Econometrics Preliminary Exam

Agricultural and Resource Economics, UC Davis

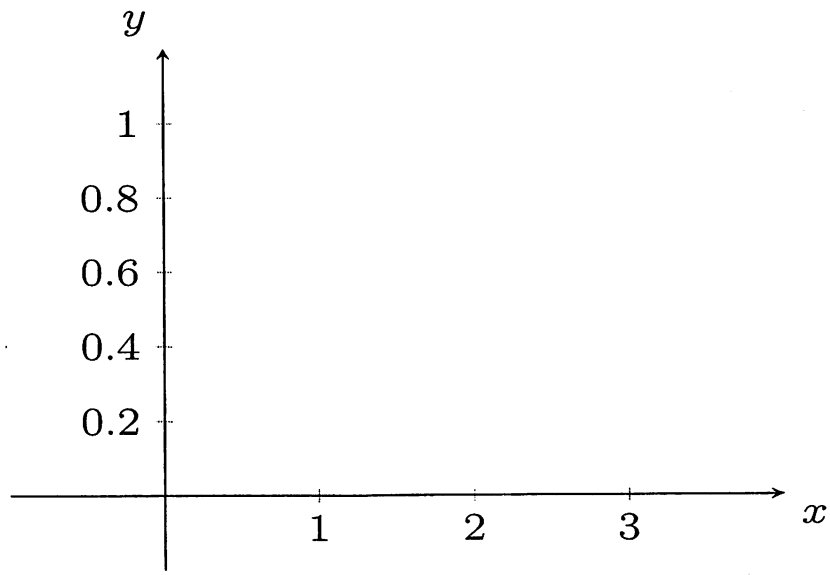
July 2022

There are **four** questions. You must answer all questions. Each question receives equal weight. Within each question, each part will receive equal weight in grading. You have 20 minutes to read the exam and then four hours to complete the exam.

I. Probability and Statistics

(a) Consider a discrete random variable $X \in \{0, 1, 2, 3\}$ following the binomial distribution with probability mass function $P(X = x) = \binom{3}{x} \cdot 0.4^x \cdot 0.6^{3-x}$ for $x = 0, 1, 2, 3$.

(i) Derive the cumulative distribution function of X and sketch it out on your answer sheet.



(ii) What is the probability mass function of random variable $Y = X^2$?

(b) Consider scalar continuous random variables X and Y, where the conditional distribution of Y given $X = x$ is normally distributed following $N(x, 1)$, for $x \in [0, 1]$, and X follows the uniform distribution $U[0, 1]$. [Hint: Some of the final answers could just be left as functions of $\phi(.)$ or $\Phi(.)$, where $\phi(.)$ is the the probability density function (pdf) of standard normal $N(0, 1)$ and $\Phi(.)$ is the cumulative distribution function (cdf) of standard normal $N(0, 1)$.]

(i) What is the joint pdf of X and Y? Are X and Y independent?

(ii) Use the law of iterated expectations to calculate $E[Y]$ and $P[Y \leqslant 0]$. Use the law of total variance to calculate $Var(Y)$. [Hint: Recall that the law of total variance is $Var[Y] = E[Var(Y|X)] + Var(E[Y|X])$.]

(c) Consider scalar continuous random variables $X_1, X_2, ..., X_n$ that are independent and identically distributed following the binomial distribution $B(10, p)$, for some $p \in (0, 1)$. The probability mass function of $B(10, p)$ is $f(x) = \binom{10}{x} \cdot p^x \cdot (1-p)^{10-x}$, for $x \in \{0, 1, 2, ..., 10\}$. [Note: a random variable with $B(m, p)$ distribution has mean mp and variance $mp(1-p)$. The combination notation $\binom{m}{x} = \frac{m!}{x!(m-x)!}$.]

(i) Derive the MLE estimator of p and call it $\hat{p}$.

(ii) Derive $E[\hat{p}]$ and $V[\hat{p}]$.

(iii) What is the limiting distribution of $\sqrt{n}(\hat{p} - p)$?

(iv) Write down the Likelihood Ratio Test (LRT) statistic for the null hypothesis $H_0 : p = 0.5$ and the alternative hypothesis $H_1 : p \neq 0.5$. Then form a decision rule such that your proposed test controls size at 5% asymptotically.

II. **Linear Regression**

Consider the model $y_i = \beta x_i^2 + e_i$, where x_i is scalar and $E[e_i|x_i] = 0$. You have an *iid* random sample of size n. Consider the following estimators:

$$\bar{\beta} = \frac{\sum_{i=1}^n x_i^2 y_i}{\sum_{i=1}^n x_i^4}$$
$$\hat{\beta} = \frac{\sum_{i=1}^n (x_i - \bar{x}) y_i}{\sum_{i=1}^n (x_i - \bar{x})^2}$$

(a) Is $\bar{\beta}$ unbiased for β? If so, prove it. If not, either (i) state additional conditions you require for unbiasedness and prove unbiasedness under those conditions, or (ii) explain why no such conditions exist.

(b) Is $\bar{\beta}$ consistent for β? If so, prove it. If not, state additional conditions you require for consistency and prove consistency under those conditions.

(c) Following on from (b), find the asymptotic distribution of $\sqrt{n}(\bar{\beta} - \beta)$ as $n \to \infty$. You may carry over any assumptions you made in (b) to show consistency. State any additional assumptions you require.

(d) Following on from (b) and (c), propose a consistent estimator for the asymptotic variance of $\bar{\beta}$ and prove that it is consistent. You may carry over any assumptions you made in (b) and (c). State any additional assumptions you require.

(e) Is $\hat{\beta}$ consistent for β? If so, prove it. If not, state additional conditions you require for consistency and prove consistency under those conditions.

(f) Under the conditions you imposed for consistency in (e), is $\hat{\beta}$ unbiased for β? If so, prove it. If not, either (i) state additional conditions you require for unbiasedness and prove unbiasedness under those conditions, or (ii) explain why no such conditions exist.

(g) Under the conditions you imposed for consistency in (b) and (e), which estimator is more efficient asymptotically, $\bar{\beta}$ or $\hat{\beta}$? If you need any additional conditions to answer this question, state them. Otherwise, you may answer in words. No mathematical proof necessary.

(h) Describe how you would construct a bootstrap confidence interval for $1/\beta$. Be specific, including about which estimator you would use and why.

III. Estimation and Testing in Linear and Nonlinear Regression Models

(a) Consider the linear regression model $y = X\beta + e$, where y and e are $n \times 1$ vectors, β a $K \times 1$ vector and X an $n \times K$ matrix (n denotes the sample size). Assume $e|X \sim N(0, \sigma^2 I_n)$. The OLS residuals are $\hat{e} = y - X\hat{\beta}$.

  (i) Show that $E(s^2) = \sigma^2$, where $s^2 = \hat{e}'\hat{e}/(n-K)$.

  (ii) Consider using the following statistic to test the null hypothesis $H_0 : \sigma^2 = 2$

$$t = \frac{\sqrt{n}(s^2 - 2)}{\sqrt{\frac{1}{n}\sum_{i=1}^{n}(\hat{e}_i^2 - s^2)^2}}$$

  Show that the asymptotic null distribution of this statistic is standard normal.

  (iii) Suppose $\sigma^2 = 2.5$ and $n = 100$. Using asymptotic theory, approximate the power of a test based on the t statistic in (ii).

(b) Consider the nonlinear regression model $y_i = h(x_i; \gamma_0) + u_i$ for $i = 1, \ldots, n$, where $h(x, \gamma)$ is a known function of x and γ and $dim(\gamma) = \ell$.

  (i) Define the nonlinear least squares estimator of γ_0.

  (ii) Provide the Wald statistic to test $H_0 : \gamma_0 = 0$. Derive its asymptotic distribution under H_0. You can rely on high-level conditions for the result, no need to provide primitive conditions.
  Note: In your derivation, you can rely on the following result: If a $k \times 1$ random variable $z_n \xrightarrow{d} N(0, V)$, then $z_n'\hat{V}^{-1}z_n \xrightarrow{d} \chi^2_k$, assuming $\hat{V} \xrightarrow{p} V$.

  (iii) For a significance level $\alpha \in (0, 1)$, how would you obtain the critical value of the Wald test? What does your derivation imply about the asymptotic rejection probability of the test under H_0 for a given α?

  (iv) Show the asymptotic behavior of the Wald statistic under $H_A : \gamma_0 = c$ for some $c \neq 0$. What does your derivation imply the asymptotic rejection probability of the Wald test under H_0 for a given significance level α?

IV. Transformation Models

(a) **Estimation of Transformation Models with Endogeneity**. For $i = 1, \ldots, n$, let $y_i = t(x_i'\gamma_0) + \epsilon_i$, where $E[\epsilon_i|x_i] \neq 0$, $t(z)$ is a known function given z, $dim(x) = dim(\gamma) = d_\gamma$.

(i) Given instruments z_i, where $E[z_i\epsilon_i] = 0$ and $dim(z_i) = d_z$, propose an estimator of γ_0 given data $\{y_i, x_i, z_i\}_{i=1}^n$. Make sure to discuss the number of instruments required for identification.

(ii) Provide sufficient conditions for the consistency of the estimator you proposed in (a) as $n \to \infty$. Provide a heuristic description of why these conditions are sufficient, but no need to provide a proof.

(iii) Derive the mean value expansion to obtain an expression for the sampling error of the estimator you proposed in (a). Provide sufficient conditions for its asymptotic normality as $n \to \infty$. State primitive conditions wherever possible and make sure to state its asymptotic distribution. Provide a heuristic description of why these conditions are sufficient, but no need to provide a proof.

(b) **Estimation of Transformation Models using Panel Data.** Now suppose you observe panel data of both outcome and regressors, such that for $i = 1, \ldots, n$ and $t = 1, \ldots, T$, $y_{it} = t(x_{it}'\gamma_0) + a_i + u_{it}$.
Note: For all of the following questions, consistency refers to asymptotic behavior as $n \to \infty$. Furthermore, you do not need to provide conditions for consistency of the estimators you propose. However, you have to formally define the estimator. If you are using a nonlinear least squares or maximum likelihood estimator, then you have to define the objective function. If you are using a generalized method-of-moments estimator, then you have to derive the moment conditions and define the objective function of the estimator.
In your answers to the following three questions, (i)-(iii), you should propose at least four different estimators.

(i) Assume that $E[u_{it}|x_{i1}, \ldots, x_{iT}, a_i] = 0$ and $E[a_i|x_{i1}, \ldots, x_{iT}] = 0$, propose two consistent estimators of γ_0.

(ii) Suppose $E[a_i|x_{i1}, \ldots, x_{iT}] \neq 0$. Assume that $E[u_{it}|x_{i1}, \ldots, x_{iT}, a_i] = 0$, propose two consistent estimators of γ_0.

(iii) Suppose $E[a_i|x_{i1}, \ldots, x_{iT}] \neq 0$ and $E[u_{it}|x_{i1}, \ldots, x_{iT}, a_i] \neq 0$. Assume that $E[u_{it}|x_{i1}, \ldots, x_{it}, a_i] = 0$, propose two consistent estimators of γ_0.

ARE/ECN 240B Reference Sheet

Notation. θ_0, Θ, y_i, x_i, $s(y_i, x_i; \theta)$ and $H(y_i, x_i; \theta)$ pertain to the objects defined in the 240B lecture notes.

Assumption ULLN 1 $\sup_{\theta\in\Theta} |\sum_{i=1}^{n} f(y_i, x_i; \theta)/n - E[f(y_i, x_i; \theta)]| \xrightarrow{p} 0$, if the following conditions hold,

(i) *(i.i.d.)* $\{y_i, x_i\}_{i=1}^{n}$ is an i.i.d. sequence of random variables;

(ii) *(Compactness)* Θ is compact;

(iii) *(Continuity)* $f(y_i, x_i; \theta)$ is continuous in θ for all $(y_i, x_i')'$;

(iv) *(Measurability)* $f(y_i, x_i; \theta)$ is measurable in $(y_i, x_i')'$ for all $\theta \in \Theta$;

(v) *(Dominance)* There exists a dominating function $d(y_i, x_i)$ such that $|f(y_i, x_i; \theta)| \leqslant d(y_i, x_i)$ for all $\theta \in \Theta$ and $E[d(y_i, x_i)] < \infty$.

Assumption ULLN 2 $\sup_{\theta\in\Theta} |\sum_{i=1}^{n} f(y_i, x_i; \theta)/n - E[f(y_i, x_i; \theta)]| \xrightarrow{p} 0$, if the following conditions hold,

(i) (*Law of Large Numbers*) $\{y_i, x_i\}$ is i.i.d., and $E[f(y_i, x_i; \theta)] < \infty$ for all $\theta \in \Theta$, which implies $\sum_{i=1}^{n} f(y_i, x_i; \theta)/n \xrightarrow{p} E[f(y_i, x_i; \theta)]$.

(ii) (*Compactness of* Θ) Θ is in a compact subset of $\mathbb{R}^k$.

(iii) *(Measurability in $(y_i, x_i')'$)* $f(y_i, x_i; \theta)$ is measurable in $(y_i, x_i')'$ for all $\theta \in \Theta$.

(iv) (*Lipschitz Continuity*) For all $\theta, \theta' \in \Theta$, there exists $g(y_i, x_i)$, such that $|f(y_i, x_i; \theta) - f(y_i, x_i; \theta')| \leqslant g(y_i, x_i)\|\theta - \theta'\|$, for some norm $\|.\|$, and $E[g(y_i, x_i)] < \infty$.

Formula for the score statistic

$$S \equiv \left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n} s_i(\hat{\theta}_R)\right)' A_{nR}^{-1} C_{nR}' \left\{\widehat{Avar}\left(C_{nR} A_{nR}^{-1} \sum_{i=1}^{n} s_i(\hat{\theta}_R)/\sqrt{n}\right)\right\}^{-1} C_{nR} A_{nR}^{-1} \left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n} s_i(\hat{\theta}_R)\right)$$