that in period 1 the value of the object to the Buyer is an integer $v \geq 2$ but in period 2 it decreases to $v-1-\varepsilon$ with $0<\varepsilon<1$.

- **(f)** Find the backward-induction solution.
- **(g)** Are there any Nash equilibria that differ from the backward-induction solution? If Yes, find one, if No explain your answer.

Version 3 (n-stage game with $n \geq 3$). In stage 1 the Seller announces a price $p_1 \in \mathbb{N}$, which the Buyer accepts or rejects. If the Buyer accepts then she gets the object and pays p_1 to the Seller. If the Buyer rejects, then in stage 2 the Seller makes a second offer $p_2 \in \mathbb{N}$, which the Buyer accepts or rejects. If the Buyer accepts then she gets the object and pays p_2 to the Seller. If the Buyer rejects, then in stage 3 the Seller makes a third offer $p_3 \in \mathbb{N}$, which the Buyer accepts or rejects. If the Buyer accepts then she gets the object and pays p_3 to the Seller. If the Buyer rejects, then we proceed to stage 4, etc. If in the last stage (stage n) if the Buyer rejects the last offer then the game ends with no sale. Assume that in period 1 the value of the object to the Buyer is and integer $v \geq n+1$ and the value in period $t>1$ it $v-(t-1)-\varepsilon$ with $0<\varepsilon<1$.

- **(h)** Find the backward-induction solution.

Question 2: Price Risk and Output

Consider the case of a firm that uses two inputs in the production of its output. Specifically, for the production function

$$q = f(k, \ell) = \sqrt{\min\{k, \ell\}},$$

where q represents output of some commodity and k and ℓ represent input of capital and labor, respectively, the firm is

$$\mathcal{F} = \{y \in \mathbb{R}^3 \mid y_2 \leq 0, y_3 \leq 0, y_1 \leq f(-y_2, -y_3)\}.$$

The producer is an expected-utility maximizer, with strictly increasing cardinal utility index $u(x)$.

(a) Argue that the firm satisfies decreasing returns to scale:

(b) Suppose that the prices of capital and labor are, respectively, $r > 0$ and $w > 0$. Solve the cost minimization problem

$$\min_{k,\ell}\{rk + w\ell : f(k, \ell) \geq q\},$$

and determine the optimal input demands and the cost function.

(c) Suppose that $r = w = 1/4$, so that the total cost of producing q units is $c(q) = \frac{1}{2}q^2$, and that the price is known to be $\bar{P} > 0$ per unit. Find the optimal output level, $\bar{q}$, and resulting profit.

(d) Suppose now that the price the individual face, P, is a random variable and can be 0 or $2\bar{P}$ with probability 1/2. Write the first-order condition that characterizes the optimal output level, $\hat{q}$. Do you expect $\hat{q}$ to be smaller or larger than $\bar{q}$? Why?

(e) Suppose, in particular, that $u(x) = \log(5/8 + x)$ and $\bar{P} = 1$. Argue that $\hat{q} = 1/2$, by showing that this value satisfies the first-order condition found in the previous part. Find $\bar{q}$ and compare it to $\hat{q}$. Is your intuition of the previous part confirmed?

(f) Suppose that someone offers the producer a future contract that locks her price at $\bar{P}$. Write an equation that characterizes the maximum value that she would be willing to pay for this future.

Question 3: Dynamic General Equilibrium

Consider an exchange economy with society $\mathcal{I} = \{1, \dots, I\}$. There are $L + K$ commodities and trade takes place in two periods:

1. In the morning, L commodities are traded. Individual i is endowed with ω^i_ℓ units of commodity $\ell = 1, \dots, L$, and her consumption is x^i_ℓ. The price per unit of commodity ℓ is p_ℓ.

2. In the afternoon, the other K commodities are traded. The endowment and consumption of commodity $k = 1, \dots, K$ by individual i are ψ^i_k and y^i_k, respectively. The price per unit of commodity k is q_k.

3. In the evening, each agent consumes. If agent i has purchased the bundle $x = (x_1, \dots, x_L)$ in the morning and the bundle $y = (y_1, \dots, y_K)$ in the afternoon, her utility in the evening is $u^i(x) + v^i(y)$.

In the morning, besides trading the corresponding commodities, individual i chooses an amount m^i of nominal savings. If positive, m^i becomes nominal wealth in the afternoon; if negative, it is nominal debt that the agent must honor.

In the afternoon, given the prices $q = (q_1, \dots, q_K)$ and her savings m^i, agent i solves the problem

$$\max_{y \in \mathbb{R}^K_+} \left\{ v^i(y) \, : \, q \cdot y \leq q \cdot \psi^i + m^i \right\}, \tag{1}$$

where $\psi^i = (\psi^i_1, \dots, \psi^i_K)$. In the morning, given $p = (p_1, \dots, p_L)$ and anticipating q, she solves

$$\max_{(x,m) \in \mathbb{R}^L_+ \times \mathbb{R}} \left\{ u^i(x) + V^i(m, q) \, : \, p \cdot x + m \leq p \cdot \omega^i \right\}, \tag{2}$$

where $\omega^i = (\omega^i_1, \dots, \omega^i_L)$ and

$$V^i(m, q) = \max_{y \in \mathbb{R}^K_+} \left\{ v^i(y) \, : \, q \cdot y \leq q \cdot \psi^i + m \right\}.$$

The minister of finance of this economy is worried that the agents may be acting silly. She would prefer it if, instead of solving the two problems (2) and (1), agent i solves the intertemporal problem

$$\max_{(x,y,m) \in \mathbb{R}^L_+ \times \mathbb{R}^K_+ \times \mathbb{R}} \left\{ u^i(x) + v^i(y) \, : \, p \cdot x + m \leq p \cdot \omega^i \text{ and } q \cdot y \leq q \cdot \psi^i + m \right\}. \tag{3}$$

Not having taken the second-year GE course in his Ph.D., the dean of the most prominent economics department in the economy is worried that in the model he learned in the first-year course, people were assumed to solve the static problem

$$\max_{(x,y) \in \mathbb{R}^L_+ \times \mathbb{R}^K_+} \left\{ u^i(x) + v^i(y) \, : \, p \cdot x + q \cdot y \leq p \cdot \omega^i + q \cdot \psi^i \right\}, \tag{4}$$

as if all trade took place at the same time.

There are three definitions of equilibrium for this economy.

1. The definition that the dean understands is:[1] a tuple $(p, q, \mathbf{x}, \mathbf{y})$, where $\mathbf{x} = (x^1, \dots, x^I)$ and $\mathbf{y} = (y^1, \dots, y^I)$, is a *static equilibrium* if
   - for each individual, the pair (x^i, y^i) solves the static problem (4);
   - $\sum_i x^i = \sum_i \omega^i$ and $\sum_i y^i = \sum_i \psi^i$.
2. The minister would wish that the following was the definition of equilibrium: a tuple $(p, q, \mathbf{x}, \mathbf{y}, \mathbf{m})$, where $\mathbf{m} = (m^1, \dots, m^I)$, is an *intertemporal equilibrium* if
   - for each individual the triple (x^i, y^i, m^i) solves the intertemporal problem (3);
   - $\sum_i x^i = \sum_i \omega^i$, $\sum_i y^i = \sum_i \psi^i$, and $\sum_i m^i = 0$.
3. The actual definition of equilibrium is: a tuple $(p, q, \mathbf{x}, \mathbf{y}, \mathbf{m})$ is a *dynamic equilibrium* if
   - for each individual the pair (x^i, m^i) solves the morning problem (2), and the bundle y^i solves the afternoon problem (1);
   - $\sum_i x^i = \sum_i \omega^i$, $\sum_i y^i = \sum_i \psi^i$, and $\sum_i m^i = 0$.

The definition of efficiency, on the other hand, does not change: allocation $(\mathbf{x}, \mathbf{y})$ is efficient if there does not exist another allocation $(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})$ such that $u^i(\tilde{x}^i) + v^i(\tilde{y}^i) \geq u^i(x^i) + v^i(y^i)$ for all i, with strict inequality for some.

Assume that for all i, the functions u^i and v^i are of class $\mathbb{C}^2$, differentiably strictly increasing and differentiably strictly concave.

(a) Argue that if an interior allocation $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$ is efficient, then there exist vectors $\delta \in \mathbb{R}^L_{++}$ and $\gamma \in \mathbb{R}^K_{++}$ and a profile of scalars $(\mu^1, \dots, \mu^I) \in \mathbb{R}^I_{++}$ such that

$$\mu^i Du^i(\bar{x}^i) = \delta \ \text{ and } \ \mu^i Dv^i(\bar{y}^i) = \gamma \tag{5}$$

for all i.[2]

(b) Let $(p, q, \mathbf{x}, \mathbf{y})$ be a static equilibrium with interior allocation. Argue that allocation $(\mathbf{x}, \mathbf{y})$ satisfies the necessary condition for efficiency written in part (a).

(c) Let $(p, q, \mathbf{x}, \mathbf{y}, \mathbf{m})$ be an intertemporal equilibrium with interior allocation. Argue that allocation $(\mathbf{x}, \mathbf{y})$ satisfies the necessary condition for efficiency written in part (a).

(d) Let $(p, q, \mathbf{x}, \mathbf{y}, \mathbf{m})$ be a dynamic equilibrium with interior allocation. Argue that allocation $(\mathbf{x}, \mathbf{y})$ satisfies the necessary condition for efficiency written in part (a).[3]

[1] This is the definition that we learned in the course.

[2] In these expressions, D denotes the differentiation operator: given a function $f : X \to \mathbb{R}$, $Df(x)$ is its gradient, evaluated at the point $x \in X$.

[3] You may want to invoke the Envelope Theorem, which appears below.

Suppose now that the sets $D \subseteq \mathbb{R}^K$ and $\Omega \subseteq \mathbb{R}$ are open and finite-dimensional, and that $v : D \times \Omega \to \mathbb{R}$ and $g : D \times \Omega \to \mathbb{R}$, and consider the following problem: given $m \in \Omega$, let

$$V(m) = \max_{y \in D} \{v(y, m) : g(y, m) = 0\}.$$

Suppose that the second-order conditions are satisfied. Then,

Theorem (The Envelope Theorem). *Function V is continuously differentiable and*

$$V'(m) = \frac{\partial \mathcal{L}}{\partial m}(y(m), m),$$

where

$$\mathcal{L}(y, m) = v(y, m) + \eta g(y, m)$$

is the Lagrangean function of V.

ANSWER KEYS Micro Prelim August 12, 2021

QUESTION 1 ANSWER KEYS

(a) No. For any $p_B \geq 1$ (no matter whether such a p_B is rational or irrational for the Buyer), $p_S > p_B$ gives zero payoff to the Seller, $p_S = p_B$ gives him a positive payoff equal to p_B and $p_S < p_B$ gives him a payoff of $p_S < p_B$; thus $p_S = p_B$ gives the largest payoff to the Seller when the Buyer chooses $p_B \geq 1$. If $p_B = 0$ then any strategy of the Seller gives the Seller zero payoff.

(b) No. Even a $\hat{p}_B > v$ is not strictly dominated, because if $p_S < v$ then the Buyer's payoff is $v - p_S > 0$, for any $p_B \geq p_S$ (thus also with $\hat{p}_B$) and zero for any $p_B < p_S$.

(c) Yes: $p_S = 0$ is weakly dominated by any $p_S > 0$.

(d) Yes: any $p_B > v$ is weakly dominated by $p_B = v$.

(e) Every pair (p_S, p_B) with $p_S = p_B \leq v$ is a Nash equilibrium. The only other Nash equilibria are the pairs $(p_S, 0)$ with $p_S \geq v$.

Pairs (p_S, p_B) with $p_S < p_B$ are not Nash equilibria because the Seller can increase his payoff from 0 to p_B by switching to $p_S = p_B$; pairs (p_S, p_B) with $p_S > p_B \geq 1$ are not Nash equilibria because the Seller can increase his payoff from 0 to p_B by switching to $p_S = p_B$; pairs $(p_S, 0)$ with $v > p_S > 0$ are not Nash equilibria because the Buyer can increase her payoff by switching to $p_B = p_S$.

(f) In stage 2 the Buyer will accept any $p_2 \leq v - 2$ and reject any other offer. Thus at the beginning of period 2 the Seller will offer $p_2 = v - 2$ (and the Buyer will accept). Hence in period 1 the Seller knows that he can guarantee himself a payoff of $v - 2$ in period 2, leaving the Buyer with a payoff of $v - 1 - \varepsilon - (v - 2) = 1 - \varepsilon$ (recall that $0 < 1 - \varepsilon < 1$). In period 1 the Buyer will reject any $p_1 \geq v$, and will accept any $p_1 \leq v - 1$. Hence the Seller will offer $p_1 = v - 1$ and the Buyer will accept; the payoffs are $v - 1$ for the Seller and 1 for the Buyer. The backward-induction strategies are as detailed above.

(g) Yes there are many other Nash equilibria, based on incredible threats by the Buyer. For example the Buyer's strategy (which is not credible) is to reject in every period any price greater than 1, in which case the best reply for the Seller is to offer 1 in period 1 (and to plan to offer 1 again in period 2).

(h) The reasoning here is an extension of the reasoning in part (f). In period $t \geq 2$ the Buyer will accept any $p_t \leq v - t$ and reject any other offer; thus at the beginning of period *t* the Seller would offer $p_t = v - t$. Hence in period 1 the Seller will offer $p_1 = v - 1$ and the Buyer will accept; the payoffs are $v - 1$ for the Seller and 1 for the Buyer. The backward-induction strategies are as detailed above.

Question 2: Price Risk and Output

Consider the case of a firm that uses two inputs in the production of its output. Specifically, for the production function

$$q = f(k, \ell) = \sqrt{\min\{k, \ell\}},$$

where q represents output of some commodity and k and ℓ represent input of capital and labor, respectively, the firm is

$$\mathcal{F} = \{y \in \mathbb{R}^3 \mid y_2 \leq 0, y_3 \leq 0, y_1 \leq f(-y_2, -y_3)\}.$$

The producer is an expected-utility maximizer, with strictly increasing cardinal utility index $u(x)$.

(a) Argue that the firm satisfies decreasing returns to scale:

Answer: This is immediate, since the production function is homogeneous of degree $1/2 < 1$. □

(b) Suppose that the prices of capital and labor are, respectively, $r > 0$ and $w > 0$. Solve the cost minimization problem

$$\min_{k,\ell}\{rk + w\ell : f(k, \ell) \geq q\},$$

and determine the optimal input demands and the cost function.

Answer: The optimal demands are $k(r, w, q) = \ell(r, w, q) = q^2$; the cost function is

$$c(r, w, q) = (r + w)q^2.$$ □

(c) Suppose that $r = w = 1/4$, so that the total cost of producing q units is $c(q) = \frac{1}{2}q^2$, and that the price is known to be $\bar{P} > 0$ per unit. Find the optimal output level, $\bar{q}$, and resulting profit.

Answer: The producer's utility index is irrelevant: she solves

$$\max_q\{\bar{P}q - \tfrac{1}{2}q^2\}.$$

The optimal output level is $\bar{q} = \bar{P}$ and maximized profits are $\bar{\pi} = \frac{1}{2}\bar{P}^2$. □

(d) Suppose now that the price the individual face, P, is a random variable and can be 0 or $2\bar{P}$ with probability 1/2. Write the first-order condition that characterizes the optimal output level, $\hat{q}$. Do you expect $\hat{q}$ to be smaller or larger than $\bar{q}$? Why?

Answer: In this case, the utility index matters: the producer maximizes the expected utility of the (random) profits she makes, by solving

$$\max_q\{\tfrac{1}{2}u(-\tfrac{1}{2}q^2) + \tfrac{1}{2}u(2\bar{P}q - \tfrac{1}{2}q^2)\}.$$

The first-order condition of this problem is that

$$u'(-\tfrac{1}{2}\hat{q}^2)\hat{q} = u'(2\bar{P}\hat{q} - \tfrac{1}{2}\hat{q}^2)(2\bar{P} - \hat{q}). \quad (*)$$

Whether $\hat{q}$ is greater or smaller than $\bar{q}$ depends on the producer's attitude towards risk. If she is risk-neutral, since the expected price is $\bar{P}$ she will produce the same amount in both cases. On other hand, for any output level q, the variance of the profits equals q^2 times the variance of the price. Since this is increasing in q, the producer will choose $\hat{q} < \bar{q}$ if, and only if, she is risk-averse. □

(e) Suppose, in particular, that $u(x) = \log(5/8 + x)$ and $\bar{P} = 1$. Argue that $\hat{q} = 1/2$, by showing that this value satisfies the first-order condition found in the previous part. Find $\bar{q}$ and compare it to $\hat{q}$. Is your intuition of the previous part confirmed?

Answer: For this particular index and value,

$$u'(x) = \frac{1}{5/8 + x},$$

so the first-order condition $(*)$ becomes

$$\frac{\hat{q}}{5/8 - \hat{q}^2/2} = \frac{2 - \hat{q}}{5/8 + 2\hat{q} - \hat{q}^2/2}.$$

In order to confirm that $\hat{q} = 1/2$, it suffices to substitute:

$$\frac{1/2}{5/8 - 1/8} = 1 = \frac{2 - 1/2}{5/8 + 1 - 1/8}.$$

Since u is strictly concave, namely since the producer is risk-averse, the intuition of part (d) holds true: $\bar{q} = \bar{P} = 1 > \hat{q}$. □

(f) Suppose that someone offers the producer a future contract that locks her price at $\bar{P}$. Write an equation that characterizes the maximum value that she would be willing to pay for this future.

Answer: If she pays Γ in order to lock the price at $\bar{P}$, she will produce $\bar{q}$, her profits will be $\bar{\pi} - \Gamma$ and her utility will be $u(\bar{\pi} - \Gamma)$. Otherwise, she produces $\hat{q}$ and faces random profits Π, which take the values $-\frac{1}{2}\hat{q}^2$ or $2\bar{P}\hat{q} - \frac{1}{2}\hat{q}^2$ with equal probabilities. The resulting expected utility is

$$\mathrm{E}[u(\Pi)] = \tfrac{1}{2}u(-\tfrac{1}{2}\hat{q}^2) + \tfrac{1}{2}u(2\bar{P}\hat{q} - \tfrac{1}{2}\hat{q}^2).$$

Since index u is strictly increasing, the most the producer is willing to pay for the future is $\bar{\Gamma}$ such that $u(\bar{\pi} - \Gamma) = \mathrm{E}[u(\Pi)]$. □

Question 3: Dynamic General Equilibrium

Consider an exchange economy with society $\mathcal{I} = \{1, \ldots, I\}$. There are $L+K$ commodities and trade takes place in two periods:

1. In the morning, L commodities are traded. Individual i is endowed with ω^i_ℓ units of commodity $\ell = 1, \ldots, L$, and her consumption is x^i_ℓ. The price per unit of commodity ℓ is p_ℓ.
2. In the afternoon, the other K commodities are traded. The endowment and consumption of commodity $k = 1, \ldots, K$ by individual i are ψ^i_k and y^i_k, respectively. The price per unit of commodity k is q_k.
3. In the evening, each agent consumes. If agent i has purchased the bundle $x = (x_1, \ldots, x_L)$ in the morning and the bundle $y = (y_1, \ldots, y_K)$ in the afternoon, her utility in the evening is $u^i(x) + v^i(y)$.

In the morning, besides trading the corresponding commodities, individual i chooses an amount m^i of nominal savings. If positive, m^i becomes nominal wealth in the afternoon; if negative, it is nominal debt that the agent must honor.

In the afternoon, given the prices $q = (q_1, \ldots, q_K)$ and her savings m^i, agent i solves the problem

$$\max_{y \in \mathbb{R}^K_+} \left\{ v^i(y) \, : \, q \cdot y \leq q \cdot \psi^i + m^i \right\}, \tag{1}$$

where $\psi^i = (\psi^i_1, \ldots, \psi^i_K)$. In the morning, given $p = (p_1, \ldots, p_L)$ and anticipating q, she solves

$$\max_{(x,m) \in \mathbb{R}^L_+ \times \mathbb{R}} \left\{ u^i(x) + V^i(m, q) \, : \, p \cdot x + m \leq p \cdot \omega^i \right\}, \tag{2}$$

where $\omega^i = (\omega^i_1, \ldots, \omega^i_L)$ and

$$V^i(m, q) = \max_{y \in \mathbb{R}^K_+} \left\{ v^i(y) \, : \, q \cdot y \leq q \cdot \psi^i + m \right\}.$$

The minister of finance of this economy is worried that the agents may be acting silly. She would prefer it if, instead of solving the two problems (2) and (1), agent i solves the intertemporal problem

$$\max_{(x,y,m) \in \mathbb{R}^L_+ \times \mathbb{R}^K_+ \times \mathbb{R}} \left\{ u^i(x) + v^i(y) \, : \, p \cdot x + m \leq p \cdot \omega^i \text{ and } q \cdot y \leq q \cdot \psi^i + m \right\}. \tag{3}$$

Not having taken the second-year GE course in his Ph.D., the dean of the most prominent economics department in the economy is worried that in the model he learned in the first-year course, people were assumed to solve the static problem

$$\max_{(x,y) \in \mathbb{R}^L_+ \times \mathbb{R}^K_+} \left\{ u^i(x) + v^i(y) \, : \, p \cdot x + q \cdot y \leq p \cdot \omega^i + q \cdot \psi^i \right\}, \tag{4}$$

as if all trade took place at the same time.

There are three definitions of equilibrium for this economy.

1. The definition that the dean understands is: a tuple $(p, q, \mathbf{x}, \mathbf{y})$, where $\mathbf{x} = (x^1, \ldots, x^I)$ and $\mathbf{y} = (y^1, \ldots, y^I)$, is a *static equilibrium* if
   - for each individual, the pair (x^i, y^i) solves the static problem (4);
   - $\sum_i x^i = \sum_i \omega^i$ and $\sum_i y^i = \sum_i \psi^i$.

2. The minister would wish that the following was the definition of equilibrium: a tuple $(p, q, \mathbf{x}, \mathbf{y}, \mathbf{m})$, where $\mathbf{m} = (m^1, \ldots, m^I)$, is an *intertemporal equilibrium* if
   - for each individual the triple (x^i, y^i, m^i) solves the intertemporal problem (3);
   - $\sum_i x^i = \sum_i \omega^i$, $\sum_i y^i = \sum_i \psi^i$, and $\sum_i m^i = 0$.
3. The actual definition of equilibrium is: a tuple $(p, q, \mathbf{x}, \mathbf{y}, \mathbf{m})$ is a *dynamic equilibrium* if
   - for each individual the pair (x^i, m^i) solves the morning problem (2), and the bundle y^i solves the afternoon problem (1);
   - $\sum_i x^i = \sum_i \omega^i$, $\sum_i y^i = \sum_i \psi^i$, and $\sum_i m^i = 0$.

The definition of efficiency, on the other hand, does not change: allocation $(\mathbf{x}, \mathbf{y})$ is efficient if there does not exist another allocation $(\tilde{\mathbf{x}}, \tilde{\mathbf{y}})$ such that $u^i(\tilde{x}^i) + v^i(\tilde{y}^i) \geq u^i(x^i) + v^i(y^i)$ for all i, with strict inequality for some.

Assume that for all i, the functions u^i and v^i are of class $\mathbb{C}^2$, differentiably strictly increasing and differentiably strictly concave.

(a) Argue that if an interior allocation $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$ is efficient, then there exist vectors $\delta \in \mathbb{R}^L_{++}$ and $\gamma \in \mathbb{R}^K_{++}$ and a profile of scalars $(\mu^1, \ldots, \mu^I) \in \mathbb{R}^I_{++}$ such that

$$\mu^i Du^i(\bar{x}^i) = \delta \quad \text{and} \quad \mu^i Dv^i(\bar{y}^i) = \gamma \tag{5}$$

for all i.

Answer: If $(\bar{\mathbf{x}}, \bar{\mathbf{y}})$ is efficient, it must solve

$$\max_{\mathbf{x},\mathbf{y}} \left\{ u^1(x^1) + v^1(y^1) : \begin{array}{c} \forall i \geq 2, u^i(x^i) + v^i(y^i) \geq u^i(\bar{x}^i) + v^i(\bar{y}^i) \\ \sum_i x^i = \sum_i \omega^i \\ \sum_i y^i = \sum_i \psi^i \end{array} \right\}.$$

The Lagrangean of this problem is

$$u^1(x^1) + v^1(y^1) + \sum_{i \geq 2} \mu^i [u^i(x^i) + v^i(y^i) - \bar{w}^i] + \delta \cdot \sum_i (\omega^i - x^i) + \gamma \cdot \sum_i (\psi^i - y^i),$$

where $\bar{w}^i = u^i(\bar{x}^i) + v^i(\bar{y}^i)$ for each $i \geq 2$. The first-order conditions of this problem imply that

$$Du^1(\bar{x}^1) = \delta \quad \text{and} \quad Dv^1(\bar{y}^1) = \gamma,$$

while

$$\mu^i Du^i(\bar{x}^i) = \delta \quad \text{and} \quad \mu^i Dv^i(\bar{y}^i) = \gamma,$$

for all $i \geq 2$. Letting $\mu^1 = 1$ completes the argument. □

(b) Let $(p, q, \mathbf{x}, \mathbf{y})$ be a static equilibrium with interior allocation. Argue that allocation $(\mathbf{x}, \mathbf{y})$ satisfies the necessary condition for efficiency written in part (a).

Answer: As in the course, the first-order conditions of problem (4) are that

$$Du^i(x^i) = \lambda^i p \quad \text{and} \quad Dv^i(y^i) = \lambda^i q. \tag{$*$}$$

Thus, we just need to let $\mu^1 = 1/\lambda^i$, $\delta = p$ and $\gamma = q$. □

(c) Let $(p, q, \mathbf{x}, \mathbf{y}, \mathbf{m})$ be an intertemporal equilibrium with interior allocation. Argue that allocation $(\mathbf{x}, \mathbf{y})$ satisfies the necessary condition for efficiency written in part (a).

Answer: The Lagrangean of problem (3) is

$$u^i(x) + v^i(y) + \lambda^i(p \cdot \omega^i - p \cdot x - m) + \eta^i(q \cdot \psi^i + m - q \cdot y),$$

so the first-order conditions are that

$$Du^i(x^i) = \lambda^i p, \; Dv^i(y^i) = \eta^i q \;\text{ and }\; \lambda^i = \eta^i. \tag{$**$}$$

These are equivalent to $(*)$, so the result follows from (b). □

(d) Let $(p, q, \mathbf{x}, \mathbf{y}, \mathbf{m})$ be a dynamic equilibrium with interior allocation. Argue that allocation $(\mathbf{x}, \mathbf{y})$ satisfies the necessary condition for efficiency written in part (a).

Answer: The Lagrangean of problem (1) is

$$\mathcal{L}^i(y; m, q) = v^i(y) + \eta^i(q \cdot \psi^i + m - q \cdot y),$$

so its first-order conditions are that

$$Dv^i(y^i) = \eta^i q.$$

The Lagrangean of (2) is

$$u^i(x) + V^i(m, q) + \lambda^i(p \cdot \omega^i - p \cdot x - m),$$

so at its solution

$$Du^i(x^i) = \lambda^i p \;\text{ and }\; \frac{\partial V^i}{\partial m}(m^i, q) = \lambda^i.$$

On the other hand, from the envelope theorem we know that

$$\frac{\partial V^i}{\partial m}(m^i, q) = \frac{\partial \mathcal{L}^i}{\partial m}(y^i; m^i, q) = \eta^i.$$

The last two equations imply that $\lambda^i = \eta^i$, so these conditions are equivalent to $(**)$ and the result follows from part (c). □