

PRELIMINARY EXAMINATION FOR THE Ph.D. DEGREE

Directions: Answer all questions. Feel free to impose additional structure on the problems below, but please state your assumptions clearly. Point totals for each question are given in parentheses.

1. (10) Briefly discuss the following statements (keep your answers short and concise):
 - (a) Provide an intuitive - but concise - explanation for how the existence and uniqueness of the value function as defined by the Bellman equation associated with the standard growth model was established. In your answer, be sure to identify the metric space used in the analysis.
 - (b) Within the context of the representative agent consumption-based capital asset pricing model, discuss the factors that affect the equilibrium level of the yield on risk-free, one-period bonds.

ANSWER: For part (a), it is critical to identify the metric space: it is the space of continuous, bounded functions with the sup norm used as the metric. This is important because this space is a complete metric space which implies that a Cauchy convergent sequence will converge to a point (in this case, a function - the value function) in the space. Then the operator defined by the Bellman equation can be shown to satisfy Blackwell's sufficiency conditions for a contraction mapping. This implies that iterating on an initial value function through the use of the operator will produce a Cauchy convergent sequence which, because the metric space is complete, will converge to a unique fixed point. For part (b) the starting point for a good answer would be the Euler equation for real bonds evaluated at the market clearing levels of consumption:

$$U'(c_t) = \beta E_t [U'(c_{t+1})] (1 + r_t)$$

This implies that the real interest rate reflects the scarcity of consumption today vis-a-vis consumption tomorrow. If consumption today is relatively scarce, so that the marginal utility of consumption is higher than average, then the real interest rate will be high as agents attempt to borrow to increase their consumption. Given the representative agent assumption, the equilibrium interest rate must increase so that equilibrium bond sales are zero. A good answer would also point out that, since $U'(c_{t+1})$ is a non-linear function, higher moments in the distribution for consumption can also affect the equilibrium real interest rate. For instance, if the growth rate of consumption is log linearly distributed with mean of g and *i.i.d.* then it was shown the real interest rate can be written as:

$$r = \rho + \gamma g - \gamma (1 + \gamma) \frac{\sigma_c^2}{2}$$

where $\beta = \frac{1}{1+\rho}$, γ denotes agent's relative risk aversion parameter (assuming CRRA preferences) and σ_c^2 is the variance of consumption. Hence, real interest rates will be lower in an economy, *ceteris paribus*, that has a higher variance of consumption growth. The high variance reduces the certainty equivalence of future consumption which results in a lower equilibrium interest rate.

2. (20) Consider a simple, representative agent RBC model in which output, y_t , is produced via a standard Cobb-Douglas production function:

$$y_t = z_t k_t^\alpha h_t^{1-\alpha}$$

where k_t denotes beginning-of-period capital, h_t is labor, and z_t is an *i.i.d.* technology shock. The depreciation rate of capital is 100%. In each period, agents make consumption and labor decisions in order to maximize lifetime expected utility:

$$E_0 \left[\sum_{t=0}^{\infty} \beta^t U(c_t, h_t) \right]$$

Within this environment, consider two variations defined by the functional form for $U(\cdot)$.

(a) In Economy A, agents have preferences given by:

$$U(c_t, h_t) = \ln c_t - \frac{1}{2} h_t^2$$

In this economy, do the following

- i. Express the maximization problem as social planner problem and write down the associated Bellman equation.
- ii. Solve for the equilibrium policy functions describing consumption, investment and labor.

(b) In Economy B, agents have preferences given by:

$$U(c_t, h_t) = \ln \left(c_t - \frac{1}{2} h_t^2 \right)$$

- i. Express the maximization problem as social planner problem and write down the associated Bellman equation.
- ii. Solve for the equilibrium policy functions describing consumption, investment and labor.

(c) Compare the equilibrium behavior in both economies and provide an explanation for the differences.

ANSWER: For both economies, the Bellman equation associated with the social planner problem can be written as:

$$V(z_t, k_t) = \max_{(c_t, h_t, k_{t+1})} \left\{ \begin{array}{l} U(c_t, h_t) + \beta E[V(z_{t+1}, k_{t+1})] \\ + \lambda_t (z_t k_t^\alpha h_t^{1-\alpha} - c_t - k_{t+1}) \end{array} \right\}$$

where the state variables are (z_t, k_t) and λ_t denotes a Lagrange multiplier on the resource constraint. The associated necessary conditions, after applying the envelope theorem are:

$$U_{c,t} = \alpha \beta E \left[U_{c,t} \frac{y_{t+1}}{k_{t+1}} \right]$$

$$U_{h,t} + U_{c,t} (1 - \alpha) \frac{y_t}{h_t} = 0$$

where $U_{c,t}$ denotes the marginal utility of consumption and $U_{h,t}$ denotes the marginal disutility of labor supply. For Economy A, these conditions become

$$\frac{1}{c_t} = \alpha \beta E \left[\frac{1}{c_{t+1}} \frac{y_{t+1}}{k_{t+1}} \right]$$

$$h_t^2 = (1 - \alpha) \frac{y_t}{c_t}$$

The first condition is similar to what we saw in class and, as a consequence, it is reasonable to conjecture that the optimal policy function for consumption will be linear in output. That is, conjecture that $c_t = \theta y_t$ with θ an unknown constant (of course this implies $k_{t+1} = (1 - \theta) y_t$). Using this in the Euler equation for capital yields: $\theta = (1 - \alpha \beta)$ which verifies the conjecture. The immediate implication, from the labor-leisure marginal condition is that labor is constant in Economy A. That is, we have

$$h_{A,t} = \frac{(1 - \alpha)}{(1 - \alpha \beta)}$$

Then the policy function for consumption is:

$$c_{A,t} = (1 - \alpha \beta) \left[\frac{1 - \alpha}{1 - \alpha \beta} \right]^{\frac{1-\alpha}{2}} z_t k_t^\alpha$$

The critical thing to note is that, with these preferences, the income and substitution effects of a technology shock cancel with respect to labor decisions; this is the reason for the constant labor supply. Hence, consumption is affected directly by the current productivity shock but there is no endogenous effect via a labor response.

For Economy B, this is not the case. These preferences imply that there is no income effect on labor supply. To see this, first express the necessary conditions using the functional form for preferences which, upon simplification, becomes:

$$\frac{1}{c_t - \frac{1}{2}h_t^2} = \alpha\beta E \left[\left(\frac{1}{c_{t+1} - \frac{1}{2}h_{t+1}^2} \right) \frac{y_{t+1}}{k_{t+1}} \right]$$

$$h_t^2 = (1 - \alpha) y_t$$

The labor-leisure condition can be used in the Euler equation for capital to eliminate labor in the marginal utility of consumption. In addition, the form of the Euler equation suggests that the same conjecture for the consumption policy function can be used: $c_t = \eta y_t$ where, again, η is an unknown constant. Using this conjecture in the Euler equation yields $\eta = (1 - \alpha\beta)$ - the same constant as in Economy B. But note that labor is not constant in this economy because there is no income effect. Using the production function in the labor-leisure condition and simplifying yields the optimal policy function for labor in Economy B

$$h_{B,t} = [(1 - \alpha) z_t k_t^\alpha]^{\frac{1}{1+\alpha}}$$

Note that the elasticity of labor supply response with respect to a technology shock is decreasing in α - why is this? Using this in the policy function for consumption and simplifying yields:

$$c_{B,t} = (1 - \alpha\beta) (1 - \alpha)^{\frac{1-\alpha}{1+\alpha}} z_t^{\frac{2}{1+\alpha}} k_t^{\frac{2\alpha}{1+\alpha}}$$

Since $\alpha < 1$, the exponent on the technology shock is greater than unity which shows that the consumption response with respect to a technology shock in Economy B is greater than that in Economy A. This is due to the endogenous labor response which affects output.

3. (20) Consider a discrete time variation of the Sidrauski monetary model with a constant population. Specifically, assume that the representative agent's lifetime utility is given by:

$$\sum_{t=0}^{\infty} \beta^t \left[U(c_t) + V\left(\frac{M_t}{P_t}\right) \right]$$

where $U(\cdot)$ and $V(\cdot)$ are concave, twice-differentiable functions, c_t denotes consumption and M_t is money chosen in period t . Each period, agents use beginning of period nominal balances, the revenue from sales of output and a lump-sum monetary transfer to purchase consumption, investment and new money. In contrast to the Sidrauski model, both capital and money are used as inputs into the production process. Letting y_t denote output, the production function is given by:

$$y_t = \left(1 - z \left(\frac{M_t}{P_t} \right) \right) f(k_t)$$

where $z'(\cdot) < 0$, $z''(\cdot) > 0$, $z(0) = 1$, $\lim_{M_t/P_t \rightarrow \infty} z(M_t/P_t) = 0$. The function $f(k_t)$ has standard properties. The money supply in this economy is growing at the constant rate $\mu > 0$ and capital depreciates at the constant rate of $\delta < 1$.

- (a) Derive and interpret the necessary conditions associated with the agent's maximization problem.
- (b) Define a steady-state equilibrium in this economy. Use the equations defining the steady-state equilibrium to establish the relationship between steady-state consumption and real balances (*i.e.* $d\bar{c}/d\bar{m}$) and steady-state capital and real balances (*i.e.* $d\bar{k}/d\bar{m}$) where $(\bar{c}, \bar{m}, \bar{k})$ denote steady-state values.

- (c) Now consider the relationship between \bar{m} and the growth rate of money (μ). Does this economy exhibit superneutrality? Provide an intuitive explanation for your answer (Do NOT try to establish an analytic result as this gets messy and will depend on functional forms.)

Answer:

The maximization problem can be written as the following dynamic programming problem:

$$W(k_t, M_{t-1}) = \max \left\{ \begin{array}{l} U(c_t) + V\left(\frac{M_t}{P_t}\right) + \beta W(k_{t+1}, M_t) + \\ \lambda_t \left[f(k_t) \left(1 - z\left(\frac{M_t}{P_t}\right)\right) + \frac{M_{t-1}}{P_t} + \frac{T_t}{P_t} + k_t(1 - \delta) - c_t - \frac{M_t}{P_t} - k_{t+1} \right] \end{array} \right\}$$

where the vector of states is (k_t, M_{t-1}) and T_t represents the lump-sum monetary transfer. After applying the envelope theorem, the necessary conditions are:

$$U'(c_t) = \beta \left\{ U'(c_{t+1}) \left[f'(k_{t+1}) \left(1 - z\left(\frac{M_{t+1}}{P_{t+1}}\right)\right) + 1 - \delta \right] \right\} \quad (1)$$

$$U'(c_t) \left[f(k_t) z' \left(\frac{M_t}{P_t}\right) + 1 \right] = V' \left(\frac{M_t}{P_t}\right) + \beta U'(c_{t+1}) \frac{P_t}{P_{t+1}} \quad (2)$$

Note that the marginal product of capital is affected by the quantity of real balances (eq.(1)), also, the term $f(k_t) z' \left(\frac{M_t}{P_t}\right)$ in the second expression reflects the marginal effect that money has on the output (capturing transactions costs). Recall that $z'(\cdot) < 0$.

A steady-state is defined by a constant level of consumption, capital and real balances that satisfy eqs. (1) and (2) and the resource constraint. In addition, a monetary steady state implies that inflation will be equal to the monetary growth rate. Define $\theta = 1/(1 + \mu)$, these steady-state values are the solution to:

$$1 = \beta \left[f'(\bar{k}) (1 - z(\bar{m})) + 1 - \delta \right] \quad (3)$$

$$\frac{V'(\bar{m})}{U'(\bar{c})} = f(\bar{k}) z'(\bar{m}) + 1 - \beta\theta \quad (4)$$

$$f(\bar{k}) (1 - z(\bar{m})) = \bar{c} + \delta\bar{k} \quad (5)$$

From this it is apparent that this economy will not exhibit superneutrality: changes in θ (i.e. inversely related to the monetary growth rate) will affect the marginal product of capital and therefore the steady-state level of the capital stock. And this will influence the steady-state level of consumption. Taking the total differential of eq.(3) and rearranging yields (dropping the arguments of the functions for expositional purposes):

$$\frac{dk}{dm} = \frac{f'z'}{f''(1-z)} > 0$$

Hence, real balances and capital will be positively related. This makes sense: if real balances fall, then the net MPK of capital falls. Consequently, the capital stock must fall to maintain equality of the net MPK and agents' discount factor and depreciation. We can then use this in the resource constraint, eq. (5). Taking the total differential of this expression we have:

$$\frac{dc}{dm} = [f'(1-z) - \delta] \frac{dk}{dm} - f'z' > 0$$

The term in brackets is positive as can be seen from eq. (3). The assumption that $z' < 0$ and the previous result establishes that consumption real balances will be positively related. The final step is to solve for $\frac{dm}{d\theta}$. This can be done by taking the total differential of eq. (4) and then using the previous results. But things get messy algebraically and there is no general result. Typically, an increase in the monetary growth rate will cause a fall in steady-state real balances since the implied inflation tax increases. This story will hold in this economy but, as already demonstrated, capital and consumption will also fall. But it is also possible for real balances to increase if the marginal effect on production outweighs the implied inflation tax. In either situation, however, superneutrality (a feature in the basic Sidrauski setup) does not hold.

4. (20) Thanasis Q1

5. (20) Thanasis Q2

(a)

6. (10) Thanasis Q3

**MACROECONOMICS PRELIM, JUNE 2016
ANSWER KEY FOR QUESTIONS 4-6**

Question 4

a) The flows in and out of unemployment must satisfy:

$$\lambda(1 - u) = \theta q(\theta)[1 - F(R)]u.$$

Hence, steady state unemployment (or the Beveridge Curve) is given by

$$u = \frac{\lambda}{\lambda + \theta q(\theta)[1 - F(R)]}.$$

Clearly, equilibrium unemployment depends, not only on the market tightness, but also on the reservation value R : If R is high firms and workers are very picky, and very few matches result into jobs, hence, unemployment will be high (for any given θ).

b) The value functions for the worker are as follows:

$$rU = z + \theta q(\theta) \left[F(R)U + \int_R^1 W(x)dF(x) - U \right], \quad (1)$$

$$rW(x) = w(x) + \lambda[U - W(x)], \quad (2)$$

where it is understood that (2) holds for all $x \geq R$.

c) For the firm, we have:

$$rV = -c + q(\theta) \left[F(R)V + \int_R^1 J(x)dF(x) - V \right], \quad (3)$$

$$rJ(x) = x - w(x) - \lambda J(x), \quad (4)$$

where, again, it is understood that (4) holds for all $x \geq R$. Also, in equilibrium it will be true that $V = 0$, due to free entry.

d) Simply set $V = 0$ in (3). The condition is

$$\int_R^1 J(x)dF(x) = \frac{c}{q(\theta)}, \quad (5)$$

and it states that the expected value of a job will be equal to the expected cost of filling the vacancy. The only thing that differs from the baseline model is that here the productivity of the job is stochastic.

e) As in most bargaining problems we have solved, for any $x \geq R$, the solution must satisfy

$$\beta J(x) = (1 - \beta)[W(x) - U].$$

We can solve for J and W from parts (b), (c) and substitute the results in the expression above. This will give us

$$w(x) = \beta x + (1 - \beta)rU, \quad (6)$$

which is similar to the expression we found in the baseline model. The last step is to get rid of U . Notice that, after some standard manipulations, we can re-write (1) as

$$rU = z + \theta q(\theta) \int_R^1 [W(x) - U] dF(x). \quad (7)$$

Plug this expression into (6) to get

$$\begin{aligned} w(x) &= \beta x + (1 - \beta)z + \theta q(\theta) \int_R^1 (1 - \beta)[W(x) - U] dF(x) \\ &= \beta x + (1 - \beta)z + \theta q(\theta) \int_R^1 \beta J(x) dF(x). \end{aligned}$$

Exploiting condition (5), we obtain the following Wage Curve:

$$w(x) = \beta x + (1 - \beta)z + \beta c\theta,$$

which looks very similar to the WC of the baseline model (except from the idiosyncratic productivity term).

f) Simply substitute the wage from the WC above into (4) and solve with respect to J . It can be easily shown that

$$J(x) = \frac{1}{r + \lambda} [(1 - \beta)(x - z) - \beta c\theta]. \quad (8)$$

Now we have a closed-form solution for the value function J .

g) For job creation, simply combine parts (d) and (f) as suggested in the question. In other words, substitute J from (8) into (5). We find

$$\int_R^1 \frac{(1 - \beta)(x - z) - \beta c\theta}{r + \lambda} dx = \frac{c}{q(\theta)}.$$

h) For job destruction (as hinted in the question) it makes sense to look at the point where jobs are “destroyed” or not worth keeping. That point, of course, is $x = R$,

and, by construction, it must be true that $J(R) = 0$. Evaluating (8) at $x = R$, and setting it equal to zero, yields the JD condition:

$$R = z + \frac{\beta c \theta}{1 - \beta}.$$

i) We have 4 (steady state) equilibrium objects, $u, \theta, R, w(x)$, and 4 equilibrium conditions, the BC, the JC curve, the JD curve and the WC. Notice that the JD and JC curves are a block that contains only θ and R . If these two variables were determined, then automatically u is determined through the BC, and the wage is determined through the WC.

We now study the existence and uniqueness of θ, R . The JR curve is very easy. It has positive slope in the (θ, R) space, and when $\theta = 0$, it gives $R = z$.

The JC curve is slightly more involved, but there is a clever observation that can make this proof much easier (if one does not make this observation, he/she could still derive all the results, but it would take a little more time). Since in equilibrium we have $\beta c \theta = (1 - \beta)(R - z)$, we can impose this condition into the JC and significantly simplify it. More precisely, the JC can be re-written as

$$\frac{1 - \beta}{r + \lambda} \int_R^1 (x - R) dx = \frac{c}{q(\theta)}.$$

This expression is more useful, because here the term θ shows up only on the RHS and the term R shows up only on the LHS, and so using the Implicit Function Theorem will be much easier. In fact, we can make things even easier by calculating this integral (we have assumed that F is uniform in $[0, 1]$; that's as easy as it gets). Hence, the JC simplifies to:

$$\frac{1 - \beta}{r + \lambda} \left(\frac{1}{2} + \frac{R^2}{2} - R \right) = \frac{c}{q(\theta)}.$$

First, notice that when $\theta = 0$, we have $R = 1 > z$. This means that the JC curve “cuts” the R -axis at a point which is higher than the one where the JD curve cuts it. Moreover, it is not hard to show that the JC curve cuts the θ -axis at some finite and positive θ . Given these facts, and the continuity of these functions, existence of equilibrium is established.

Can we also establish uniqueness? We have already seen that JD is strictly increasing. To see the sign of JC apply total differentiation on the last JC condition (the simplest one). We can see that

$$\frac{1 - \beta}{r + \lambda} (R - 1) R'(\theta) = \frac{-c q'(\theta)}{q^2(\theta)}.$$

The multiplier of the term $R'(\theta)$ on the LHS is negative (because R must be smaller than 1), and the RHS is positive, because $q'(\theta) < 0$ and the whole term is multiplied by negative 1. Hence, $R'(\theta) < 0$, which concludes the proof of uniqueness.

Question 5

a) Write down the Social Planner's problem, letting $a \in (0, 1)$ denote the Pareto weight that the planner assigns on the typical agent of country 1. The planner wishes to maximize the expression

$$\sum_{t=0}^{\infty} [a\beta_1^t \ln(c_t^1) + (1-a)\beta_2^t \ln(c_t^2)],$$

subject to $c_t^1 + c_t^2 = e$ for all t , by choosing sequences of consumptions for the typical agent of each country. The Lagrangian function for this problem is

$$L = \sum_{t=0}^{\infty} [a\beta_1^t \ln(c_t^1) + (1-a)\beta_2^t \ln(c_t^2)] + \sum_{t=0}^{\infty} \mu_t (e - c_t^1 - c_t^2),$$

where μ_t is the Lagrangian multiplier for period t (there will be infinitely many such multipliers). The first order conditions with respect to c_t^1 and c_t^2 , respectively, are given by:

$$\begin{aligned} a\beta_1^t &= \mu_t c_t^1, \\ (1-a)\beta_2^t &= \mu_t c_t^2, \end{aligned}$$

and combining these we obtain the following relationship between consumption in the two countries:

$$c_t^1 = \frac{a}{1-a} \left(\frac{\beta_1}{\beta_2} \right)^t c_t^2. \quad (9)$$

But we also know that the consumptions must obey the feasibility constraint: $c_t^1 + c_t^2 = e$, for all t . Combining these facts, we can obtain the following result: For all t , the consumption of the typical agent in the two countries is given by:

$$\begin{aligned} c_t^1(a) &= (1 - \gamma_t) e, \\ c_t^2(a) &= \gamma_t e, \end{aligned}$$

where

$$\gamma_t \equiv \frac{1}{1 + \frac{a}{1-a} \left(\frac{\beta_1}{\beta_2} \right)^t}.$$

It is easy to see that $\gamma_t \in (0, 1)$, for all t . But it is also easy to check that γ_t is increasing in t and that, as t grows infinitely large, $\gamma_t \rightarrow 1$. That is, the agents who are more patient will eventually become so wealthy that they will consume the whole endowment available in this economy. Notice that we can make all these statements without even knowing the equilibrium yet (we have not solved for the "correct" a).

To fully characterize equilibrium, we need to find which value of a each type of

agent can “afford”. To that end, define the usual transfer functions for agent i , using as prices the Lagrangian multipliers μ_t (this is the so-called Negishi method). We have

$$t^i(a) = \sum_{t=0}^{\infty} \mu_t [c_t^i(a) - e_i],$$

where

$$\mu_t = \frac{(1-a)\beta_2^t + a\beta_1^t}{e}.$$

As is usual in these cases, we have an extra degree of freedom, which means we can work with either type of agent. Hence, let’s focus on agents in country 2. After combining the last two equations and a little bit of algebra, we can show that

$$t^2(a) = \frac{1-a}{1-\beta_2} - \frac{a\beta_1}{1-\beta_1^2} - \frac{(1-a)\beta_2}{1-\beta_2^2}.$$

Of course, the value of a associated with equilibrium is the one for which $t^2(a) = 0$, so that the type 2 agents do not need any transfer in order to afford that specific allocation. Solving $t^2 = 0$ with respect to a yields the unique solution:¹

$$\hat{a} = \frac{1-\beta_1^2}{1-\beta_1^2 + \beta_1(1-\beta_2^2)}.$$

After a little more algebra, we find that

$$\hat{c}_t^2 = \left[1 + \frac{1-\beta_1^2}{\beta_1(1-\beta_2^2)} \left(\frac{\beta_1}{\beta_2} \right)^t \right]^{-1} e.$$

Of course, $\hat{c}_t^1 = e_t - \hat{c}_t^2$.

b) Based on Negishi’s Theorem, we know that the equilibrium prices will be given by the Lagrangian multiplier, again, evaluated at the “correct” $a = \hat{a}$. It is, therefore, easy to see that

$$\hat{p}_t = \mu_t(\hat{a}) = \frac{(1-\hat{a})\beta_2^t + \hat{a}\beta_1^t}{e}.$$

c) It is very easy to see that in early periods the typical agent 1 consumes more than the typical agent 2. For instance, set $t = 0$. In that period, the consumption for the typical agent in country 2 is

$$\hat{c}_0^2 = \frac{1}{1 + \frac{1-\beta_1^2}{\beta_1(1-\beta_2^2)}} e.$$

¹ If you have reached this point and you are wondering whether this result is correct, there is a nice test that can give you a strong indication of the fact that the result is correct. Setting $\beta_1 = \beta_2 = \beta$ in this expression, will give you $\hat{a} = 1/(1 + \beta)$, which is exactly what we saw in class for the symmetric case.

and this is lower than what agents in country 1 consume, because the multiplier of e in the last expression is smaller than $1/2$. Hence, for early periods, $\hat{c}_t^1 > \hat{c}_t^2$. But we have also seen that \hat{c}_t^2 is increasing in t , and, actually, as $t \rightarrow \infty$, $\hat{c}_t^2 \rightarrow e$. Hence, there must be some t in between, where $\hat{c}_t^1 = \hat{c}_t^2$. This t solves

$$\frac{1 - \beta_1^2}{\beta_1(1 - \beta_2^2)} \left(\frac{\beta_1}{\beta_2} \right)^t = 1,$$

which implies that the t we are looking for is

$$t^* = \frac{\ln \left(\frac{\beta_1(1 - \beta_2^2)}{1 - \beta_1^2} \right)}{\ln \beta_1 - \ln \beta_2}.$$

In the exam some students took the idea that t needs to be an integer very seriously. That's fine and they got all the credit, but what was more important here is to realize that there exists a point in time, t^* , such that country 1's consumption is higher than country 2's consumption for $t < t^*$.

Question 6

a) The Planner's problem is indeed much easier, because one only needs to describe the allocations, and not the prices of all commodities (which are infinite sequences). What allows us to use this technique here, is the fact that in this environment both Welfare Theorems hold. So we know that the competitive allocation and the Planners allocation will coincide. After characterizing the Planners allocation, we can construct the whole competitive equilibrium, like we did in class.

b) Using any technique you like, you can arrive at the following Euler condition (which is necessary for the dynamic maximization):

$$\frac{1}{k_t^a - k_{t+1}} = \frac{a\beta k_{t+1}^{a-1}}{k_{t+1}^a - k_{t+2}^a}.$$

If you impose the guess I gave you as a hint, and after a little bit of algebra, you will find that

$$k_{t+1} = gk_t^a = a\beta k_t^a.$$

Notice that this term is in $(0, 1)$. This result simply means that each period agents should invest a part equal to g of the output and eat the remaining $1 - g$.

Also, since we know that in equilibrium $k_t = x_t$, and x_0 is given, we can fully characterize the whole capital stock allocation. In particular, for all $T > 0$, we have

$$k_T = (a\beta)^{1+a+\dots+a^T-1} (x_0^a)^T.$$

c) Regardless of the initial condition, this economy will always converge to a steady state. To find it, take the limit as $T \rightarrow \infty$ in the equation above. We get

$$k^* = \lim_{T \rightarrow \infty} k_T = (a\beta)^{\frac{1}{1-a}}.$$

Regarding the rental rate of capital in the long run, we know that $r_t = F_K(k_t, 1)$. Therefore, as $t \rightarrow \infty$, we have $r^* = 1/\beta$. Similarly, $w_t = F_N(k_t, 1)$. Therefore, as $t \rightarrow \infty$, we have $w^* = (1-a)(a\beta)^{a/(1-a)}$.

d) If you write down the AD problem of the household and take the first order conditions, you will find an expression very similar to the one obtained in class:

$$\frac{p_{t+1}}{p_t} = \frac{1}{r_{t+1}}.$$

As we usually do, let us treat consumption at $t = 0$ as the numeraire, i.e. set $p_0 = 1$. Then, $p_1 = r_1^{-1}$, $p_2 = r_2^{-1}p_1 = (r_1 r_2)^{-1}$, and so on. In general, we have

$$p_T = \prod_{t=1}^T \frac{1}{r_t},$$

and since $r_t = F_K(k_t, 1)$, we can write

$$p_T = \prod_{t=1}^T \frac{1}{ak_t^{a-1}},$$

where the sequence $\{k_t\}_{t=1}^T$ has been described in detail in part (b).