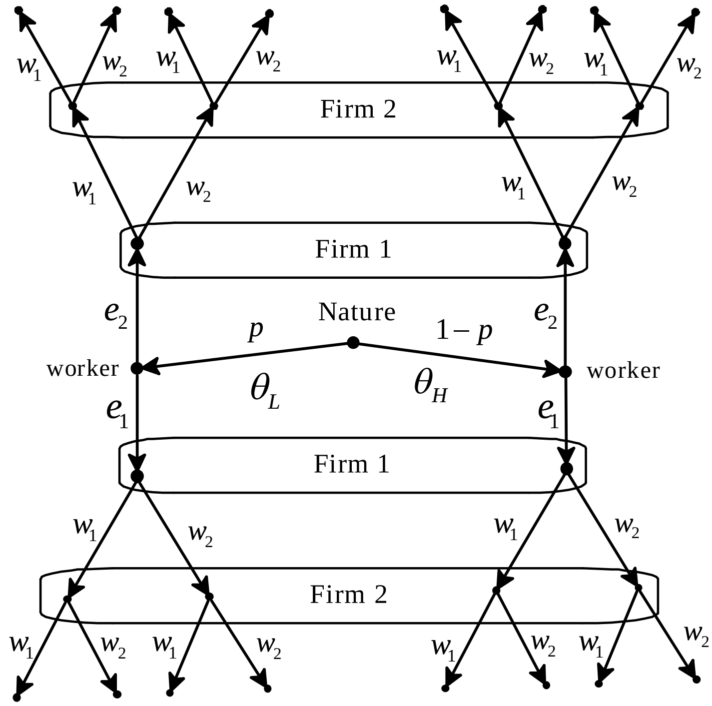
Micro Prelim August 18, 2022
QUESTION 1 ANSWER KEYS

(a) The game is as follows (after both firms offer the same wage then one could add a move by Nature, with equal probability, determining which offer is accepted by the worker):



(b) A pure strategy for the worker is a function $e:\{\theta_L,\theta_H\}\to[0,\bar{e}]$, a pure strategy for firm i is a function $s_i:[0,\bar{e}]\to[0,\infty]$

(c) A pooling equilibrium is as follows:

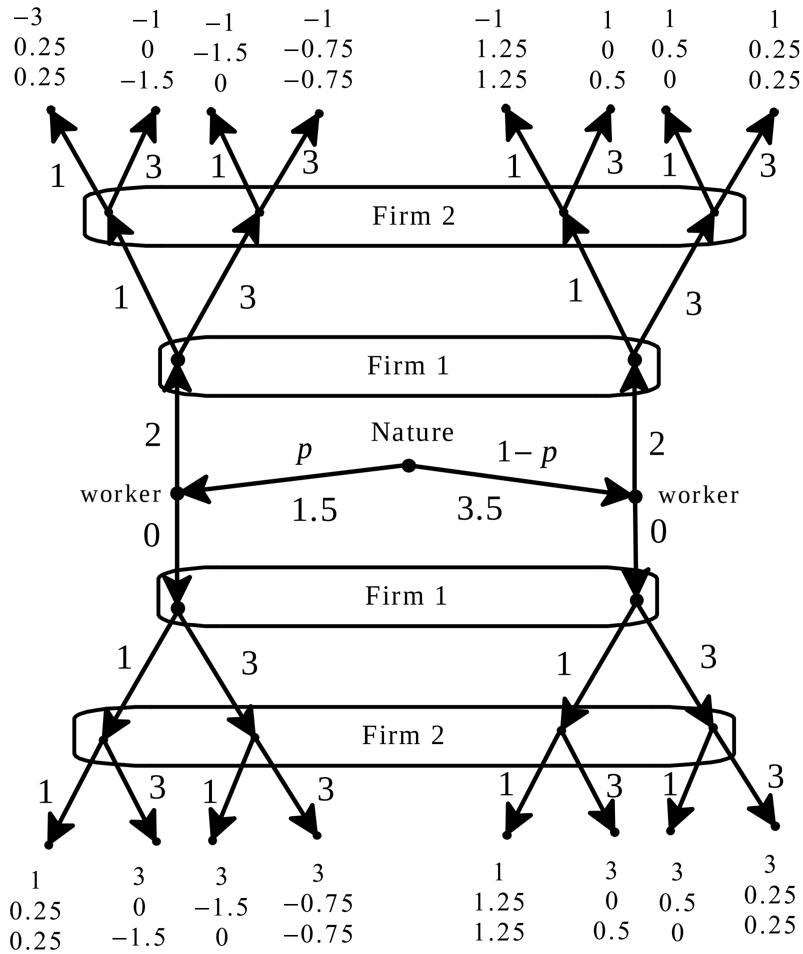
- both types of workers choose $e = 0$,
- for every $e > 0$, both firms assign probability 1 to the worker being of type θ_L and offer the same wage $w=\theta_L$ (for the firm that moves second, say firm 2, probability 1 is assigned to the node that follows Nature's choice of θ_L then the observed choice of e and then $w_1=\theta_L$),
- after observing $e = 0$, both firms assign probability p to the worker being of type θ_L and probability $(1-p)$ to the worker being of type θ_H and offer a wage $w=p\theta_L+(1-p)\theta_H$ (for the firm that moves second, say firm 2, probability p is assigned to the node that follows Nature's choice of θ_L then the observed choice of $e = 0$ and then $w_1=p\theta_L+(1-p)\theta_H$, and probability $(1-p)$ is assigned to the node that follows Nature's choice of θ_H then the observed choice of $e = 0$ and then $w_1=p\theta_L+(1-p)\theta_H$).

Bayesian updating is satisfied by construction (it only applies to the information sets after the education choice $e = 0$). Sequential rationality for the worker of type θ is satisfied, because with $e = 0$ her payoff is $\theta_L-c(0,\theta)=\theta_L$ (since $c(0,\theta)=0$) and, given the strategies of the firms, if she switched to any $e > 0$ her payoff would be to $\theta_L-c(e,\theta)<\theta_L$ (since, for $e > 0$, $c(e,\theta)>c(0,\theta)=0$). Sequential rationality for firm i is satisfied because:

(1) at an information set following a choice of education $e > 0$, given its beliefs the firm's expected profit is $\frac{1}{2}(\theta_L-\theta_L)+\frac{1}{2}0=0$ and if it offers a wage $w>\theta_L$ then it expected profit is $\theta_L-w<0$ and if it offers a wage $w<\theta_L$ its payoff is 0 (since the worker accepts the offer of θ_L by the other firm),

(2) at the information set following the choice e = 0, given its beliefs the firm's expected profit is $\frac{1}{2}\left[p\theta_L + (1-p)\theta_H - (p\theta_L + (1-p)\theta_H)\right] + \frac{1}{2}0 = 0$ and if it offers a wage $w > p\theta_L + (1-p)\theta_H$ then its expected profit is $p\theta_L + (1-p)\theta_H - w < 0$ and if it offers a wage $w < p\theta_L + (1-p)\theta_H$ then its profit is 0 (because the worker accepts the offer of the other firm).

(d) The payoffs are as shown in the figure below:



A separating weak sequential equilibrium is as follows:

- worker's strategy: choose e = 0 if of type θ_L and e = 2 if of type θ_H ,
- firm i's strategy: offer w = 1 if the worker chose e = 0 and offer w = 3 if the worker chose e = 2,
- firm i's beliefs: probability 1 on the left-most node of the information set in the lower part of the game and probability 1 on the right-most node of the information set in the upper part of the game.

The beliefs satisfy Bayesian updating.

Sequential rationality for the worker: the payoff of type θ_L is 1 – 0 = 1 and if she switched to e = 2 her payoff would be 3 – 4 = – 1; the payoff of type θ_H is 3 – 2 = 1 and if she switched to e = 0 her payoff would be 1 – 0 = 1.

Sequential rationality for firm i: (1) at the lower information set its expected payoff is $\frac{1}{2}(1.5 - 1) + \frac{1}{2}0 = 0.25$ and if it switched to w_i = 3 then its payoff would be 1.5 – 3 = – 1.5; (2) at the upper information set its payoff is $\frac{1}{2}(3.5 - 3) + \frac{1}{2}0 = 0.25$ and if it switched to w_i = 1 then its payoff would be 0.

Question 2 Answer keys

a. Let $\boldsymbol{p} = (p_1, p_2, p_3, p_4)$ be the price vector consisting of prices for commodities 1 to 4. Further, denote by w the wealth level of a consumer. Assume $(\boldsymbol{p}, w) >> \mathbf{0}$. Derive the Walrasian demand functions for the following utility functions:

(i) $u(x_1, x_2, x_3, x_4) = \min\left\{\sqrt{x_1 x_2}, \sqrt{x_3 x_4}\right\}$

Suppose she spends wealth w' on commodities 1 and 2 and wealth w'' on commodities 3 and 4 with $w' + w'' = w$. Then (x_1, x_2) maximizes $\sqrt{x_1 x_2}$ subject to $p_1 x_1 + p_2 x_2 = w'$ and (x_3, x_4) maximizes $\sqrt{x_3 x_4}$ subject to $p_3 x_3 + p_4 x_4 = w''$. Using the indirect utility functions for these problems, the utility of the consumer is

$$\min\left\{\frac{w'}{2\sqrt{p_1 p_2}}, \frac{w''}{2\sqrt{p_3 p_4}}\right\}.$$

It is maximized when

$$\frac{w'}{2\sqrt{p_1 p_2}} = \frac{w''}{2\sqrt{p_3 p_4}}.$$

This is equivalent to

$$\frac{w'}{w''} = \frac{\sqrt{p_1 p_2}}{\sqrt{p_3 p_4}}.$$

From this equation and the fact that $w' + w'' = w$, it follows that

$$w' = \frac{\sqrt{p_1 p_2}}{\sqrt{p_1 p_2} + \sqrt{p_3 p_4}} w$$

and

$$w'' = \frac{\sqrt{p_3 p_4}}{\sqrt{p_1 p_2} + \sqrt{p_3 p_4}} w.$$

We can now conclude

$$\begin{aligned}
x_1(\boldsymbol{p}, w) &= \frac{w'}{2p_1} = \frac{\sqrt{p_2}}{2\sqrt{p_1}(\sqrt{p_1 p_2} + \sqrt{p_3 p_4})} w \\
x_2(\boldsymbol{p}, w) &= \frac{w'}{2p_2} = \frac{\sqrt{p_1}}{2\sqrt{p_2}(\sqrt{p_1 p_2} + \sqrt{p_3 p_4})} w \\
x_3(\boldsymbol{p}, w) &= \frac{w''}{2p_3} = \frac{\sqrt{p_4}}{2\sqrt{p_3}(\sqrt{p_1 p_2} + \sqrt{p_3 p_4})} w \\
x_4(\boldsymbol{p}, w) &= \frac{w''}{2p_4} = \frac{\sqrt{p_3}}{2\sqrt{p_4}(\sqrt{p_1 p_2} + \sqrt{p_3 p_4})} w
\end{aligned}$$

(ii) $u(x_1, x_2, x_3, x_4) = \sqrt{x_1 x_2} + \sqrt{x_3 x_4}$

Similar to the previous problem, let w' denote the amount of wealth spent on commodities 1 and 2 and w'' the amount of wealth spent on 3 and 4 with $w' + w'' = w$. Then (x_1, x_2) maximizes $\sqrt{x_1 x_2}$ subject to $p_1 x_1 + p_2 x_2 = w'$ and (x_3, x_4) maximizes $\sqrt{x_3 x_4}$ subject to $p_3 x_3 + p_4 x_4 = w''$. Her utility is

$$\frac{w'}{2\sqrt{p_1 p_2}} + \frac{w''}{2\sqrt{p_3 p_4}}.$$

The problem becomes now choosing w' and w'' such as to maximize her utility subject to the constraint $w' + w'' = w$. If $p_1 p_2 < p_3 p_4$, then $w' = w$ and $w'' = 0$. Then $x_1 = \frac{w}{2p_1}$, $x_2 = \frac{w}{2p_2}$, $x_3 = 0$, and $x_4 = 0$. If $p_1 p_2 > p_3 p_4$, then $w' = 0$ and $w'' = w$. In this case, $x_1 = 0$, $x_2 = 0$, $x_3 = \frac{w}{2p_3}$ and $x_4 = \frac{w}{2p_4}$. If $p_1 p_2 = p_3 p_4$ then $x_1 = \frac{w'}{2p_1}$, $x_2 = \frac{w'}{2p_2}$, $x_3 = \frac{w''}{2p_3}$ and $x_4 = \frac{w''}{2p_4}$ for any $w', w'' \geq 0$ with $w' + w'' = w$.

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b. Consider a consumer who prefers any bundle of commodities $\boldsymbol{x} = (x_1, x_2)$ satisfying $x_1 > 0$ and $x_2 > 1$ over any bundle that is not satisfying these inequalities. Her utility function over bundles with $x_1 > 0$ and $x_2 > 1$ is given by

$$u(x_1, x_2) = \ln(x_1 + 1) + \ln(x_2 - 1).$$

We denote by $p_1 > 0$ and $p_2 > 0$ the prices of commodities 1 and 2, respectively, and by $w > 0$ the wealth of the consumer.

(i) For which vectors $(p_1, p_2, w) >> 0$ does the consumer consume strict positive amounts of both commodities?

For an interior solution, we must have

$$\frac{p_1}{p_2} = \frac{x_2 - 1}{x_1 + 1} = MRS(\boldsymbol{x}),$$

which implies

$$p_1 x_1 + p_1 = p_2 x_2 - p_2.$$

Using the budget equation, we solve for

$$\begin{aligned} x_1(\boldsymbol{p}, w) &= \frac{w - p_1 - p_2}{2p_1} \\ x_2(\boldsymbol{p}, w) &= \frac{w + p_1 + p_2}{2p_2} \end{aligned}$$

The consumer consumes strict positive amounts of commodities 1 and 2 if and only if $w > p_1 + p_2$ and $(\boldsymbol{p}, w) >> \mathbf{0}$.

(ii) Derive the Walrasian demand function.

If $w > p_1 + p_2$ and $(\boldsymbol{p}, w) >> 0$, then the Walrasian demand function is given by the equations above. If $p_1 + p_2 \geq w \geq p_2$ and $(\boldsymbol{p}, w) >> 0$, then $x_1(\boldsymbol{p}, w) = 0$ and $x_2(\boldsymbol{p}, w) = \frac{w}{p_2}$.

(iii) Derive the indirect utility function.

If $w > p_1 + p_2$ and $(\boldsymbol{p}, w) >> \mathbf{0}$, then the indirect utility function is

$$\begin{aligned} v(\boldsymbol{p}, w) &= \ln\left(\frac{w + p_1 - p_2}{2p_1}\right) + \ln\left(\frac{w + p_1 - p_2}{2p_2}\right) \\ &= 2\ln(w + p_1 - p_2) - \ln(2p_1) - \ln(2p_2). \end{aligned}$$

If $p_1 + p_2 \geq w \geq p_2$ and $(\boldsymbol{p}, w) >> \mathbf{0}$, then it is

$$\begin{aligned} v(\boldsymbol{p}, w) &= \ln\left(\frac{w}{p_2} - 1\right) \\ &= \ln(w - p_2) - \ln(p_2). \end{aligned}$$

(iv) Consider now n consumers. Consumer $i \in \{1, ..., n\}$ has utility function

$$u^i(x_1^i, x_2^i) = a^i \ln(x_1^i + b^i) + \ln(x_2^i - 1)$$

with $a^i, b^i > 0$. Which restrictions do we need to place on a^i and b^i such that aggregate demands for commodities 1 and 2 are determined by prices p_1 and p_2, the sum $\sum_{i=1}^n w^i$, and does not depend on the distribution of wealth? How is this answer related to the Gorman form?

From previous considerations, we know that if consumer i buys strict positive amounts of both goods then

$$\frac{p_1}{p_2} = a^i \frac{x_2^i - 1}{x_1^i + b^i},$$

which is equivalent to

$$p_1 x_1^i + p_1 b^i = a^i p_2 x_2^i - a^i p_2.$$

Together with the budget equations, we can solve for Walrasian demand functions. If consumer i consumes strict positive amounts of both commodities, then i's Walrasian demand function is given

$$x_1^i(\boldsymbol{p}, w^i) = \frac{\frac{a^i}{a^i+1} w^i - \frac{a^i}{a^i+1}\left(\frac{b^i}{a^i} p_1 + p_2\right)}{p_1},$$

and

$$x_2^i(\boldsymbol{p}, w^i) = \frac{\frac{1}{a^i+1} w^i + \frac{1}{a^i+1}\left(b^i p_1 + a^i p_2\right)}{p_2}.$$

Summing over all consumers $i = 1, ..., n$, we obtain aggregate demand for commodity 1

$$\sum_{i=1}^n x_1^i(\boldsymbol{p}, w^i) = \frac{\sum_{i=1}^n \left(\frac{a^i}{a^i+1}\right) w^i - \sum_{i=1}^n \left(\frac{a^i}{a^i+1}\right)\left(\frac{b^i}{a^i} p_1 + p_2\right)}{p_1}.$$

If $a^i = a$ for all $i = 1, ..., n$, then aggregate demand for commodity 1 is

$$\sum_{i=1}^n x_1^i(\boldsymbol{p}, w^i) = \frac{\left(\frac{a}{a+1}\right) \sum_{i=1}^n w^i - \left(\frac{a}{a+1}\right)\left(\frac{\sum_{i=1}^n b^i}{a} p_1 + n p_2\right)}{p_1}.$$

Aggregate demand for commodity 1 is determined by prices and aggregate wealth but does not depend on the distribution of wealth. If $a^i > a^j$ for any two consumers i and j, then an income redistribution from consumer i to consumer j would increase aggregate demand for commodity 1.

Analogous arguments hold for the aggregate demand of commodity 2. We conclude that if prices and wealth levels are such that all consumers buy strict positive amounts of both commodities, then aggregate demand is determined by prices and aggregate wealth if and only if all consumers have identical parameters a^i. Parameters b^i are allowed to differ across consumers.

We know from class/textbook (Proposition 4.B.1. of MWG) that a necessary and sufficient condition for aggregate demand to depend on prices and aggregate wealth only is that indirect utilities are of the Gorman form, i.e.,

$$v^i(\boldsymbol{p}, w^i) = A(\boldsymbol{p})w^i + B^i(\boldsymbol{p}).$$

Our consumers' utility functions are of the Stone-Geary class. Using exponentiation with the power $\frac{1}{a^i+1}$ as a monotone transformation yields utility functions

$$(x_1^i + b^i)^{\alpha^i}(x_2^i - 1)^{(1-\alpha^i)},$$

where $\alpha^i := \frac{a^i}{a^i+1} \in (0, 1)$.

Walrasian demand functions can be written, respectively, as

$$\begin{aligned} x_1^i(\boldsymbol{p}, w^i) &= -b^i + \frac{1-\alpha^i}{p_1}\left(w^i + b^i p_1 - p_2\right) \\ x_2^i(\boldsymbol{p}, w^i) &= 1 + \frac{\alpha^i}{p_2}\left(2^i + b^i p_1 - p_2\right). \end{aligned}$$

The indirect utility function is

$$\begin{aligned} v^i(\boldsymbol{p}, w^i) &= \left(\frac{\alpha^i}{p_1}\left(w^i + b^i p_1 - p_2\right)\right)^{\alpha^i}\left(\frac{1-\alpha^i}{p_2}\left(w^i + b^i p_1 - p_2\right)\right)^{(1-\alpha^i)} \\ &= \left(w^i + b^i p_1 - p_2\right)\left(\frac{\alpha^i}{p_1}\right)^{\alpha^i}\left(\frac{1-\alpha^i}{p_2}\right)^{1-\alpha^i} \end{aligned}$$

If $a^i = a$ for all $i = 1, ..., n$, then $\alpha^i = \alpha$ for all $i = 1, ..., n$. In such a case, it is now clear that we can write the indirect utility functions as

$$v^i(\boldsymbol{p}, w^i) = A(\boldsymbol{p})w^i + B^i(\boldsymbol{p}).$$

Thus, when $a^i = a$ for all $i = 1, ..., n$, then consumers have indirect utilities of the Gorman form and the proposition mentioned above applies.

Question 3: The Core of an Economy

(a) Define the *weak core* of exchange economy $\{I, \mathbf{u}, \mathbf{w}\} = \{I, (u^i, w^i)_{i \in I}\}$ as the set of its allocations $\mathbf{x}$ such that there do not exist $\mathcal{H} \subseteq I$ and $(\hat{x}^i)_{i \in \mathcal{H}}$ for which $\sum_{i \in \mathcal{H}} \hat{x}^i = \sum_{i \in \mathcal{H}} w^i$ and $u^i(\hat{x}^i) > u^i(x^i)$ for all $i \in \mathcal{H}$. Argue that:

   i. the core is a subset of the weak core; and

   ii. if all preferences are continuous and strictly monotone, the core and the weak core are the same set.

(b) Given an exchange economy $\{I, \mathbf{u}, \mathbf{w}\}$, prove the following:

   i. If $\mathbf{w}$ is efficient, then it is a core allocation.

   ii. If each u^i is strongly quasiconcave and $\mathbf{w}$ is efficient, then $\mathbf{w}$ is the only core allocation.

(c) Consider a two-person exchange economy

$$\{I = \{1, 2\}, \mathbf{u} = (u^1, u^2), \mathbf{w} = (w^1, w^2)\},$$

and suppose that (p, x^1, x^2) is a competitive equilibrium. Argue that if (x^1, x^2) is *not* in the core of the economy, then it must be Pareto inefficient.

Answer: 1. (a) It suffices to show that the complement of the weak core is a subset of the complement of the core. Let allocation $\mathbf{x}$ *not be* in the weak core of the economy. By definition, there exist $\mathcal{H} \subseteq I$ and $(\hat{x}^i)_{i \in \mathcal{H}}$ for which $\sum_{i \in \mathcal{H}} \hat{x}^i = \sum_{i \in \mathcal{H}} w^i$ and $U^i(\hat{x}^i) > U^i(x^i)$ for all $i \in \mathcal{H}$. The latter implies, obviously, $U^i(\hat{x}^i) \geq U^i(x^i)$ for all $i \in \mathcal{H}$, with strict inequality for some. But this implies that the allocation is not in the core of the economy, as needed.

(b) Again, it's easier to show that the complement of the core is a subset of the complement of the weak core. If $\mathbf{x}$ isn't in the core, there exist $\mathcal{H} \subseteq I$ and $(\hat{x}^i)_{i \in \mathcal{H}}$ for which $\sum_{i \in \mathcal{H}} \hat{x}^i = \sum_{i \in \mathcal{H}} w^i$ and $U^i(\hat{x}^i) \geq U^i(x^i)$ for all $i \in \mathcal{H}$, with strict inequality for some $i' \in \mathcal{H}$. By monotonicity and continuity of $u^{i'}$, we can find $\bar{x}^{i'} < \hat{x}^{i'}$ such that $u^{i'}(\bar{x}^{i'}) > u^{i'}(x^{i'})$. Defining, for every $i \in \mathcal{H} \setminus \{i'\}$,

$$\bar{x}^i = \hat{x}^i + \frac{1}{I-1}(\hat{x}^{i'} - \bar{x}^{i'}) > \hat{x}^i,$$

we get, by strict monotonicity, that $u^i(\bar{x}^i) > u^i(\hat{x}^i) \geq u^i(x^i)$. By construction,

$$\sum_{i \in \mathcal{H}} \bar{x}^i = \bar{x}^{i'} + \sum_{i \in \mathcal{H} \setminus \{i'\}} \left[\hat{x}^i + \frac{1}{I-1}(\hat{x}^{i'} - \bar{x}^{i'})\right] = \sum_{i \in \mathcal{H}} \hat{x}^i = \sum_{i \in \mathcal{H}} w^i,$$

so it follows that $\mathbf{x}$ isn't in the weak core either.

2. (a) If coalition $\mathcal{H}$ had an objection $(x^i)_{i \in \mathcal{H}}$, we could construct an objection for the grand coalition, I, by simply completing the allocation with $x^i = w^i$ for all $i \notin \mathcal{H}$.

(b) Suppose that $\mathbf{x}$ is another allocation in the core. By construction, the allocation constructed by letting $\hat{x}^i = 1/2(w^i + x^i)$ for each i is feasible too. Since x is in the core, $u^i(x^i) \geq u^i(w^i)$, which implies that $u^i(\hat{x}^i) \geq u^i(w^i)$, by quasiconcavity. Since $x \neq (w^i)_{i \in I}$, there exists some

i for whom $x^i \neq w^i$. For such i, by strict quasiconcavity, the previous inequality is strict: $u^i(\hat{x}^i) > u^i(w^i)$.

Existence of $\hat{\mathbf{x}}$ contradicts the fact that $\mathbf{w}$ is Pareto efficient, though.

3. Since

$$x^i \in \arg\max_x \{u^i(x) : p \cdot x \leq p \cdot w^i\}$$

for both i, it must be true that $u^i(x^i) \geq u^i(w^i)$. Then, since there are only two people in the economy, for (x^1, x^2) to not be in the core, it must be blocked by the grand coalition. $\square$