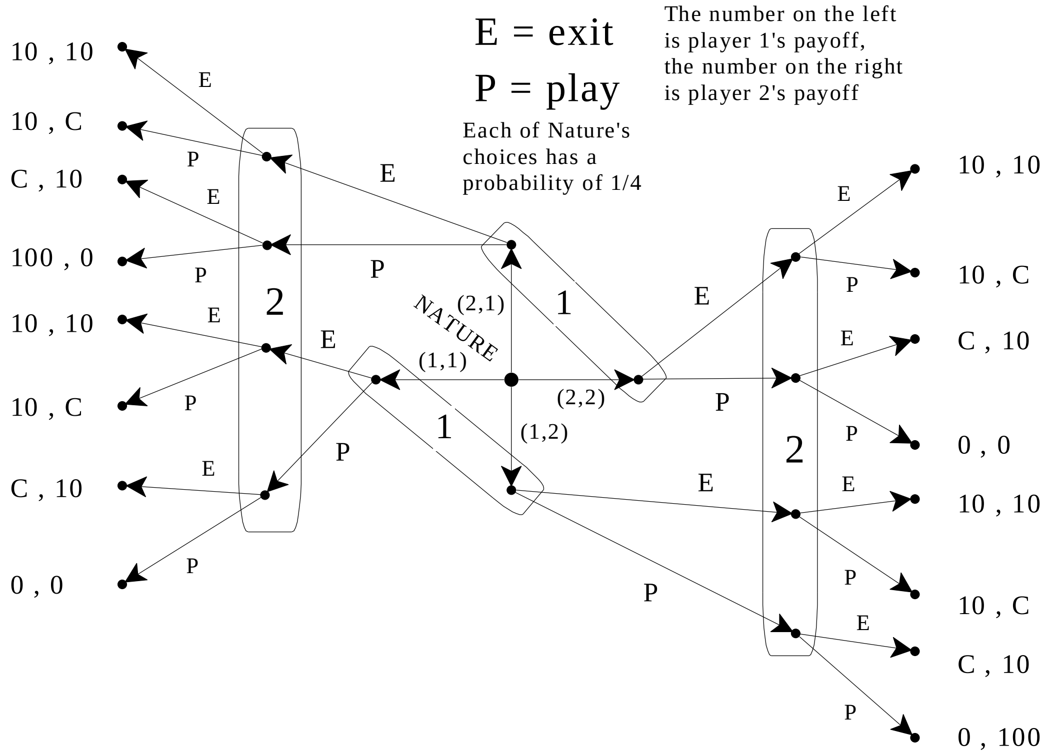
Micro Prelim June 21, 2022
QUESTION 1 ANSWER KEYS

(a.1) The extensive game is as follows:



(a.2) For each player a strategy is a choice of action for each number the player gets to see. The strategic-form is as follows:

Player 2

Player 1	E always	If 1 E, if 2 P	if 1 P, if 2 E	P always
E always	10 , 10	10 , 5	10 , 5	10 , 0
If 1 E, if 2 P	5 , 10	5 , 5	30 , 5	30 , 0
if 1 P, if 2 E	5 , 10	5 , 30	5 , 5	5 , 25
P always	0 , 10	0 , 30	25 , 5	25 , 25

(a.3) For each player "P always" is strictly dominated by "if 1 E, if 2 P" and "if 1 P, if 2 E" 1 is strictly dominated by "E always".

(a.4) Deleting the strictly dominated strategies we are left with:

Player 2

Player 1	E always	If 1 E, if 2 P
E always	10 , 10	10 , 5
If 1 E, if 2 P	5 , 10	5 , 5

Now the strategy "if 1 E, if 2 P" is strictly dominated by "E always". Thus there is a unique Nash equilibrium: (E always, E always).

(b) (E always, E always) is a Nash equilibrium also in this case. "E always" is the unique best reply to the strategy "E always" by the other player. Indeed, if, say, Player 1 chooses "E always" then for Player 2 "E always" guarantees a payoff of 10, while any other strategy would involve playing P at some information sets, implying an expected payoff less than 10 (0 at those information sets and 10 at those information sets, if any, where the strategy involves playing E).

(c) Assume that Player 1's strategy is "If 1 Exit, if 2 Play". Then at her left information set (where she knows that she has a 1) Player 2 must assign probability ½ to each of the two middle nodes, so that the expected payoff from Play is $\frac{C}{2}$ and the expected payoff from Exit is 10. The two are equal if and only if C = 20. Thus it is rational for Player 2 to mix at that information set if and only if C = 20. The same is true at Player 2's information set on the right (where she knows that she has a 2). Thus the answer is: C = 20.

(d) Let C = 20. Let $q_1 \in (0,1)$ be the probability with which Player 2 plays Exit when she has a 1 and $q_2 \in (0,1)$ be the probability with which Player 2 plays Exit when she has a 2. Then for Player 1 at his top information set (where he has a 2) Play is at least as good as Exit if and only if

$$\underbrace{\tfrac{1}{2}[20q_1 + 100(1-q_1)] + \tfrac{1}{2}20q_2}_{\text{expected payoff from Play}} \geq \underbrace{10}_{\substack{\text{expected}\\ \text{payoff}\\ \text{from Exit}}} \quad \text{equivalent to} \quad 2 \geq 2q_1 - 0.5q_2 \text{ (true for all } q_1, q_2 \in (0,1)\text{)}.$$

For Player 1 at his bottom information set (where he has a 1) Exit is at least as good as Play iff

$$\underbrace{10}_{\substack{\text{expected}\\ \text{payoff}\\ \text{from Exit}}} \geq \underbrace{\tfrac{1}{2}[20q_1] + \tfrac{1}{2}20q_2}_{\text{expected payoff from Play}} \quad \text{which is equivalent to} \quad 1 \geq q_1 + q_2$$

Thus, for all $q_1 \in (0,1)$ and $q_2 \in (0,1)$ such that $q_1 + q_2 \leq 1$ the following strategy profile (together with the beliefs specified above) is a weak sequential equilibrium: Player 1's strategy is "If 1 Exit, if 2 Play", Player 2's strategy is "if 1 Exit with probability q_1, if 2 Exit with probability q_2".

(e.1) The partitions are as follows:

Player 1:

(1,1)	(1,2)	(1,3)	(1,4)
(2,1)	(2,2)	(2,3)	(2,4)
(3,1)	(3,2)	(3,3)	(3,4)
(4,1)	(4,2)	(4,3)	(4,4)

Player 2:

(1,1)	(1,2)	(1,3)	(1,4)
(2,1)	(2,2)	(2,3)	(2,4)
(3,1)	(3,2)	(3,3)	(3,4)
(4,1)	(4,2)	(4,3)	(4,4)

(e.2) The common knowledge partition is the trivial one containing the set of all states.

(e.3) *E* = the union of the third row and the second column: {(3,1), (3,2), (3,3), (3,4), (1,2), (2,2), (4,2)}.

Question 2 Answer Keys

I always wondered why some people buy ugly but expensive handbags. It is as if the price, as a special characteristic of the commodity, contributes to the utility of the commodity.

Consider a utility function of the form

$$u(\boldsymbol{x}, \boldsymbol{p}) \;=\; \prod_{\ell=1}^{L} x_\ell^{\sqrt{p_\ell}}. \tag{1}$$

where $\boldsymbol{x} = (x_1, ..., x_L) \in \mathbb{R}_+^L$ is the consumption vector and $\boldsymbol{p} = (p_1, ..., p_L) >> 0$ is the price vector. This utility function resembles the Cobb-Douglas utility function except that its parameters are the prices.

Denote the consumer's wealth by w and assume $w > 0$. Let's consider the problem of maximizing this utility function subject to the budget constraint, i.e.,

$$\max_{\boldsymbol{x} \in \mathbb{R}^L} u(\boldsymbol{x}, \boldsymbol{p}) \;=\; \prod_{\ell=1}^{L} x_\ell^{\sqrt{p_\ell}} \tag{2}$$

subject to

$$\boldsymbol{p} \cdot \boldsymbol{x} \;\leq\; w \tag{3}$$

$$\boldsymbol{x} \;\geq\; \mathbf{0} \tag{4}$$

a. Use the Lagrangian or the Kuhn-Tucker-Lagrangian approach to solve the constrained maximization problem. Derive step-by-step the Walrasian demand functions.

The utility function is monotone in $\boldsymbol{x}$. Thus, we ignore the nonnegativity constraints.

It is useful to consider the log-monotone transformation of the utility function. The Lagrangian is

$$\mathcal{L}(x_1, \ldots, x_L, \lambda) = \sum_{\ell=1}^{L} \sqrt{p_\ell} \log x_\ell - \lambda \left(\sum_{\ell=1}^{L} p_\ell x_\ell - w \right)$$

The first-order necessary conditions for any $\ell \in \{1, ..., L\}$:

$$\frac{\partial \mathcal{L}(\boldsymbol{x}, \lambda)}{\partial x_\ell} = \frac{\sqrt{p_\ell}}{x_\ell} - \lambda p_\ell \stackrel{!}{=} 0 \tag{5}$$

The MRS of any two goods (ℓ, k):

$$\frac{\frac{\sqrt{p_\ell}}{x_\ell}}{\frac{\sqrt{p_k}}{x_k}} = \frac{p_\ell}{p_k} \Rightarrow x_k = x_\ell \frac{\sqrt{p_\ell}}{\sqrt{p_k}} \tag{6}$$

Plug into the budget constraint:

$$p_\ell x_\ell + \sum_{k\neq\ell}^{L} p_k x_k = p_\ell x_\ell + \sum_{k\neq\ell}^{L} p_k x_\ell \frac{\sqrt{p_\ell}}{\sqrt{p_k}} = w \tag{7}$$

$$\Rightarrow \sqrt{p_\ell} x_\ell \left(\sqrt{p_\ell} + \sum_{k\neq\ell}^{L} \sqrt{p_k} \right) = \sqrt{p_\ell} x_\ell \left(\sum_{k=1}^{L} \sqrt{p_k} \right) = w \tag{8}$$

$$\Rightarrow x_\ell^* = \frac{w}{\sqrt{p_\ell}\left(\sum_{k=1}^{L} \sqrt{p_k}\right)} = \frac{w}{p_\ell} \frac{\sqrt{p_\ell}}{\left(\sum_{k=1}^{L} \sqrt{p_k}\right)} \tag{9}$$

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b. Show that Walrasian demand functions are homogeneous of degree zero.

For any $\ell \in 1, ..., L$, $\alpha > 0$

$$x_\ell(\alpha \boldsymbol{p}, \alpha w) = \frac{\alpha w}{\alpha p_\ell} \frac{\sqrt{\alpha p_\ell}}{\left(\sum_{k=1}^{L} \sqrt{\alpha p_k}\right)} = \frac{w}{p_\ell} \frac{\sqrt{\alpha}\sqrt{p_\ell}}{\sqrt{\alpha}\left(\sum_{k=1}^{L} \sqrt{p_k}\right)} = \frac{w}{p_\ell} \frac{\sqrt{p_\ell}}{\left(\sum_{k=1}^{L} \sqrt{p_k}\right)} = x_\ell(\boldsymbol{p}, w)$$

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c. State the indirect utility function and simplify as much as you can.

$$v(\boldsymbol{p}, w) = \prod_{\ell=1}^{L} \left(\frac{w}{p_\ell} \frac{\sqrt{p_\ell}}{\left(\sum_{k=1}^{L} \sqrt{p_k}\right)} \right)^{\sqrt{p_\ell}} \tag{10}$$

$$= \frac{w^{\sum_{\ell=1}^{L} \sqrt{p_\ell}}}{\prod_{\ell=1}^{L} \sqrt{p_\ell}^{\sqrt{p_\ell}} \left(\sum_{k=1}^{L} \sqrt{p_k}\right)^{\sum_{\ell=1}^{L} \sqrt{p_\ell}}} \tag{11}$$

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d. Check whether the indirect utility function is homogeneous of degree zero.

$$v(\alpha \boldsymbol{p}, \alpha w) = \frac{(\alpha w)^{\sum_{i=1}^{L} \sqrt{\alpha p_i}}}{\prod_{\ell=1}^{L} \sqrt{\alpha p_i}^{\sqrt{\alpha p_\ell}} \left(\sum_{\ell=1}^{L} \sqrt{\alpha p_\ell}\right)^{\sum_{\ell=1}^{L} \sqrt{\alpha p_\ell}}} \tag{12}$$

$$= \frac{\alpha^{\sum_{\ell=1}^{L} \sqrt{\alpha p_\ell}} (w)^{\sum_{\ell=1}^{L} \sqrt{\alpha p_\ell}}}{\sqrt{\alpha}^{\sum_{\ell=1}^{L} \sqrt{\alpha p_\ell}} \left(\prod_{\ell=1}^{L} \sqrt{p_\ell}^{\sqrt{\alpha p_\ell}}\right) \sqrt{\alpha}^{\sum_{\ell=1}^{L} \sqrt{\alpha p_\ell}} \left(\sum_{\ell=1}^{L} \sqrt{p_\ell}\right)^{\sum_{\ell=1}^{L} \sqrt{\alpha p_\ell}}} \tag{13}$$

$$= \frac{w^{\sum_{\ell=1}^{L} \sqrt{\alpha p_\ell}}}{\prod_{\ell=1}^{L} \sqrt{p_\ell}^{\sqrt{\alpha p_\ell}} \left(\sum_{\ell=1}^{L} \sqrt{p_\ell}\right)^{\sum_{\ell=1}^{L} \sqrt{\alpha p_\ell}}} \tag{14}$$

$$= \left(\frac{w^{\sum_{\ell=1}^{L} \sqrt{p_\ell}}}{\prod_{\ell=1}^{L} \sqrt{p_\ell}^{\sqrt{p_\ell}} \left(\sum_{\ell=1}^{L} \sqrt{p_\ell}\right)^{\sum_{\ell=1}^{L} \sqrt{p_\ell}}} \right)^{\sqrt{\alpha}} \tag{15}$$

So, it is not homogeneous of degree zero, which is different from "standard" indirect utility functions.

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e. We all noticed that inflation is back. For the sake of concreteness, consider inflation as a proportional rise of all prices and wealth. Would consumers with such kind of utility functions be happy about inflation?

What a timely utility function! The indirect utility increases with inflation.

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f. Now consider the expenditure minimization problem

$$\min_{\boldsymbol{x}\in\mathbb{R}^L} \boldsymbol{p}\cdot\boldsymbol{x} \tag{16}$$

subject to

$$u(\boldsymbol{x},\boldsymbol{p}) = \prod_{\ell=1}^{L} x_\ell^{\sqrt{p_\ell}} \geq \bar{u} \tag{17}$$

$$\boldsymbol{x} \geq \boldsymbol{0} \tag{18}$$

for $\bar{u} > 0$. Solve step-by-step for the Hicksian demand functions.

The Lagrangian to solve the expenditure minimization problem is

$$\mathcal{L}(x_1,\ldots,x_L,\lambda) = \sum_{\ell=1}^{L} p_\ell x_\ell - \lambda\left(\prod_{\ell=1}^{L} x_\ell^{\sqrt{p_\ell}} - \bar{u}\right)$$

The first-order necessary conditions for any $i \in \{1,...,L\}$:

$$\frac{\partial\mathcal{L}(\boldsymbol{x},\lambda)}{\partial x_\ell} = p_\ell - \lambda\frac{\sqrt{p_\ell}}{x_\ell}\prod_{k=1}^{L} x_k^{\sqrt{p_k}} \overset{!}{=} 0 \tag{19}$$

Combining with the constraint on $\bar{u}$, we can get the solution:

$$h_\ell(\boldsymbol{p},\bar{u}) = x_\ell^* = \bar{u}^{\frac{1}{\sum_{k=1}^{L}\sqrt{p_k}}}\prod_{k\neq\ell}\left(\frac{\sqrt{p_k}}{\sqrt{p_\ell}}\right)^{\frac{\sqrt{p_k}}{\sum_{m=1}^{L}\sqrt{p_m}}}$$

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g. Is the Hicksian demand function homogeneous of degree zero in prices? Explain.

The Hicksian demand is not homogeneous of degree zero with respect to prices.

$$h_\ell(\alpha \boldsymbol{p}, \bar{u}) = \bar{u}^{\frac{1}{\sum_{k=1}^{L} \sqrt{\alpha p_k}}} \prod_{k \neq \ell} \left(\frac{\sqrt{\alpha p_k}}{\sqrt{\alpha p_\ell}} \right)^{\frac{\sqrt{\alpha p_k}}{\sum_{m=1}^{L} \sqrt{\alpha p_m}}} \tag{20}$$

$$= \bar{u}^{\frac{1}{\sqrt{\alpha} \sum_{k=1}^{L} \sqrt{p_k}}} \prod_{k \neq \ell} \left(\frac{\sqrt{p_k}}{\sqrt{p_\ell}} \right)^{\frac{\sqrt{p_k}}{\sum_{m=1}^{L} \sqrt{p_m}}} \tag{21}$$

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h. Write down the expenditure function.

$$e(\boldsymbol{p}, \bar{u}) = \sum_{\ell=1}^{L} p_\ell x_\ell \tag{22}$$

$$= \sum_{\ell=1}^{L} p_\ell \bar{u}^{\frac{1}{\sum_{k=1}^{L} \sqrt{p_k}}} \prod_{k \neq \ell} \left(\frac{\sqrt{p_k}}{\sqrt{p_\ell}} \right)^{\frac{\sqrt{p_k}}{\sum_{m=1}^{L} \sqrt{p_m}}} \tag{23}$$

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i. Is it homogeneous of degree one in prices? Explain.

The expenditure function is not homogeneous of degree 1 w.r.t. prices because Hicksian demand is not homogeneous of degree zero w.r.t. prices.

Question 3 Answer Keys

Before proceeding, let us verify the properties that were taken for granted:

1. Note that the domain of the maximization program (1) can be written as

$$\{z \in \mathbb{R}^L \mid w^i + z \geq 0 \text{ and } p \cdot z \leq 0\} \cap \{z \in \mathbb{R}^L \mid \forall \ell, -n \leq z_l \leq n\}.$$

   The first of these two sets is non-empty, convex and closed while the second is non-empty, convex and compact. Since both sets contain the origin, their intersection is non-empty. The intersection is also compact, so by Weierstrass the program has at least one solution, since u^i is continuous. Finally, the solution is unique since the intersection is convex and u^i is strictly quasi-concave.

   Continuity is straightforward from Berge's theorem, since the domain of program (1) is a continuous correspondence and its objective function is continuous.

   That Z^n is well defined and continuous follows immediately.

2. By definition, for all i and ℓ, $|z^{i,n}(p)| \leq n$, so $|\sum_i z^{i,n}(p)| \leq \sum_i |z^{i,n}(p)| \leq n \cdot I$ by construction.

3. The argument is standard, as in class: the domain of the problem is non-empty, convex and compact, and the objective function is continuous and concave, so it follows that the set of solutions is non-empty, compact and convex. As z varies, the domain of the problem is a constant correspondence, so it is continuous. Since the objective function is continuous, by Berge the set of solutions is an upper hemicontinuous correspondence.

4. The first claim is straightforward, from properties 2 and 3: the product of two non-empty, compact and convex sets retains the three properties. Similarly, upper hemicontinuity is preserved when a correspondence is defined by Cartesian products of the images of finitely many correspondences that exhibit the property. Finally, $P^n(z) \subseteq \Delta$ and, by (3), $\{Z^n(p)\} \subseteq \mathbb{C}^n$, so $\Gamma^n(p,z) \subseteq \Delta \times \mathbb{C}^n$.

Now, the proof is as follows:

Answer: (a) Since, by property 4, Γ is a non-empty-, compact- and convex-valued upper hemicontinuous correspondence mapping a non-empty, compact and convex set into itself, the result follows from Kakutani's fixed point theorem.

(b) By construction $p \cdot Z^n(p) = 0$. By (a), then, $\bar{p}^n \cdot \bar{z}^n = \bar{p}^n \cdot Z^n(\bar{p}^n) = 0$.

For the second claim, suppose by way of contradiction that $\bar{z}_1^n > 0$. Then, consider

$$p = \tfrac{1}{2}[\bar{p}^n + (1, 0, \ldots, 0)]$$

Then, $p \in \Delta$ and

$$p \cdot \bar{z}^n = \tfrac{1}{2}\bar{p}^n \cdot \bar{z}^n + \tfrac{1}{2}\bar{z}_1^n = \tfrac{1}{2}\bar{z}_1^n > 0 = \bar{p}^n \cdot \bar{z}^n,$$

contradicting the fact that $\bar{p}^n \in P^n(\bar{z}^n)$, according to (a).

(c) By definition of the truncated functions, Eq. (1), $w^i + \bar{z}^{i,n} \geq 0$, so

$$\bar{z}^{i,n} \geq -w^i, \tag{$*$}$$

which immediately implies that $\bar{z}^n \leq -\sum_i w^i$. By (b), also, $\bar{z}^{i,n} + \sum_{j \neq i} \bar{z}^{j,n} \leq 0$, so, by $(*)$

$$\bar{z}^{i,n} \leq -\textstyle\sum_{j \neq i} \bar{z}^{j,n} \leq \sum_{j \neq i} w^j.$$

(d) Since each $\bar{p}^n \in \Delta$, sequence $(\bar{p}^n)_{n=1}^{\infty}$ is bounded. By (b) and (c), so are all the other components of the sequence. Existence of a convergent subsequence follows by Bolzano-Weierstrass.

(e) That $\bar{p} \in \Delta$ follows from closedness of Δ. The other two claims follow from the fact that limits preserve weak inequalities.

(f) This follows simply from the fact that the sequence is bounded.

(g) Suppose not. Then, there is some z^* such that $w^i + z^* \geq 0$, $\bar{p}^n \cdot z^* = 0$ and $u^i(w^i + z^*) > u^i(w^i + \bar{z}^{i,n})$. Construct $z = \alpha \bar{z}^{i,n} + (1 - \alpha) z^*$, for $0 < \alpha < 1$. By construction, $w^i + z \geq 0$ and $\bar{p}^n \cdot z = 0$, while, by strict quasi-concavity of u^i,

$u^i(w^i + z) > u^i(w^i + \bar{z}^{i,n})$. By (f), $|\bar{z}^{i,n}_\ell| < n$ for all ℓ, so if α is close enough to 1, $|z_\ell| \leq n$ for all ℓ. But this contradicts the fact that $\bar{z}^{i,n} = z^{i,n}(\bar{p}^n)$ is the solution to Program (1).

(h) It follows from (g) that $\bar{z}^n = \sum_i \bar{z}^{i,n} = \sum_i z^i(\bar{p}^n) = Z(\bar{p}^n)$, and hence, by convergence and continuity, that[7]

$$\bar{z} = \lim_{n\to\infty} \bar{z}^n = Z(\lim_{n\to\infty} \bar{p}^n) = Z(\bar{p}).$$

By (g), since limits preserve weak inequalities, $\bar{z} \leq 0$.

(i) Suppose not: $\bar{z} < 0$. Then, define $z = \bar{z}^1 - \bar{z} > \bar{z}^1$. By strict monotonicity, $u^1(w^1 + z) > u^1(w^1 + \bar{z}^1)$. By (e), $\bar{p} \cdot z = \bar{p} \cdot \bar{z}^1 - \bar{p} \cdot \bar{z} = 0$. But this is impossible, since, by (g) and continuity,

$$\bar{z}^1 = \lim_{n\to\infty} z^{1,n} = \lim_{n\to\infty} z^1(\bar{p}^n) = z^1(\bar{p})$$

so

$$\bar{z}^1 = \arg\max_{z\in\mathbb{R}^L} \left\{ u^1(w^1 + z) : w^1 + z \geq 0 \text{ and } \bar{p} \cdot z = 0 \right\}. \qquad \square$$

[7] Before the following, we need to argue that $\bar{p} \in \Delta^\circ$. This follows from boundary behavior: if $\bar{p} \in \Delta^\partial$, then $Z_\ell(\bar{p}^n)$ is unbounded above for some ℓ, so $\bar{z}^n$ could not be convergent.