Question 3: The Core of an Economy

(a) Define the *weak core* of exchange economy $\{I, \mathbf{u}, \mathbf{w}\} = \{I, (u^i, w^i)_{i\in I}\}$ as the set of its allocations $\mathbf{x}$ such that there do not exist $\mathcal{H} \subseteq I$ and $(\hat{x}^i)_{i\in\mathcal{H}}$ for which $\sum_{i\in\mathcal{H}} \hat{x}^i = \sum_{i\in\mathcal{H}} w^i$ and $u^i(\hat{x}^i) > u^i(x^i)$ for all $i \in \mathcal{H}$. Argue that:

   i. the core is a subset of the weak core; and

   ii. if all preferences are continuous and strictly monotone, the core and the weak core are the same set.

(b) Given an exchange economy $\{I, \mathbf{u}, \mathbf{w}\}$, prove the following:

   i. If $\mathbf{w}$ is efficient, then it is a core allocation.

   ii. If each u^i is strongly quasiconcave and $\mathbf{w}$ is efficient, then $\mathbf{w}$ is the only core allocation.

(c) Consider a two-person exchange economy

$$\{I = \{1, 2\}, \mathbf{u} = (u^1, u^2), \mathbf{w} = (w^1, w^2)\},$$

and suppose that (p, x^1, x^2) is a competitive equilibrium. Argue that if (x^1, x^2) is *not* in the core of the economy, then it must be Pareto inefficient.

Answer:

1. (a) It suffices to show that the complement of the weak core is a subset of the complement of the core. Let allocation $\mathbf{x}$ *not be* in the weak core of the economy. By definition, there exist $\mathcal{H} \subseteq I$ and $(\hat{x}^i)_{i\in\mathcal{H}}$ for which $\sum_{i\in\mathcal{H}} \hat{x}^i = \sum_{i\in\mathcal{H}} w^i$ and $U^i(\hat{x}^i) > U^i(x^i)$ for all $i \in \mathcal{H}$. The latter implies, obviously, $U^i(\hat{x}^i) \geq U^i(x^i)$ for all $i \in \mathcal{H}$, with strict inequality for some. But this implies that the allocation is not in the core of the economy, as needed.

   (b) Again, it's easier to show that the complement of the core is a subset of the complement of the weak core. If $\mathbf{x}$ isn't in the core, there exist $\mathcal{H} \subseteq I$ and $(\hat{x}^i)_{i\in\mathcal{H}}$ for which $\sum_{i\in\mathcal{H}} \hat{x}^i = \sum_{i\in\mathcal{H}} w^i$ and $U^i(\hat{x}^i) \geq U^i(x^i)$ for all $i \in \mathcal{H}$, with strict inequality for some $i' \in \mathcal{H}$. By monotonicity and continuity of $u^{i'}$, we can find $\bar{x}^{i'} < \hat{x}^{i'}$ such that $u^{i'}(\bar{x}^{i'}) > u^{i'}(x^{i'})$. Defining, for every $i \in \mathcal{H} \setminus \{i'\}$,

$$\bar{x}^i = \hat{x}^i + \frac{1}{I-1}(\hat{x}^{i'} - \bar{x}^{i'}) > \hat{x}^i,$$

   we get, by strict monotonicity, that $u^i(\bar{x}^i) > u^i(\hat{x}^i) \geq u^i(x^i)$. By construction,

$$\sum_{i\in\mathcal{H}} \bar{x}^i = \bar{x}^{i'} + \sum_{i\in\mathcal{H}\setminus\{i'\}} \left[\hat{x}^i + \frac{1}{I-1}(\hat{x}^{i'} - \bar{x}^{i'})\right] = \sum_{i\in\mathcal{H}} \hat{x}^i = \sum_{i\in\mathcal{H}} w^i,$$

   so it follows that $\mathbf{x}$ isn't in the weak core either.

2. (a) If coalition $\mathcal{H}$ had an objection $(x^i)_{i\in\mathcal{H}}$, we could construct an objection for the grand coalition, I, by simply completing the allocation with $x^i = w^i$ for all $i \notin \mathcal{H}$.

   (b) Suppose that $\mathbf{x}$ is another allocation in the core. By construction, the allocation constructed by letting $\hat{x}^i = 1/2(w^i + x^i)$ for each i is feasible too. Since x is in the core, $u^i(x^i) \geq u^i(w^i)$, which implies that $u^i(\hat{x}^i) \geq u^i(w^i)$, by quasiconcavity. Since $x \neq (w^i)_{i\in I}$, there exists some

i for whom $x^i \neq w^i$. For such i, by strict quasiconcavity, the previous inequality is strict: $u^i(\hat{x}^i) > u^i(w^i)$.

Existence of $\hat{\mathbf{x}}$ contradicts the fact that $\mathbf{w}$ is Pareto efficient, though.

3. Since

$$x^i \in \arg\max_x \{u^i(x) : p \cdot x \leq p \cdot w^i\}$$

for both i, it must be true that $u^i(x^i) \geq u^i(w^i)$. Then, since there are only two people in the economy, for (x^1, x^2) to not be in the core, it must be blocked by the grand coalition. □