ANSWER KEYS Micro Prelim June 22, 2020

QUESTION 1 ANSWER KEYS

(a) The strategic form is as follows:

		Country 2: C	Country 2: D
Country 1	C	$\frac{x}{2}$, $1-\frac{x}{2}$	x, $1-x$
	D	$1-x$, x	$\frac{1-x}{2}$, $\frac{1+x}{2}$

(b) For D to dominate C for Country 1 we need $1-x \geq \frac{x}{2}$ (that is, $x \leq \frac{2}{3}$) **and** $\frac{1-x}{2} \geq x$ (that is, $x \leq \frac{1}{3}$). Thus

- If $x < \frac{1}{3}$ D strictly dominates C and if $x = \frac{1}{3}$ D weakly (but not strictly) dominates C.
- If $x > \frac{2}{3}$ C strictly dominates D and if $x = \frac{2}{3}$ C weakly (but not strictly) dominates D.

For values of x strictly between $\frac{1}{3}$ and $\frac{2}{3}$ Country 1 does not have a dominant strategy.

(c) For C to dominate D we need $1-\frac{x}{2} \geq 1-x$ (always true) **and** $x \geq \frac{1+x}{2}$ (that is, $x = 1$).

For D to dominate C we need $1-\frac{x}{2} \leq 1-x$ (that is, $x = 0$) **and** $x \leq \frac{1+x}{2}$ (always true). Thus

- If $x = 0$ D weakly dominates C.
- If $x = 1$ C weakly dominates D.

For any value of x strictly between 0 and 1 Country 2 does not have a dominant strategy.

(d) If $x \leq \frac{1}{3}$ (D,D) is a Nash equilibrium and if $x \geq \frac{2}{3}$ (C,C) is a Nash equilibrium. For values of x strictly between $\frac{1}{3}$ and $\frac{2}{3}$ there is no pure-strategy Nash equilibrium.

(e) Let $x \in \left(\frac{1}{3}, \frac{2}{3}\right)$. To find the mixed-strategy Nash equilibrium, let p be the probability with which Country 1 chooses C and q the probability with which Country 2 chooses C. Then it must be that

$$\tfrac{x}{2} q + x(1-q) = (1-x)q + \tfrac{1-x}{2}(1-q), \text{ that is, } \quad q = 3x-1$$

$$\left(1-\tfrac{x}{2}\right) p + x(1-p) = (1-x)p + \tfrac{1+x}{2}(1-p), \text{ that is, } \quad p = 1-x$$

Thus the mixed-strategy Nash equilibrium is as follows:

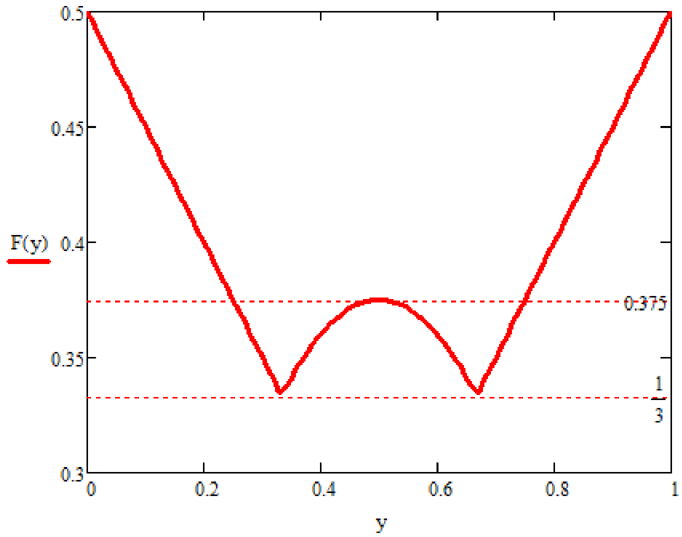
$$\text{Country 1's strategy: } \begin{pmatrix} C & D \\ 1-x & x \end{pmatrix}, \quad \text{Country 2's strategy: } \begin{pmatrix} C & D \\ 3x-1 & 2-3x \end{pmatrix}$$

At the mixed-strategy equilibrium the expected payoffs are (note that this is a constant-sum game):

$$\text{Country 1's payoff: } \frac{3x(1-x)}{2}, \quad \text{Country 2's payoff: } 1-\frac{3x(1-x)}{2}$$

(f) The equilibrium payoff function of Player 1 is

$$\Pi_1(x) = \begin{cases} \dfrac{1-x}{2} & \text{if } x \leq \frac{1}{3} \\ \dfrac{3x(1-x)}{2} & \text{if } \frac{1}{3} < x < \frac{2}{3} \\ \dfrac{x}{2} & \text{if } \frac{2}{3} \leq x \leq 1 \end{cases}. \quad \text{Its graph is as follows:}$$



(g) (g.1) A pure strategy of Country 1 is a pair (x, f) where $x \in [0,1]$ and $f : [0,1] \to \{C, D\}$ is a function that specifies the location of the attack as a function of the value of *x*.

A pure strategy of Country 2 is a function $g : [0,1] \to \{C, D\}$ that specifies the location to defend as a function of the value of *x*.

(g.2) Let $\Delta_{[0,1]}$ be the set of probability distributions over [0,1] and $\Delta_{\{C,D\}}$ the set of probability distributions over $\{C, D\}$. A behavioral strategy of Country 1 is a pair (p, f) where $p \in \Delta_{[0,1]}$ ($p(x)$ is the probability of choosing $x \in [0,1]$) and $f : [0,1] \to \Delta_{\{C,D\}}$ is a function that specifies the probabilities of attacking areas *C* and *D* as a function of the value of *x*.

A behavioral strategy of Country 2 consists of a function $g : [0,1] \to \Delta_{\{C,D\}}$ that specifies the probabilities of defending areas *C* and *D* as a function of the value of *x*.

Question 2 Answer Keys

a) Roger lives a simple life: For breakfast, he eats eggs with coffee, and for dinner he eats hot dogs with beer. In between he watches Fox News and earns his money with maintaining a couple of thousand twitter bots. Since he likes everything to be in order and simple, he puts his income into two pots: One with money for breakfast and one with money for dinner. Eggs and coffee are paid only from the breakfast pot; hot dogs and beer only from the dinner pot. That is,

$$\begin{aligned} p_1 x_1 + p_2 x_2 &\leq w_B \\ p_3 x_3 + p_4 x_4 &\leq w_D \end{aligned}$$

with $p_1, p_2, p_3, p_4, w_B, w_D > 0$, where subscripts $1, 2, 3, 4, B$, and D refer to eggs, coffee, hot dogs, beer, breakfast, and dinner, respectively. As usual, p_i is the price of one unit of commodity i, x_i is the quantity consumed of commodity i, and w_B and w_D is the amount of money in his breakfast or dinner pot, respectively.

His utility function is given by

$$u(x_1, x_2, x_3, x_4) = \left(x_1^e x_2^c + x_3^h x_4^b\right)^a$$

with $e, c, h, b, a > 0$.

aa) Given w_B and w_D, derive step-by-step Roger's Walrasian demand functions for eggs, coffee, hot dogs, and beer. Verify also second-order conditions.

Since utility functions are unique up to monotone transformation, we can consider the equivalent utility function

$$\tilde{u}(x_1, x_2, x_3, x_4) = x_1^e x_2^c + x_3^h x_4^b$$

The utility function is the sum of two Cobb-Douglas utility functions. Since the constraints for each term are independent as well, we can find the constrained max of the function by looking at the constraint max of each term. Thus, it just boils down to deriving twice demand functions for Cobb-Douglas utility functions on $\mathbb{R}^2_+$.

Since the Cobb-Douglas utility function is strictly increasing, the budget constraints hold with equality at the solution. Moreover, the min of each Cobb-Douglas utility function is attained at $x_i = 0$. So at the max we must have $x_i > 0$ for both factors in each Cobb-Douglas utility function.

Taking into account that we have to answer ab) as well, we use the Lagrange approach. I.e.,

$$\max_{x_1, x_2 \in \mathbb{R}^2_+} L(x_1, x_2, \lambda_B) = x_1^e x_2^c - \lambda_B (p_1 x_1 + p_2 x_2 - w_B).$$

Side remark: Since in ab) we need the marginal utility of money, it will be useful not to use the usual log transformation of Cobb-Douglas utilities. Monotone transformations have to be applied to the utility function above, not separately to each of the two Cobb-Douglas utilities. While for demand functions, it wouldn't matter, it does matter for interpreting in part ab) the Lagrange multiplier as marginal utility of breakfast or dinner money, respectively.

Derive first-order conditions:

$$\begin{aligned} e x_1^{e-1} x_2^c &= \lambda_B p_1 \\ c x_1^e x_2^{c-1} &= \lambda_B p_2 \\ p_1 x_1 + p_2 x_2 &= w_B \end{aligned}$$

Rewrite the first two conditions a little:

$$\begin{aligned} e x_1^e x_2^c &= \lambda_B p_1 x_1 \\ c x_1^e x_2^c &= \lambda_B p_2 x_2 \end{aligned}$$

Add up them up to get

$$(e+c) x_1^e x_2^c = \lambda_B (p_1 x_1 + p_2 x_2)$$

Substitute the third condition into the r.h.s.

$$(e+c) x_1^e x_2^c = \lambda_B w_B$$

and solve for λ_B

$$\lambda_B = \frac{e+c}{w_B} x_1^e x_2^c$$

Plug in above and solve for the Walrasian demand functions:

$$\begin{aligned} x_1 &= \frac{e}{e+c} \frac{w_B}{p_1} \\ x_2 &= \frac{c}{e+c} \frac{w_B}{p_2} \end{aligned}$$

Analogously we obtain

$$\begin{aligned} x_3 &= \frac{h}{h+b} \frac{w_D}{p_3} \\ x_4 &= \frac{b}{h+b} \frac{w_D}{p_4} \end{aligned}$$

We are also asked to verify second-order conditions. Consider the bordered Hessian. I.e., we need to show that

$$\det \begin{pmatrix} 0 & p_1 & p_2 \\ p_1 & -e x_1^{e-2} x_2^c & ec x_1^{e-1} x_2^{c-1} \\ p_2 & ec x_1^{e-1} x_2^{c-1} & -c x_1^e x_2^{c-2} \end{pmatrix} > 0$$

Use Sarrus' rule and focus just on the sign of each term. Then you see that indeed the determinant must be strictly positive. Analogously for the bordered Hessian at dinner.

ab) While watching Fox News, Roger heard about the government shifting money earmarked for fighting drugs to the construction of the border wall. He suddenly thought whether it would be better for him to move one dollar from his breakfast pot to the dinner pot. Find a condition on the primitives (i.e., parameters e, c, h, b, a, prices p_1, p_2, p_3, p_4, and budgets w_B and w_D) under which moving a dollar from his breakfast pot and putting it in the dinner pot is better for him.

Recall the interpretation of Lagrange multipliers. They give us the marginal utility of relaxing the breakfast or dinner budget constraint. The question boils down to which one is bigger. The marginal utility to dinner money is larger than the marginal utility to breakfast money iff

$$\begin{aligned} \lambda_D &\geq \lambda_B \\ \frac{h+b}{w_D} x_1^h x_2^b &\geq \frac{e+c}{w_B} x_1^e x_2^c \\ \frac{h+b}{w_D}\left(\frac{h}{h+b}\frac{w_D}{p_3}\right)^h \left(\frac{b}{h+b}\frac{w_D}{p_4}\right)^b &\geq \frac{e+c}{w_B}\left(\frac{e}{e+c}\frac{w_B}{p_1}\right)^e \left(\frac{c}{e+c}\frac{w_B}{p_2}\right)^c \\ w_D^{h+b-1}\frac{h+b}{(h+b)^{h+b}}\left(\frac{h}{p_3}\right)^h \left(\frac{b}{p_4}\right)^b &\geq w_B^{e+c-1}\frac{e+c}{(e+c)^{e+c}}\left(\frac{e}{p_1}\right)^e \left(\frac{c}{p_2}\right)^c \quad (1) \end{aligned}$$

ac) Suppose that the primitives are such that it is better for Roger to move a dollar from his breakfast pot to his dinner pot. Suppose further that both $e + c \geq 1$ and $h + b \geq 1$. Would it be better for Roger to skip breakfast altogether and just spend all the money on dinner?

Observe that when $e+c \geq 1$ and $h+b \geq 1$, then preferences are quasiconvex in meals. Observe further that the l.h.s. of inequality (1) monotonically increases in w_D and the r.h.s. monotonically decreases in w_B. Thus, if inequality (1) holds, it also holds after shifting one dollar from the breakfast pot to the dinner pot, and also after shifting a second dollar from the breakfast pot to the dinner pot etc. In other words, along successive shifts, the ''bang for a buck'' of dinner money continues to be larger than the ''bang for a buck'' of breakfast money. Thus, it makes sense for Roger to spend all his money on dinner only.

b) Verify for the case of Cobb-Douglas utility functions on $\mathbb{R}^2_+$ that the Slutsky substitution matrix is negative semidefinite and symmetric.

`We can calculate the Slutsky substitution matrix with the Hick demand functions or Walrasian demand functions. Since we derived already the Walrasian demand functions, we may use them here as well.`

`The Slutsky substitution matrix is`

$$
\begin{aligned}
&S(p_1, p_2, w_B) \\
&= \begin{pmatrix} \frac{\partial x_1(p_1,p_2,w_B)}{\partial p_1} + \frac{\partial x_1(p_1,p_2,w_B)}{\partial w_B} x_1(p_1, p_2, w_B) & \frac{\partial x_1(p_1,p_2,w_B)}{\partial p_2} + \frac{\partial x_1(p_1,p_2,w_B)}{\partial w_B} x_2(p_1, p_2, w_B) \\ \frac{\partial x_2(p_1,p_2,w_B)}{\partial p_1} + \frac{\partial x_2(p_1,p_2,w_B)}{\partial w_B} x_1(p_1, p_2, w_B) & \frac{\partial x_2(p_1,p_2,w_B)}{\partial p_2} + \frac{\partial x_2(p_1,p_2,w_B)}{\partial w_B} x_2(p_1, p_2, w_B) \end{pmatrix} \\
&= \begin{pmatrix} -\frac{e}{e+c}\frac{c}{e+c}\frac{w_B}{p_1^2} & \frac{e}{e+c}\frac{c}{e+c}\frac{w_B}{p_1 p_2} \\ \frac{e}{e+c}\frac{c}{e+c}\frac{w_B}{p_1 p_2} & -\frac{e}{e+c}\frac{c}{e+c}\frac{w_B}{p_2^2} \end{pmatrix}
\end{aligned}
$$

`Clearly, this matrix is symmetric since the off-diagonal elements are equal.`

`Note that both main diagonal elements are nonpositive. Moreover, the determinant is nonnegative. I.e.,`

$$
\det S(p_1, p_2, w) = -\frac{e}{e+c}\frac{c}{e+c}\frac{w_B}{p_1^2} \cdot - \left(\frac{e}{e+c}\frac{c}{e+c}\frac{w_B}{p_2^2} \right) - \left(\frac{e}{e+c}\frac{c}{e+c}\frac{w_B}{p_1 p_2} \right)^2 = 0
$$

`Thus, it is negative semidefinite.`

Question 3 Answer keys

In this question you will argue that the set of competitive equilibrium prices of a competitive economy has essentially no structure other than *closedness*.

Consider a two-commodity world, let prices be normalized to the sphere

$$\mathcal{S} = \{p \in \mathbb{R}^2_{++} \mid \|p\| = 1\},$$

fix $\varepsilon > 0$, and denote

$$\mathcal{S}_\varepsilon = \{p \in \mathcal{S} \mid p_1 \geqslant \varepsilon \text{ and } p_2 \geqslant \varepsilon\}.$$

Fix an arbitrary set $E \subseteq \mathcal{S}_\varepsilon$ and suppose that it is closed. Define the function $Z : \mathcal{S} \to \mathbb{R}^2$ as follows:

(i) for commodity 1,

$$Z_1(p) = \min_{\hat{p}} \{\|\hat{p} - p\| : \hat{p} \in E\}; \tag{1}$$

(ii) and for commodity 2,

$$Z_2(p) = -\frac{p_1}{p_2} Z_1(p). \tag{2}$$

With this construction:

(a) Argue that Z is defined for all $p \in \mathcal{S}$.

Answer: Since $\mathcal{S}$ is bounded and $E \subseteq \mathcal{S}$ is closed, it is immediate that E is compact. By Weierstrass's theorem, the program in Eq. (1) always has a solution, since the Euclidean norm is continuous. □

(b) Argue that Z is continuous and satisfies Walras's law.

Answer: That Z is continuous follows from Berge's Theorem, since the Euclidean norm is continuous and the domain correspondence in the program of Eq. (1) is trivially continuous, since it is a constant.

For Walras's law:

$$p \cdot Z(p) = p_1 Z_1(p) + p_2 Z_2(p) = p_1 Z_1(p) - \frac{p_1 p_2}{p_2} Z_1(p) = 0.$$ □

(c) Argue that there exists an exchange economy

$$\{\mathcal{I}, (u^i, w^i)_{i \in \mathcal{I}}\}$$

where each $u^i : \mathbb{R}^2_+ \to \mathbb{R}$ is continuous, locally non-satiated and quasi-concave and such that for all $p \in \mathcal{S}_\varepsilon$,

$$\textstyle\sum_i [x^i(p) - w^i] = Z(p), \tag{$*$}$$

where

$$x^i(p) = \operatorname{argmax}_x \left\{u^i(x) : p \cdot x \leqslant p \cdot w^i\right\}.$$

Answer: This follows immediately from the SMD theorem, given the conclusion of the previous two parts. □

(d) Conclude that every $p \in E$ is an equilibrium price vector for that economy.

Answer: By the definition of Z_1 in Eq. (1), $Z_1(p) = 0$ if, and only if, $p \in E$. It follows that $Z(p) = 0$ for all $p \in E$. $\square$

(e) Use the analysis above to state a theorem to formalize the claim that "that the set of competitive equilibrium prices of a competitive economy has essentially no structure other than closedness".

Answer: The previous arguments prove the following theorem: *For any closed set* $E \subseteq \mathcal{S}$, *there exists an exchange economy*

$$\{\mathcal{I}, (u^i, w^i)_{i \in \mathcal{I}}\}$$

where each $u^i : \mathbb{R}^2_+ \to \mathbb{R}$ *is continuous, locally non-satiated and quasi-concave and such that* E *is a subset of its set of competitive equilibrium prices.* $\square$

(f) Suppose that instead of Eq. (1), we let $Z_1(p) = 1$ for all $p \in \mathcal{S}$, with Z_2 still defined by Eq. (2). Argue that the same conclusion of part (c) still applies, but explain why the fact that there is no p for which $Z(p) = 0$ is *not* a counter-example to the Arrow-Debreu existence theorem studied in class.

Answer: The conclusion of part (c) still applies to this function, since it is continuous and satisfies Walras's law so the SMD theorem can still be invoked. But this is not a counter-example to the existence theorem, since Eq. $(*)$ only considers prices in $\mathcal{S}_\varepsilon$. By boundary behavior, the economy that generates Z must have equilibrium prices in $\mathcal{S} \setminus \mathcal{S}_\varepsilon$.

Note, incidentally, that for the same reason we cannot imply in part (d) that *only* the prices in set E are competitive equilibrium prices of that economy. $\square$