Ph. D. Preliminary examination in Industrial Organization, June 2005

Answers to questions 1 and 2

1. **(i)** Let F be the c.d.f., that is, $F(x) = \int_0^x f(t)dt$. Then

$$D_1(p_1, p_2) = \begin{cases} 0 & \text{if } p_1 > r \\ N & \text{if } p_1 \le r \text{ and } p_2 > r \\ \frac{N}{2} & \text{if } p_1 = p_2 \le r \\ F(p_2 - p_1)N + \frac{1}{2}[1 - F(p_2 - p_1)]N & \text{if } p_1 < p_2 \le r \\ \frac{1}{2}[1 - F(p_1 - p_2)]N & \text{if } p_2 < p_1 \le r \end{cases}$$

$$D_2(p_1, p_2) = \begin{cases} 0 & \text{if } p_2 > r \\ N & \text{if } p_2 \le r \text{ and } p_1 > r \\ \frac{N}{2} & \text{if } p_1 = p_2 \le r \\ F(p_1 - p_2)N + \frac{1}{2}[1 - F(p_1 - p_2)]N & \text{if } p_2 < p_1 \le r \\ \frac{1}{2}[1 - F(p_2 - p_1)]N & \text{if } p_1 < p_2 \le r \end{cases}$$

(ii) Fix a firm i. If the other firm charges r, $p_i = r$ yields a profit of $\frac{rN}{2}$. For this to be a Nash equilibrium it is necessary and sufficient that firm i cannot increase its profits by choosing a price $p_i < r$. If the firm charges $p_i < r$ then its profits will be:

$$p_i F(r - p_i) N + \frac{1}{2} [1 - F(r - p_i)] N p_i$$

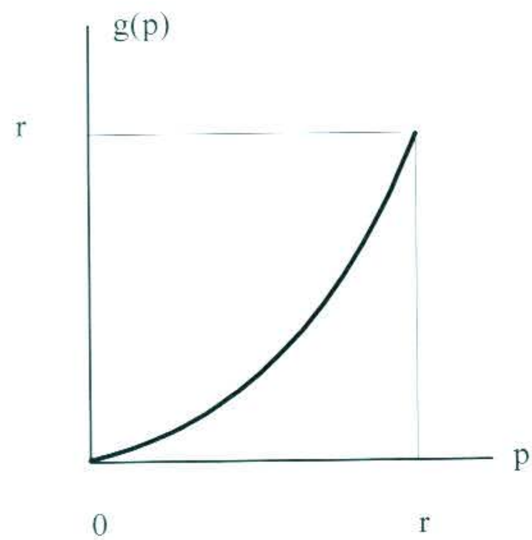
Thus we need

$$\frac{rN}{2} \ge p_i F(r - p_i) N + \frac{1}{2} [1 - F(r - p_i)] N p_i \qquad \textit{for all } p_i \le r$$

i.e.

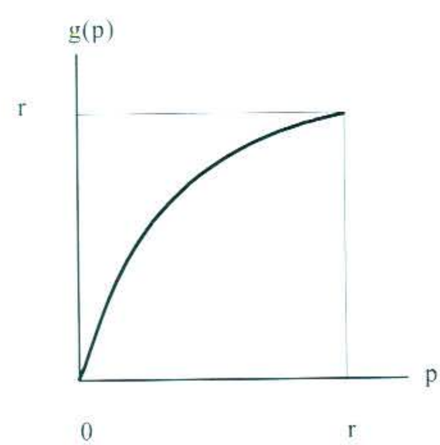
$$r \ge p_i [1 + F(r - p_i)] \qquad \textit{for all } p_i \le r. \tag{1.1}$$

Let us drop the subscript i and define the RHS of (1.1) as $g(p)$. Thus $g(p) = p + pF(r-p)$. The $g(0) = 0$ and $g(r) = r$. Furthermore, $g'(p) = 1 + F(r-p) - pf(r-p)$. Thus $g'(0) = 1 + F(r) > 0$ and $g'(r) = 1 - rf(0)$. We shall consider only the following cases. CASE 1: $g''(p) \geq 0$ for all $p \in [0,r]$; CASE 2: $g''(p) \leq 0$ for all $p \in [0,r]$. CASE 2 implies $g'(r) > 0$ so that $g(p)$ looks like



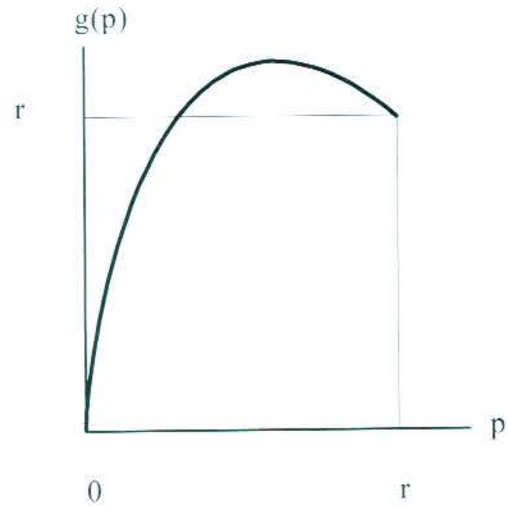
and therefore (1.1) is satisfied and (r,r) is a Nash equilibrium.

In CASE 1, if $g'(r) \geq 0$ i.e. $f(0) \leq \frac{1}{r}$ then $g(p)$ looks like



and therefore (1.1) is satisfied, hence (r,r) is a Nash equilibrium.

If, on the other hand, $g'(r) < 0$ i.e. $f(0) > \frac{1}{r}$ then $g(p)$ looks like



In this case there is a $p \in (0,r)$ such that $g(p) > r$ and therefore (1.1) is violated and (r,r) is not a Nash equilibrium.

Thus, under the assumption that $g(p)$ is either concave or convex, a necessary and sufficient condition for (r,r) to be a Nash equilibrium is

$$g'(r) \geq 0 \text{ i.e. } f(0) \leq \frac{1}{r}$$

(iii) Let $p_2 = 120$. Then $\pi_1(p_1, 120) = p_1 D_1(p_1)$. Thus $\pi_1(120,120) = 120\frac{N}{2} = 60N$, while $\pi_1(100,120)$ $= 100\left[F(20)N + \frac{1}{2}(1-F(20))N\right] = 100(0.7N) = 70N$. Thus (120,120) is not a Nash equilibrium.

(iv) If f is constant, then it must be $f(x) = \frac{1}{r}$ for all x. Then $F(x) = \frac{x}{r}$ so that the function $g(p)$ of part (ii) becomes $g(p) = p\left(1+\frac{r-p}{r}\right)$. Thus $g''(p) = -\frac{2}{r}$, i.e. $g(p)$ is concave. Hence, by the results of part (ii), (r,r) is a Nash equilibrium if and only if $f(0) \leq \frac{1}{r}$ which is of course true. So (r,r) is a Nash equilibrium in this case.

(v) By continuity, $F(x) < 1$ for sufficiently small x. Then if firm 2 charges 0, firm 1 gets zero profits if it also charges 0, but positive profits if it charges a little bit more than zero (its demand is positive since $F(x) < 1$ for x small).

(vi) Intuitively, the Bertrand paradox corresponds to the case where all the mass is concentrated at 0. One might be able to show the Bertrand paradox as a limit result: consider a family f_t of density functions such that, as $t \to \infty$, the smallest x at which $F_t(x) = 1$ tends to zero. Then the Nash equilibrium might tend to zero.

2. PART I. (a) Let P(Q) be the inverse demand function with $P'(Q) < 0$. Fix an arbitrary value of q_1, say $\hat{q}_1$. Let $\hat{q}_2$ be the output level of firm 2 that maximizes the profit of firm 2, given $\hat{q}_1$. Then it must be that at $(\hat{q}_1, \hat{q}_2)$ the first and second-order conditions are satisfied:

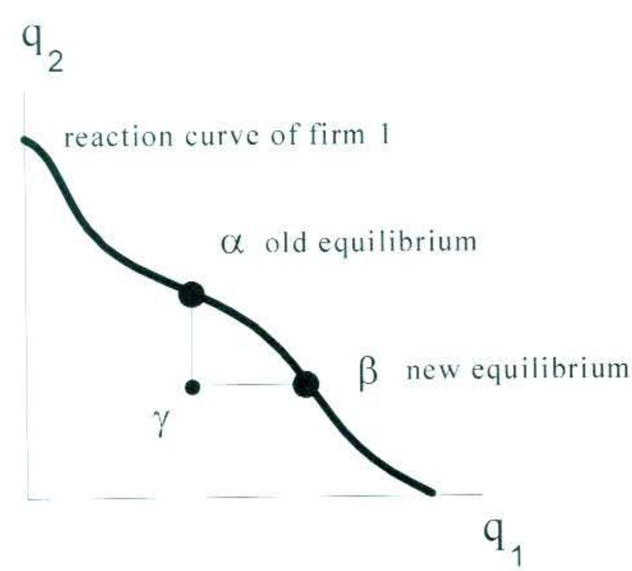
F.O.C. : $\frac{\partial \pi_2}{\partial q_2}(\hat{q}_1, \hat{q}_2) = P(\hat{q}_1 + \hat{q}_2) + \hat{q}_2 P'(\hat{q}_1 + \hat{q}_2) - c = 0$

S.O.C. : $\frac{\partial^2 \pi_2}{\partial q_2^2}(\hat{q}_1, \hat{q}_2) = 2P'(\hat{q}_1 + \hat{q}_2) + \hat{q}_2 P''(\hat{q}_1 + \hat{q}_2) < 0$

Applying the implicit function theorem in a neighborhood of $(\hat{q}_1, \hat{q}_2, c)$ we get q_2 as a function of c with

$\frac{dq_2}{dc} = -\frac{-1}{2P'(\hat{q}_1 + \hat{q}_2) + \hat{q}_2 P''(\hat{q}_1 + \hat{q}_2)} < 0$ (because of the S.O.C.). Thus a decrease in the optimal q_2 for firm 2 , given $\hat{q}_1$, requires an ***increase*** in c. Hence the marginal cost of firm 2 has gone up.

(b)



Claim: $\pi_1(\alpha) < \pi_1(\beta)$.

Proof.: $\pi_1(\alpha) < \pi_1(\gamma)$ because q_1 is the same (hence firm 1's costs are the same) while q_2 is lower at γ (hence P is higher and thus firm 1's revenue is higher).

$\pi_1(\gamma) < \pi_1(\beta)$ because it is a movement towards reaction curve (q_2 is the same; β is on the reaction curve of both firms).

Hence $\pi_1(\alpha) < \pi_1(\beta)$, that is, *firm 1's profits are higher at the new equilibrium.*

Similarly, let π_2^N denote the new profit function of firm 2 (the one that gives rise to reaction curve B) and π_2^O be the old profit function (the one that gives rise to reaction curve A). Then:

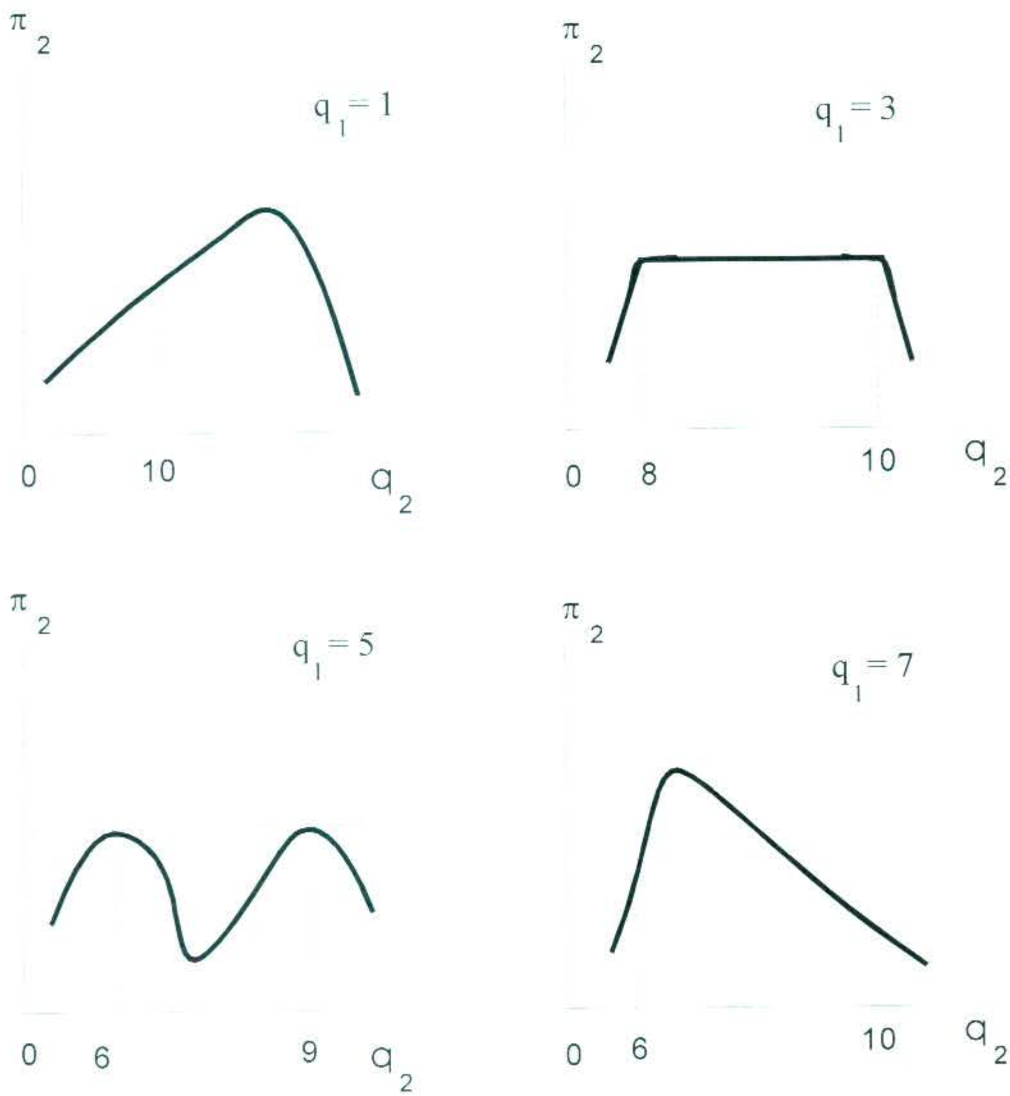
$\pi_2^O(\gamma) < \pi_2^O(\alpha)$ because from α to γ it is a movement away from 2's old reaction curve (curve A).

$\pi_2^O(\beta) < \pi_2^O(\gamma)$ because, from γ to β, q_2 is the same, q_1 has increased hence P has decreased.

Thus $\pi_2^O(\beta) < \pi_2^O(\alpha)$. Since costs have gone up, $\pi_2^O(\beta) > \pi_2^N(\beta)$ (true for every point, hence, in particular, it is true at point β).

Thus $\pi_2^N(\beta) < \pi_2^O(\alpha)$, that is, *firm 2's profits are lower at the new equilibrium.*

PART II.


π 2
q 1 = 1
0
10
q 2
π 2
q 1 = 3
0
8
10
q 2
π 2
q 1 = 5
0
6
9
q 2
π 2
q 1 = 7
0
6
10
q 2

Suggested Answer to Problem 5

5) See Train, *Optimal Regulation*, section 2.3 for the graphical answers. The problem for the firm is

$$\max\ \pi = R - C = PQ - wL - r(K+W) - F \text{ s.t. } \pi \le kPQ \text{ and } W \ge 0 \tag{1}$$

where W is waste. You can solve the problem without waste, but it is easiest to see what happens when $MR = 0$ with W in the Lagrangian. The Lagranian is

$$\begin{aligned} \mathcal{L} &= \pi - \lambda(\pi - kPQ) && (2) \\ &= (1-\lambda)\pi + \lambda kPQ && (3) \end{aligned}$$

Note the constraint can be written as $P \le \frac{AC}{1-k}$.

First order conditions:

$$\frac{d\mathcal{L}}{dL} = (1-\lambda)\left(MRP_L - w\right) + \lambda k \cdot MRP_L = 0 \tag{4}$$

$$\frac{d\mathcal{L}}{dK} = (1-\lambda)\left(MRP_K - r\right) + \lambda k \cdot MRP_K = 0 \tag{5}$$

$$\left\{W = 0 \text{ and } \frac{d\mathcal{L}}{dW} < 0\right\} \text{ or } \left\{W > 0 \text{ and } \frac{d\mathcal{L}}{dW} = 0\right\} \tag{6}$$

$$\begin{aligned} \frac{d\mathcal{L}}{dW} &= -(1-\lambda)r \Rightarrow \{W = 0 \text{ and } \lambda < 1\} \\ & \quad \text{or } \{(1-\lambda)r = 0 \text{ and } W > 0\} \\ & \Rightarrow \{W > 0 \text{ and } \lambda = 1\} && (7) \end{aligned}$$

In the former case, $\lambda < 1$ and (4) implies that $MR > 0$. To see this, solve (4) for MRP_L: $MRP_L = \frac{w(1-\lambda)}{1-\lambda(1-k)} \Rightarrow MR > 0$. In the latter case, $\lambda = 1$ and (4) implies that $MR = 0$.

first begins to bind, as k falls Q rises. To make the argument global as $k \to 0$, assume that profit is concave in Q, and so $MR'(Q) < MC'(Q)$. Then the denominator stays negative (and actually gets larger in magnitude) as k falls and Q rises.

(c) The firm will always produce in the elastic region, but does not expand output into the inelastic region of demand. We have shown above that $MR \geq 0$, never negative. The firm stops expanding output at the point of unit elasticity (because that is where the top of the revenue hill is) and either wastes or uses input inefficiently to bring actual profit down to the allowed level. See figure 2.10 and related discussion in *Optimal Regulation*.

(d) Above we showed that $\frac{dQ}{dk} < 0$: output increases as k decreases.

(e) If $k = 0$, then the constraint implies that $\pi = 0$ and we could be anywhere on the zero profit locus, on the expansion path or off.

(f) No, because the firm doesn't expand into the inelastic region of demand (shown in (c) above). If the 2nd best is in the elastic region, we can get arbitrarily close to it by setting k very small. As long as k is positive, we have input efficiency. See figure 2.9 in *Optimal Regulation* for ROO regulation; the argument is similar for ROS regulation.

Question 6

a) $Z_1 = Z_1^0 + R_1 - a_1 = Z_1^0 + -(1+x)T - a_1 \quad \because T = \frac{-R_1}{1+x}$

$= Z_1^0 - (1+x)I(p_1, p_2) - a_1 \quad \because T = I(p_1, p_2)$

max w.r.t. a_1:

FOC is $-(1+x)\frac{\partial I}{\partial p_1}\frac{dp}{da_1} = 1$.

So $B.R_1$ is implicitly def'd by

$$\boxed{-\frac{\partial I}{\partial p_1}\cdot\frac{dp}{da_1} = \frac{1}{1+x}} \qquad (1)$$

For Z_2:

$Z_2 = Z_2^0 + R_2 - a_2 = Z_2^0 + \frac{T}{1+x} - a_2$

$= Z_2^0 + \frac{I}{1+x} - a_2$.

FOC: $$\boxed{\frac{\partial I}{\partial p_2}\cdot\frac{dp}{da_2} = 1+x} \qquad (2)$$

[See next page for rest of answer to (a)]

b) Totally differentiate (1) w.r.t. x & a_1, holding a_2 fixed:

(jump to second page following)

REST OF ANSWER TO PART (a):

a_1 & a_2 are strategic complements if $\frac{da_1^*}{da_2} > 0$. Differentiate (1) wrt a_1 & a_2;

or use the Implicit Fn Th$^{\underline{m}}$:

$$\frac{da_1^*}{da_2} = -\frac{-\frac{\partial^2 I}{\partial P_1 \partial P_2} P'(a_1) P'(a_2)}{-\frac{\partial^2 I}{\partial P_1^2} P'(a_1)^2 - \frac{\partial I}{\partial P_1} P''(a_1)} > 0$$

given the ass'ns on signs listed.

Equivalently, one can check the sign of $\frac{\partial^2 z_1}{\partial a_1 \partial a_2}$.

$$-\left(\frac{\partial^2 I}{\partial P_1^2}\left(\frac{dP}{da_1}\right)^2 + \frac{\partial I}{\partial P_1}\frac{d^2P}{da_1^2}\right)da_1 = -\frac{dx}{(1+x)^2}$$

$$\Rightarrow \frac{da_1^*}{dx} = \frac{(1+x)^{-2}}{\frac{\partial^2 I}{\partial P_1^2}\left(\frac{dP}{da_1}\right)^2 + \frac{\partial I}{\partial P_1}\frac{d^2P}{da_1^2}} > 0$$

(each term is positive on RHS)

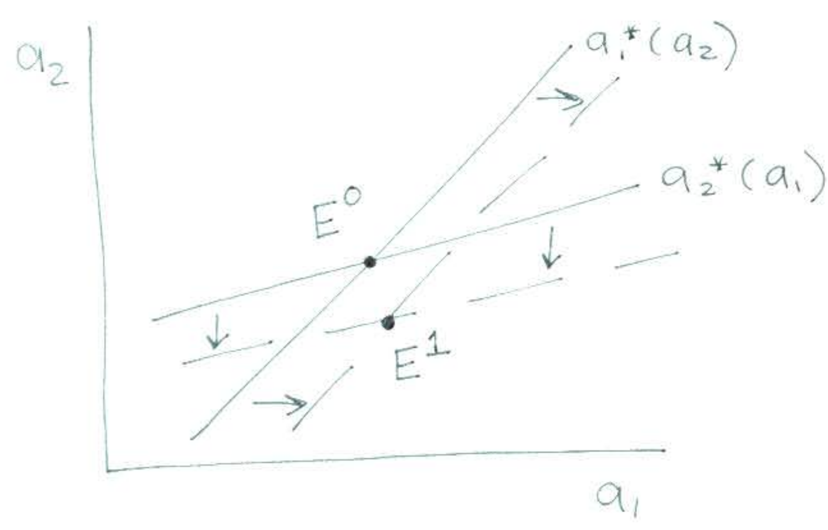
c) Do same for (2):

$$\frac{\partial^2 I}{\partial P_2}\left[p'(a_2)\right]^2 + \frac{\partial I}{\partial P_2}p''(a_2) = dx$$

$$\frac{da_2^*}{dx} = \left[\frac{\partial^2 I}{\partial P_2}p'(a_2)^2 + \frac{\partial I}{\partial P_2}p''(a_2)\right]^{-1} < 0$$

(each term is negative)

d) Using the assumptions, we have:



Move from E^0 to E^1.

As drawn, $a_1 \uparrow$ & $a_2 \downarrow$. This is what happens as long as the indirect effects are smaller than the direct effects.

If $a_2 \downarrow$ & $a_1 \uparrow$, then $I \downarrow$, and so $T \downarrow$.

e) We've shown that when the DWL of regulation is high, there is less regulation. This is Prop. 2 of Becker (QJE 1983). In his corollary to Prop 2 he argues <u>for</u> the statement; see article for details.