Prelim 2023: Answer Key

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Question 5 (*20 points*)

1. Suppose effort is observable. Show that only project 1 is financed, independently of N.

> **Solution:** If effort is observable then entrepreneurs will never shirk[1]
>
> Recall that:
>
> - Expected return of project 1 is $p_H R$
> - Expected return of project 2 is $q_H R$
>
> Denote R_j^i the payoff of project j for agent i in case of success. Then, given N, the payoff to financiers is
>
> $$R_1^F = \frac{I - N}{p_H}$$
> $$R_2^F = \frac{I - N}{q_H}$$
>
> from projects 1 and 2, respectively. Since $p_H > q_H$, then
>
> $$p_H R_1^E = p_H R - p_H R_1^F = p_H R - (I - N) > q_H R - (I - N) = q_H R - q_H R_2^F = q_H R_2^E,$$
>
> which establishes that project 2 is strictly dominated by project 1 when there is no information problem. Hence, all entrepreneurs invest in project 1, independently of their net worth N.

2. From now on, effort is not observable. Characterize the financing problem between the entrepreneur and the financiers. In particular, clearly state: i) the resource constraint; ii) the financiers' participation constraint; iii) the entrepreneur's incentive compatibility constraints.

> **Solution:** For project 1, the constraints are:
>
> - Resource constraint:
> $$R = R_1^F + R_1^E$$
> - Financier participation constraint:
> $$p_H R_1^F \geq I - N$$
> - Entrepreneur incentive compatibility:
> $$p_H R_1^E \geq p_L R_1^E + B \;\Rightarrow\; R_1^E \geq \frac{B}{\Delta p}$$

For project 2, the constraints are:

- The resource constraint:
$$R = R_2^F + R_2^E$$
- Financier participation constraint:
$$q_H R_2^F \geq I - N$$
- Entrepreneur incentive compatibility:

$$q_H R_2^E \geq q_L R_2^E + b \;\Rightarrow\; R_2^E \geq \frac{b}{\Delta q}$$

3. Divide the set of possible net worths N, $[0, I)$, into three regions, $[0, \overline{N}_q)$, $[\overline{N}_q, \overline{N}_p)$, and $[\overline{N}_p, I)$ and show that the equilibrium investment policies in these regions are "not invest," "invest in project 2," and "invest in project 1." Find the expressions for $\overline{N}_q$ and $\overline{N}_p$. Why is project 2 sometimes financed?

Solution:

Assuming that competition financiers leads to their participation to be satisfied with equality, the combination of the resource constraint, the financier's participation constraint and the entrepreneur's incentive compatibility constraint leads to

$$I - N \leq p_H \left(\underbrace{R - \frac{B}{\Delta p}}_{\equiv \rho_1} \right) \iff N \geq I - p_H \rho_1 \equiv \overline{N}_p$$

Similar calculations show that the project is feasible for E if and only if $N \geq I - q_H \rho_2 \equiv \overline{N}_q$. Note that $\overline{N}_q < \overline{N}_p$ since $p_H \rho_1 < q_H \rho_2$ by assumption.

Thus, we can partition $[0, I)$ into three regions:

- $[0, \overline{N}_q)$ where N is not enough to invest in any project. Then the optimal policy in $[0, \bar{N}_q)$ is to not invest
- $[\overline{N}_q, \overline{N}_p)$ where N is enough only to invest in project 2. Since project 2 has positive NPV, E chooses to undertake project 2 in this region
- $[\bar{N}_p, I)$ where N is enough to invest in project 1 and 2. The optimal choice is to invest in project 1 since by assumption $p_H > q_H$, which implies:
$$p_H R_1^E = p_H R - (I - N) > q_H R - (I - N) = q_H R_2^E$$

Project 2 is worse in terms of payoffs but better in terms of incentives. Intuitively the contract is doing 3 things: paying the financier, paying the entrepreneur and making sure incentives are aligned (prevent misbehavior). Project 2 pledgeable output is higher, which implies that it is less costly for the contract to prevent misbehavior, and less "skin in the game" is needed. Hence under asymmetric information there is a role for project 2 (remember that in point 1, there was no role for project 2 since there was no information problem, hence the only dimension in which project 2 was batter than project 1 was not relevant).

4. In this question *only*, suppose that the financiers cannot verify which project the entrepreneur is choosing (they only observe success/failure). Argue that nothing is altered if $N \geq \overline{N}_p$. Formally show that if $N \in [\overline{N}_q, \overline{N}_p)$, the entrepreneur doesn't get financing.

Solution:

Assume that now, financiers cannot see the project E chooses.

If $N \geq \bar{N}_p$, the contract designed for project 1 imply that if project 1 is chosen the entrepreneur will behave (not shirk). The new source of potential misbehavior is choosing to pursue project 2 when funding for project 1 was provided by the financiers. But this is not privately optimal for the entrepreneur since $p_H R + (N - I) > q_H R + (N - I)$ and $R_1^E \geq \frac{B}{\Delta p} > \frac{b}{\Delta q}$ (which guarantees that choosing project 2 and shirking is not optimal). This means that when $N \geq \bar{N}_p$, financiers know that entrepreneurs will undertake project 1 without the need of effectively observing the choice.

If $N \in [\bar{N}_q, \bar{N}_p)$, by what we discussed previously financiers will only want to give funds to entrepreneurs if it is privately optimal for entrepreneurs to pursue project 2 conditional on getting the loan from financiers and they will ask for $R_2^F = \frac{I-N}{q_H}$.

If entrepreneurs have incentives to deviate and choose project 1 and shirk, then financiers will not lend to them. If E undertakes project 2 and doesn't shirk, she gets

$$q_H \left(R - \frac{I - N}{q_H} \right)$$

If after signing the contract and getting the funding she decides to undertake project 1, she gets

$$p_H \left(R - \frac{I - N}{q_H} \right) > q_H \left(R - \frac{I - N}{q_H} \right)$$

That is, she wants to deviate and undertake project 1. However, when $N < \overline{N}_p$, we know that

$$p_H \left(R - \frac{I - N}{p_H} \right) < p_L \left(R - \frac{I - N}{p_H} \right) + B$$

hence

$$B > (p_H - p_L) \left(R - \frac{I - N}{p_H} \right) > (p_H - P_L) \left(R - \frac{I - N}{q_H} \right)$$

which implies

$$p_H \left(R - \frac{I - N}{q_H} \right) < p_L \left(R - \frac{I - N}{q_H} \right) + B$$

and therefore

$$q_H \left(R - \frac{I - N}{q_H} \right) < p_H \left(R - \frac{I - N}{q_H} \right) < p_L \left(R - \frac{I - N}{q_H} \right) + B$$

which implies that when $N \in [\overline{N}_q, \overline{N}_p)$, if E gets funding, she will undertake project 1 and shirk. But we know that in that case the project has negative NPV, so the financiers will not lend to Es with this level of wealth.

5. Suppose now that the private benefit of shirking on project 1 can be reduced from B to b by using a

monitoring technology. This technology has an implementation cost of c. Assume that

$$p_H R - c > q_H R > I$$

and

$$p_H \left(R - \frac{b}{\Delta p} \right) < I + c$$

Show that monitoring is useful if and only if

$$c < p_H \frac{B - b}{\Delta p}$$

Is there any level of N and c such that project 2 is implemented?

Solution:

Now we assume:

$$p_H R - c > q_H R > I$$

Hence it is more profitable to pay c and implement the project 1 rather than not paying c and implementing project 2.

Note that since $R - \frac{b}{\Delta p}$ is the pledgable income of the monitored project, $p_H \left(R - \frac{b}{\Delta p} \right)$ is the maximum compensation that a financier can get.

There is going to be a level of N for which financing the monitored project is possible, call it $\tilde{N}$:

$$p_H \left(\underbrace{R - \frac{b}{\Delta p}}_{\equiv \tilde{\rho}_1} \right) = I - \tilde{N} + c$$

$$\tilde{N} = I + c - p_H \tilde{\rho}_1$$

Monitoring is useful only if $\tilde{N} < \overline{N}_p$ (that is, it allows some "low" wealth Es to get funding for project 1). Formally:

$$\tilde{N} < \overline{N}_p \iff p_H \left(R - \frac{b}{\Delta p} \right) - c > p_H \left(R - \frac{B}{\Delta p} \right)$$

$$p_H \frac{B - b}{\Delta p} > c$$

Note that project 2 is going to be implemented if the required skin in the game is smaller than under project 1 monitored. Hence, we need to compare $\bar{N}_q = I + \frac{b}{\Delta q} - R$ with $\tilde{N} = I + p_H \frac{b}{\Delta p} - (p_H R - c)$

$$\bar{N}_q < \tilde{N} \iff I - q_H \left(R - \frac{b}{\Delta q} \right) < I - p_H \left(R - \frac{b}{\Delta q} \right) + c$$

Using $\Delta p = \Delta q$

$$c > (p_H - q_H) \left(R - \frac{b}{\Delta p} \right)$$

Project 2 will be undertaken by entrepreneurs with net worth $N \in [\bar{N}_q, \tilde{N})$

Question 6 (*20 points*)

We will study a way of detecting financial constraints through the behavior of cash holdings.

There are 3 periods $t = 0, 1, 2$. The firms has some initial installed capital that produces a cash flow process $\{c_0, c_1\}_{t\in\{0,1\}}$. The firm has the option to invest in a long-term project I_0 today that pays off $F(I_0)$ at time 2. Additionally, the firm expects to have another investment opportunity at time $t = 1$. If the firm invests at time 1 I_1, the technology produces $G(I_1)$ at time $t = 2$. You can think of this problem as if the firm has access to three production technologies: 1) installed capital produces $\{c_t\}_{t\in\{0,1\}}$, 2) investment in $t = 0$ produces $F(I_0)$ in $= 2$, 3) investment in $t = 1$ produces $G(I_1)$ in $t = 2$. The usual assumption on F, G: increasing, concave, continuously differentiable, and marginal product when $I \to 0$ is infinity.

Assume $\beta = 1$ so that the equilibrium interest rate is zero (the economy is populated by financiers with very large endowments), and the cost of investment goods at dates 0 and 1 is equal to 1. We will assume that the firm has access to:

- Long term debt B_0 at time $t = 0$. The firm can get B_0 at $t = 0$ and repay B_0 at $t = 2$
- Short term debt B_1 at time $t = 1$. The firm can get B_1 at time $t = 1$ and repay B_1 at $t = 2$.
- Cash C at time $t = 0$. The firm save in cash in $t = 0$ and use it in $t = 1$.
- The firm **cannot** issue equity, this means that dividends need to be weakly positive.

Investment can be liquidated at the final date $t = 2$ generating a payoff equal to $q(I_0 + I_1)$ where $q \leq 1$ and $I_0, I_1 > 0$. You can think of q as the (exogenous) resale price of capital in the final period of the firm's life. Denote the total cash flows from investment by $f(I_0) = F(I_0) + qI_0$ and $g(I_1) = G(I_1) + qI_1$. We assume that the cash flows $F(I_0)$ and $G(I_0)$ are non-verifiable and cannot be contracted upon. This means that the firm cannot pledge these funds to outside investors, but it can raise external finance by pledging the underlying productive assets as collateral. The liquidation value of assets that can be pledged to creditors is given by $(1 - \tau)qI$ where the parameter $\tau \in [0, 1)$ captures the quality of legal institutions that govern creditors' rights. For high enough τ firms might pass up on positive NPV projects due to financial constraints.

1. Let C denote the amount of cash the firm carries from period 0 to time 1, $B = (B_0, B_1)$ the debt policy, $I = (I_0, I_1)$ the investment policy. Let d_t denote the firm's dividend in period t, so that

$$\begin{aligned}
d_0 &= c_0 + B_0 - I_0 - C \\
d_1 &= c_1 + B_1 - I_1 + C \\
d_2 &= f(I_0) + g(I_1) - B_0 - B_1
\end{aligned}$$

State the firm's problem. **Hint:** Remember that dividends cannot be negative.

Solution: The problem of the firm is

$$\max d_0 + d_1 + d_2$$

subject to

$$\begin{gathered}
d_0 = c_0 + B_0 - I_0 - C \\
d_1 = c_1 + B_1 - I_1 + C \\
d_2 = f(I_0) + g(I_1) - B_0 - B_1 \\
B_0 \leq (1 - \tau) qI_0 \\
B_1 \leq (1 - \tau) qI_1 \\
d_0, d_1, d_2, C, I_0, I_1 \geq 0
\end{gathered}$$

First, we will characterize the investment level chosen by unconstrained firms.

2. State the first-order conditions of the firm's problem. Characterize the level of investment chosen by an unconstrained firm, that is, the case in which the borrowing constraints in both periods are not binding. **Hint:** Note that d_2 is always strictly positive.

Solution:

The FOCs are

$$\begin{aligned}
\partial d_0 : &\quad 1-\mu_0 \leq 0 \\
\partial d_1 : &\quad 1-\mu_1 \leq 0 \\
\partial d_2 : &\quad 1-\mu_2 \leq 0 \\
\partial I_0 : &\quad -\mu_0 + f'(I_0) + \eta_0 (1-\tau) q = 0 \\
\partial I_1 : &\quad -\mu_1 + q'(I_1) + \eta_1 (1-\tau) q = 0 \\
\partial B_0 : &\quad \mu_0 - \mu_2 - \eta_0 = 0 \\
\partial B_1 : &\quad \mu_1 - \mu_2 - \eta_1 = 0 \\
\partial C : &\quad -\mu_0 + \mu_1 \leq 0
\end{aligned}$$

Note that since the marginal product of capital is infinity when $I \to 0$, positive investment is always optimal. Moreover, the borrowing constraint implies that the firm can borrow strictly less than its production in $t=2$. Hence, $d_2 > 0$ and $\mu_2 = 1$.

Now, suppose borrowing constraints are not binding. Then $\eta_0 = \eta_1 = 0$, which implies

$$\mu_0 = \mu_1 = \mu_2 = 1$$

and

$$f'(I_0) = g'(I_1) = 1$$

A firm is financially constrained if its investment policy is distorted from the unconstrained level because of capital market frictions. From now on, focus on the case where the borrowing constraints are binding in both periods.

3. Show that the optimal cash flow policy C^* is characterized by $f'\left(\frac{c_0 - C^*}{1-q+\tau q}\right) = g'\left(\frac{c_1 + C^*}{1-q+\tau q}\right)$. Explain the trade-offs involved in the decision to hold cash.

Solution:

If constrained in both periods $\eta_0, \eta_1 > 0$ and

$$\begin{aligned}
B_0 &= (1-\tau) q I_0 \\
B_1 &= (1-\tau) q I_1
\end{aligned}$$

Moreover,

$$\begin{aligned}
\mu_0 = \mu_2 + \eta_0 > \mu_2 = 1 &\implies d_0 = 0 \\
\mu_1 = \mu_2 + \eta_1 > \mu_2 = 1 &\implies d_1 = 0
\end{aligned}$$

which implies

$$I_0 = c_0 - C + (1-\tau)\,qI_0 \implies I_0 = \frac{c_0 - C}{1-q+\tau q}$$
$$I_1 = c_1 + C + (1-\tau)\,qI_1 \implies I_1 = \frac{c_1 + C}{1-q+\tau q}$$

Then, we have

$$d_2 = f\left(\frac{c_0 - C}{1-q+\tau q}\right) + g\left(\frac{c_1 + C}{1-q+\tau q}\right) - (1-\tau)\,q\frac{c_0 - C}{1-q+\tau q} - (1-\tau)\,q\frac{c_1 + C}{1-q+\tau q}$$

Introducing this into the objective function, we get

$$\max_C f\left(\frac{c_0 - C}{1-q+\tau q}\right) + g\left(\frac{c_1 + C}{1-q+\tau q}\right) - (1-\tau)\,q\frac{c_0 - C}{1-q+\tau q} - (1-\tau)\,q\frac{c_1 + C}{1-q+\tau q}$$

The FOC is

$$-\frac{1}{1-q+\tau q}f'\left(\frac{c_0 - C}{1-q+\tau q}\right)+\frac{1}{1-q+\tau q}g'\left(\frac{c_1 + C}{1-q+\tau q}\right)+(1-\tau)\,q\frac{1}{1-q+\tau q}-(1-\tau)\,q\frac{1}{1-q+\tau q}=0$$

or

$$f'\left(\frac{c_0 - C}{1-q+\tau q}\right) = g'\left(\frac{c_1 + C}{1-q+\tau q}\right)$$

Constrained firms by definition cannot undertake all of their positive NPV projects, so holding cash is costly because it requires sacrificing some valuable investments today. The benefit of cash holdings is being able to fund future opportunities. In this sense, financial frictions give rise to an optimal cash policy C.

4. How much of its current cash flow c_0 will a constrained firm save? In other words, find an expression for $\frac{\partial C^*}{\partial c_0}$ and interpret this derivative. When will the firm save more? When will it save less?

Solution:

$$\frac{\partial C^*}{\partial c_0} = \frac{f''(I_0^*)}{f''(I_0^*) + g''(I_1^*)} = \frac{1}{1 + \frac{g''(I_1^*)}{f''(I_0^*)}}$$

If $f''() \simeq 0$, the firm wants to save little, while it wants to save a lot if $g''() \simeq 0$. The idea is that the closer f'', q'' are to zero, the less pervasive the decreasing returns are, so it is profitable to invest a lot.

5. Let $F(I) = A\ln(I)$ and $G(I) = B\ln(I)$. Derive an explicit formula for C^*. How does C^* depend on τ? Explain the intuition.

Solution:

$$C^* = \frac{\delta c_0 - c_1}{1+\delta}$$

Where $\delta \equiv \frac{B}{A}$. The cash flow sensitivity is given by $\frac{\delta}{1+\delta}$ where the parameter δ can be interpreted as the measure of the importance of future growth opportunities relative to today's opportunities.

Note that $\frac{\partial C^*}{\partial \delta} > 0$ which agrees with the intuition that a financially constrained firm will hoard more cash today if future investment opportunities are more profitable.

6. Consider the following empirical model.

$$\Delta CashHoldings_{i,t} = \alpha_0 + \alpha_1 Cashflow_{i,t} + X'_{i,t}\beta + \epsilon_{i,t}$$

Through the lens of our model:

- Should we expect a systematic relationship $Cashflow_{i,t}$ and $CashHoldings_{i,t}$ for unconstrained firms?
- Should we expect a systematic relationship $Cashflow_{i,t}$ and $CashHoldings_{i,t}$ for constrained firms?
- Empirical estimates find α_1 significant and positive for constrained firms and α_1 insignificant and positive or negative depending on details of the empirical specification for unconstrained firms. Is this in line with our theory?