

**Ph. D. Preliminary examination in Industrial Organization, September 2007**

**Answers to questions 1 and 2**

**1.**

(a) When  $P = 1 - Q$ ,  $C_1 = c_1 q_1$  and  $C_2 = c_2 q_2 - F$ , the Cournot equilibrium is  $q_1 = \frac{1 - 2c_1 + c_2}{3}$ ,

$$q_2 = \frac{1 - 2c_2 + c_1}{3}, \quad P = \frac{1 + c_1 + c_2}{3}, \quad \pi_2 = \frac{(1 - 2c_2 + c_1)^2}{9} - F. \quad \text{When } c_2 = \frac{1}{3} \text{ and } F = \frac{4}{81},$$

$$\pi_2 = \frac{\left(\frac{1}{3} + c_1\right)^2}{9} - \frac{4}{81}. \quad \text{This is an increasing function of } c_1. \quad \text{It is equal to } -\frac{3}{81} \text{ when } c_1 = 0 \text{ and}$$

$\frac{5}{81}$  when  $c_1 = \frac{2}{3}$ .  $\pi_2 = 0$  when  $c_1 = \frac{1}{3}$ . Thus in order to keep the entrant out, firm 1 must lower

its marginal cost in period 2 to  $\frac{1}{3}$  by spending  $\frac{\alpha}{3}$  dollars in period 1.

(b) If firm 1 spends  $\$x \geq \frac{\alpha}{3}$  in period 1, entry will be deterred and firm 1's profits over the two periods will be:

$$F(x) = \underbrace{\left(\frac{1 - \frac{2}{3}}{2}\right)^2}_{\text{period 1}} + \underbrace{\left[\frac{1 - \left(\frac{2}{3} - \frac{x}{\alpha}\right)}{2}\right]^2}_{\text{period 2}} - x = \frac{1}{36} + \frac{(\alpha + 3x)^2}{36\alpha^2} - x$$

The second-period cost is  $\left(\frac{2}{3} - \frac{x}{\alpha}\right)$  and this is non-negative if and only if  $x \leq \frac{2}{3}\alpha$ . Thus we are

considering  $F(x)$  only in the range  $\frac{\alpha}{3} \leq x \leq \frac{2}{3}\alpha$ . Now,

$F\left(\frac{\alpha}{3}\right) = \frac{5}{36} - \frac{1}{3}\alpha$  and  $F\left(2\frac{\alpha}{3}\right) = \frac{5}{18} - \frac{2}{3}\alpha = 2F\left(\frac{\alpha}{3}\right)$ . The function is strictly convex, hence the

maximum in the range  $\frac{\alpha}{3} \leq x \leq \frac{2}{3}\alpha$  is either  $x = \frac{\alpha}{3}$  or  $x = \frac{2}{3}\alpha$ . Clearly,  $2F\left(\frac{\alpha}{3}\right) > F\left(\frac{\alpha}{3}\right)$  if

and only if  $F\left(\frac{\alpha}{3}\right) > 0$  i.e. if and only if  $\frac{5}{36} - \frac{1}{3}\alpha > 0$ , if and only if  $\alpha < \frac{15}{36}$ . Thus if  $\alpha < \frac{15}{36}$

then the maximum in the range  $\frac{\alpha}{3} \leq x \leq \frac{2}{3}\alpha$  is achieved at  $x = \frac{2}{3}\alpha$  and  $F\left(2\frac{\alpha}{3}\right) = \frac{5}{18} - \frac{2}{3}\alpha > 0$ .

If  $\alpha > \frac{15}{36}$  then the maximum in the range  $\frac{\alpha}{3} \leq x \leq \frac{2}{3}\alpha$  is achieved at  $x = \frac{\alpha}{3}$  but  $F\left(\frac{\alpha}{3}\right) < 0$ , and

therefore the firm is better off allowing entry to take place.

If firm 1 spends \$  $x < \frac{\alpha}{3}$  in period 1, so that its period 2 marginal cost is  $c_1 = \left(\frac{2}{3} - \frac{x}{\alpha}\right) > \frac{1}{3}$ , then entry will take place in period 2 and firm 1's period 2 profits at the Cournot equilibrium will be

$$\pi_1 = \frac{(1 - 2c_1 + c_2)^2}{9} = \frac{\left[1 - 2\left(\frac{2}{3} - \frac{x}{\alpha}\right) + \frac{1}{3}\right]^2}{9} = \frac{4x^2}{9\alpha^2}. \text{ Thus its profits over the two periods will be}$$

$$\pi(x) = \underbrace{\left(\frac{1 - \frac{2}{3}}{2}\right)^2}_{\text{period 1}} + \underbrace{\frac{4x^2}{9\alpha^2}}_{\text{period 2}} - x = \frac{1}{36} + \frac{4x^2}{9\alpha^2} - x.$$

$\pi(x)$  is strictly convex,  $\pi(0) = \frac{1}{36}$ ,  $\frac{d\pi}{dx}(0) = -1 < 0$  and  $\pi\left(\frac{\alpha}{3}\right) = \frac{1}{36} + \frac{4}{81} - \frac{\alpha}{3} <$

$F\left(\frac{\alpha}{3}\right) = \frac{5}{36} - \frac{1}{3}\alpha$ . Thus the only value that we need to consider in the range  $0 \leq x < \frac{\alpha}{3}$  is  $x =$

0: if  $\pi\left(\frac{\alpha}{3}\right) > \pi(0)$  then the firm would be better off choosing an  $x$  in the range  $\frac{\alpha}{3} \leq x \leq \frac{2}{3}\alpha$ .

Now,  $\pi(0) = \frac{1}{36}$ . Thus we have the following cases:

(1)  $\alpha \geq \frac{15}{36}$  yielding  $F\left(2\frac{\alpha}{3}\right) \leq F\left(\frac{\alpha}{3}\right) \leq 0$ . In this case  $x = 0$  is the optimal choice, yielding  $\pi(0) = \frac{1}{36}$ .

(2)  $\alpha < \frac{15}{36}$ . Then we need to compare  $\pi(0) = \frac{1}{36}$  with  $F\left(2\frac{\alpha}{3}\right) = \frac{5}{18} - \frac{2}{3}\alpha$ . The former is greater than the latter if and only if  $\alpha > \frac{3}{8}$ . Thus:

if  $\alpha > \frac{3}{8}$  the optimal choice is  $x = 0$  yielding  $\pi(0) = \frac{1}{36}$ ,

if  $\alpha \leq \frac{3}{8}$  the optimal choice is  $x = \frac{2}{3}\alpha$  yielding  $F\left(2\frac{\alpha}{3}\right) = \frac{5}{18} - \frac{2}{3}\alpha$ .

Thus  $\alpha^* = \frac{3}{8}$ .

The two-period profit function of the firm is:

$$\Pi(x, \alpha) := \begin{cases} \left( \frac{1}{36} - x + \frac{4 \cdot x^2}{9 \cdot \alpha^2} \right) & \text{if } 0 \leq x < \frac{\alpha}{3} \\ \left[ \frac{1}{36} - x + \frac{1}{36} \cdot \frac{(\alpha + 3 \cdot x)^2}{\alpha^2} \right] & \text{if } \frac{\alpha}{3} \leq x \leq \frac{2 \cdot \alpha}{3} \end{cases}$$

The above analysis shows that the maximum of this function is  $x = 0$  if  $\alpha > \frac{3}{8}$  and  $x = \frac{2}{3}\alpha$  if  $\alpha < \frac{3}{8}$ . If  $\alpha = \frac{3}{8}$  then there are two maxima: one at  $x = 0$  and the other at  $x = \frac{2}{3}\alpha$ .

- (c) If firm 1 decides **not** to deter entry, it is because  $\alpha > \frac{3}{8}$ , in which case it was shown above that the optimal choice is  $x = 0$ , i.e. the firm will not reduce its marginal cost at all.
- (d) If firm 1 decides to deter entry, it is because  $\alpha \leq \frac{3}{8}$ , in which case it was shown above that the optimal choice is  $x = \frac{2}{3}\alpha$  and thus the firm will reduce its period-2 marginal cost to 0.

**2. (a)** First find the consumer  $t^*$  who is indifferent between buying and not buying:  $t^* = \left( \frac{p}{x} \right)$ .

Thus demand is  $D(p, x) = \left( 1 - \frac{p}{x} \right) N$ . The firm's profits are  $\pi(p, x) = p \left( 1 - \frac{p}{x} \right) N$ . Solving  $\frac{\partial \pi}{\partial p} = 0$

yields  $p^* = \frac{x}{2}$  and  $\pi^* = \pi(p^*, x) = \frac{x}{4} N$ . This is an increasing function of  $x$ . Thus the firm would choose the maximum value of  $x$ , namely  $x = 10$ , with a corresponding profit of  $2.5N$ .

(b) In general firms can gain by offering several products because it allows them to practice price discrimination through product differentiation. So the expectation is that the firm would prefer to offer two products rather than only one.

(c) As in part (a), demand is  $D(p, x) = \left( 1 - \frac{p}{x} \right) N$ . The firm's profits now are

$$\pi(p, x) = (p - c) \left( 1 - \frac{p}{x} \right) N. \text{ Solving } \frac{\partial \pi}{\partial p} = 0 \text{ yields } P^* = \frac{x + c}{2}, \quad Q^* = \frac{x - c}{2x} N,$$

$$\pi^* = \frac{(x - c)^2}{4x} N.$$

(d) When  $x = 4$ ,  $c = 2$ ,  $P^* = 3$ ,  $Q^* = 0.25N$ ,  $\pi^* = 0.25N$ .

(e) If cost reduction then will save 0.7 on  $0.25N$  units. Thus profits will increase by  $(0.7)(0.25N) = 0.175N$ . If quality improvement to  $x = 5$  then can sell  $0.25N$  units for a price of 3.75 (from inverse demand function with  $x = 5$ ,  $q = 0.25N$ ). Thus will get an extra 0.75 on  $0.25N$  units, for a total extra profit of  $(0.75)(0.25N) = 0.1875N$ . Thus **quality improvement is better**.

(f) If  $x = 5$  and  $c = 2$  then  $\pi^* = \frac{9}{20}N = 0.45N$ . If  $x = 4$  and  $c = 1.3$  then  $\pi^* = \frac{7.29}{16}N = 0.455625N$ . Thus **cost reduction is better**.

(g) Suppose that the firm offers two products, one of quality  $x$  at price  $p_x$  and another of quality  $y < x$  at price  $p_y < p_x$ . Then the consumer indifferent between the two products is given by the solution to

$$E - p_x + xt = E - p_y + yt. \text{ The solution is } t_{xy} = \frac{p_x - p_y}{x - y}. \text{ The consumer who is indifferent between}$$

buying product  $y$  and buying nothing is given by  $t_{y0} = \frac{p_y}{y}$ . Thus demand for the two products is

$$D_x = \left(1 - \frac{p_x - p_y}{x - y}\right)N \text{ and } D_y = \left(\frac{p_x - p_y}{x - y} - \frac{p_y}{y}\right)N. \text{ The firm's profits would thus be } \pi = p_x D_x + p_y D_y.$$

The Hessian matrix of this function is: 
$$H = \begin{pmatrix} \frac{\partial^2 \pi}{\partial p_x^2} & \frac{\partial^2 \pi}{\partial p_x \partial p_y} \\ \frac{\partial^2 \pi}{\partial p_x \partial p_y} & \frac{\partial^2 \pi}{\partial p_y^2} \end{pmatrix} = \frac{2}{x - y} N \begin{pmatrix} -1 & 1 \\ 1 & -\frac{x}{y} \end{pmatrix}. \text{ Since } \frac{\partial^2 \pi}{\partial p_x^2} < 0$$

and  $\det H > 0$  (because  $x > y$ ),  $H$  is negative definite and thus the profit maximizing prices are given by the solution to  $\frac{\partial \pi}{\partial p_x} = 0$  and  $\frac{\partial \pi}{\partial p_y} = 0$  which is  $p_x = \frac{x}{2}$  and  $p_y = \frac{y}{2}$ . Substituting these prices in the

profit function gives  $\pi = \frac{x}{4}N$ , which is maximized when  $x = 10$ , yielding the same profits as in the case of a single product of quality  $x = 10$ . Thus the firm cannot gain by offering two products rather than one. This conclusion does **not** represent a general principle.