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ANSWER KEY

PRELIMINARY EXAMINATION FOR THE Ph.D. DEGREE
MICROECONOMIC THEORY

Answer four questions (out of five)

Question 1. Variations on a theme by Herbert Scarf

Consider an economy with three consumers, labeled 1, 2 and 3, and three goods, labeled 1, 2 and 3. For i, j = 1, 2, 3, we denote by x_j^i Consumer i's consumption of good j, i. e., a superscript indicates the consumer, and a subscript the good. The utility functions of the three consumers, defined on $\Re^3_+$, are as follows.

$$u^1(x_1^1, x_2^1, x_3^1) = \min\{x_1^1, x_2^1\},$$

$$u^2(x_1^2, x_2^2, x_3^2) = \min\{x_1^2, x_3^2\},$$

$$u^3(x_1^3, x_2^3, x_3^3) = \min\{x_2^3, x_3^3\}.$$

Notation. As usual, we denote by (p_1, p_2, p_3) a price vector and by w^i a level of wealth for Consumer i, i = 1, 2, 3.

1(a). For i, j = 1, 2, 3, compute the Walrasian demands $\tilde{x}_j^i(p_1, p_2, p_3, w^i)$ for $(p_1, p_2, p_3) >> 0$ and $w^i \geq 0$.

ANSWER. We compute:

$$\tilde{x}_1^1(p_1, p_2, p_3, w^1) = \frac{w^1}{p_1 + p_2}; \tilde{x}_2^1(p_1, p_2, p_3, w^1) = \frac{w^1}{p_1 + p_2}; \tilde{x}_3^1(p_1, p_2, p_3, w^1) = 0.$$

$$\tilde{x}_1^2(p_1, p_2, p_3, w^2) = \frac{w^2}{p_1 + p_3}; \tilde{x}_2^2(p_1, p_2, p_3, w^2) = 0; \tilde{x}_3^2(p_1, p_2, p_3, w^2) = \frac{w^2}{p_1 + p_3};$$

$$\tilde{x}_1^3(p_1, p_2, p_3, w^3) = 0; \tilde{x}_2^3(p_1, p_2, p_3, w^3) = \frac{w^3}{p_2 + p_3}; \tilde{x}_3^3(p_1, p_2, p_3, w^3) = \frac{w^3}{p_2 + p_3}.$$

1(b). Is there a positive representative consumer for the unrestricted domain of price-wealth vectors $(p_1, p_2, p_3, w^1, w^2, w^3)$ where $(p_1, p_2, p_3) \in \Re^3_{++}$ and $(w^1, w^2, w^3) \in \Re^3_+$?

If YES, display its indirect utility function and check that it is indeed a positive representative consumer (checking for one of the three goods suffices). If NO, argue why not.

ANSWER. NO.

First note that the Gorman Theorem does not apply, because the preference relations are homothetic but not identical. The Eisenberg theorem does not apply either, because the wealth vectors are unrestricted. In order to show that no (positive) representative consumer exists in this case, compute the aggregate demand functions as

Good 1: $\dfrac{w^1}{p_1 + p_2} + \dfrac{w^2}{p_1 + p_3}$;

Good 2: $\dfrac{w^1}{p_1 + p_2} + \dfrac{w^3}{p_2 + p_3}$;

Good 3: $\dfrac{w^2}{p_1 + p_3} + \dfrac{w^3}{p_2 + p_3}$.

If we transfer an amount ε of wealth from Consumer 1 to Consumer 3, leaving unchanged the wealth of Consumer 2, aggregate wealth does not change, yet the aggregate demand for good 1 goes down to $\dfrac{w^1 - \varepsilon}{p_1 + p_2} + \dfrac{w^2}{p_1 + p_3}$, whereas the aggregate demand for good 3 goes up to $\dfrac{w^2}{p_1 + p_3} + \dfrac{w^3 + \varepsilon}{p_2 + p_3}$. Hence, demand depends not only on total wealth but also on its distribution, negating the possibility of a positive representative consumer when wealth levels are unrestricted.

1(c). Is there a positive representative consumer for the restricted domain of price-wealth vectors $(p_1, p_2, p_3, w^1, w^2, w^3)$ where $(p_1, p_2, p_3) \in \Re^3_{++}$ and where, for given nonnegative parameters θ^1, θ^2 and θ^3 such that $\theta^1 + \theta^2 + \theta^3 = 1$, $(w^1, w^2, w^3) \in \Re^3_+$ is restricted to satisfy $w^i = \theta^i [w^1 + w^2 + w^3], i = 1, 2, 3$?

If YES, display its indirect utility function and check that it is indeed a positive representative consumer (checking for one of the three goods suffices). If NO, argue why not.

ANSWER. YES. The Eisenberg positive representative consumer theorem applies.

In order to obtain the indirect utility function of the representative consumer, we first obtain the indirect utility functions of our three consumers.

$$v^1(p_1,p_2,p_3,w^1)=\frac{1}{p_1+p_2}w^1;\; v^2(p_1,p_2,p_3,w^2)=\frac{1}{p_1+p_3}w^2;\; v^3(p_1,p_2,p_3,w^3)=\frac{1}{p_2+p_3}w^3.$$

The statement of Eisenberg's Theorem gives the indirect utility function of the representative consumer as

$$v^\theta(p_1,p_2,p_3,w)=\left[\frac{1}{p_1+p_2}\right]^{\theta^1}\left[\frac{1}{p_1+p_3}\right]^{\theta^2}\left[\frac{1}{p_2+p_3}\right]^{\theta^3} w=\Phi w, \tag{1.1}$$

where $\Phi(p_1,p_2,p_3):=[p_1+p_2]^{-\theta^1}[p_1+p_3]^{-\theta^2}[p_2+p_3]^{-\theta^3}$.

In order to check that this is indeed a representative consumer, we first specialize to our case the aggregate consumer demands obtained in 1(b) above as follows.

Good 1: $$\frac{\theta^1 w}{p_1+p_2}+\frac{\theta^2 w}{p_1+p_3}, \tag{1.2}$$

Good 2: $$\frac{\theta^1 w}{p_1+p_2}+\frac{\theta^3 w}{p_2+p_3},$$

Good 3: $$\frac{\theta^2 w}{p_1+p_3}+\frac{\theta^3 w}{p_2+p_3},$$

where $w:=w^1+w^2+w^3$.

Now apply Roy's identity to (1.1) in order to obtain the demands of the representative consumer as follows.

Good 1:

$$\tilde{x}_1^\theta(p_1,p_2,p_3,w)=-\frac{\frac{\partial v^\theta}{\partial p_1}}{\frac{\partial v^\theta}{\partial w}}=-\frac{\frac{\partial\Phi}{\partial p_1}w}{\Phi}=-\frac{\left[-\theta^1\frac{\Phi}{p_1+p_2}-\theta^2\frac{\Phi}{p_1+p_3}\right]w}{\Phi}=\frac{\theta^1 w}{p_1+p_2}+\frac{\theta^2 w}{p_1+p_3},$$

in agreement with (1.2) for $w=w^1+w^2+w^3$.

The demands for goods 2 and 3 are similarly computed and checked.

1(d). We now move to a Jevonsian world where, for i = 1, 2, 3, instead of the money wealth amount w^i, Consumer i is endowed with an initial endowment vector $(\omega_1^i,\omega_2^i,\omega_3^i)\in\Re_+^3$ (but with a zero price-independent component of wealth, i. e., $m^i = 0$). Compute $\hat{x}^i_j(p_1, p_2, p_3)$, Consumer i's Jevonsian demand for good j, for $i, j = 1, 2, 3$.

ANSWER. We substitute $p_1\omega^i_1 + p_2\omega^i_2 + p_3\omega^i_3$ for w^i in the Walrasian demands computed in 1(b) above and obtain the Jevonsian demands as follows:

$$\hat{x}^1_1(p_1, p_2, p_3) = \frac{p_1\omega^1_1 + p_2\omega^1_2 + p_3\omega^1_3}{p_1 + p_2}; \hat{x}^1_2(p_1, p_2, p_3) = \frac{p_1\omega^1_1 + p_2\omega^1_2 + p_3\omega^1_3}{p_1 + p_2}; \hat{x}^1_3(p_1, p_2, p_3) = 0;.$$

$$\hat{x}^2_1(p_1, p_2, p_3) = \frac{p_1\omega^2_1 + p_2\omega^2_2 + p_3\omega^2_3}{p_1 + p_3}; \hat{x}^2_2(p_1, p_2, p_3) = 0; \hat{x}^2_3(p_1, p_2, p_3) = \frac{p_1\omega^2_1 + p_2\omega^2_2 + p_3\omega^2_3}{p_1 + p_3};$$

$$\hat{x}^3_1(p_1, p_2, p_3) = 0; \hat{x}^3_2(p_1, p_2, p_3) = \frac{p_1\omega^3_1 + p_2\omega^3_2 + p_3\omega^3_3}{p_2 + p_3}; \hat{x}^3_3(p_1, p_2, p_3) = \frac{p_1\omega^3_1 + p_2\omega^3_2 + p_3\omega^3_3}{p_2 + p_3}.$$

1(e). Consider an exchange economy with the above consumers, goods and utility functions for the following initial endowment vectors:

$$(\omega^1_1, \omega^1_2, \omega^1_3) = \left(\frac{6}{8}, \frac{6}{8}, \frac{6}{8}\right),$$

$$(\omega^2_1, \omega^2_2, \omega^2_3) = \left(\frac{1}{8}, \frac{1}{8}, \frac{1}{8}\right),$$

$$(\omega^3_1, \omega^3_2, \omega^3_3) = \left(\frac{1}{8}, \frac{1}{8}, \frac{1}{8}\right).$$

As usual, the social endowment of the three goods is the sum of the individual endowments.

Compute the general competitive equilibrium of this economy, specifying the equilibrium (relative) prices. We admit the possibility that one (but no more than one) price be zero at equilibrium. (Note. If you are unable to answer fully, take $p_3 = 0$, $p_1 = p_2 = 1/2$, and move on.)

ANSWER. Aggregate supply is $(\omega_1, \omega_2, \omega_3) = (6/8, 6/8, 6/8) + (1/8, 1/8, 1/8) = (1,1,1)$.

We normalize the price vector so that $p_1 + p_2 + p_3 = 1$, which yields:

$p_1\omega^1_1 + p_2\omega^1_2 + p_3\omega^1_3 = \frac{6}{8}$, $p_1\omega^2_1 + p_2\omega^2_2 + p_3\omega^2_3 = p_1\omega^3_1 + p_2\omega^3_2 + p_3\omega^3_3 = \frac{1}{8}$. Hence, from 1(d),

$$\hat{x}^1_1(p_1, p_2, p_3) = \frac{6/8}{p_1 + p_2}; \hat{x}^1_2(p_1, p_2, p_3) = \frac{6/8}{p_1 + p_2}; \hat{x}^1_3(p_1, p_2, p_3) = 0;.$$

$$\hat{x}_1^2(p_1, p_2, p_3) = \frac{1/8}{p_1 + p_3}; \hat{x}_2^2(p_1, p_2, p_3) = 0; \hat{x}_3^2(p_1, p_2, p_3) = \frac{1/8}{p_1 + p_3};$$

$$\hat{x}_1^3(p_1, p_2, p_3) = 0; \hat{x}_2^3(p_1, p_2, p_3) = \frac{1/8}{p_2 + p_3}; \hat{x}_3^3(p_1, p_2, p_3) = \frac{1/8}{p_2 + p_3}.$$

Therefore, aggregate demands are as follows:

Good 1. $\frac{6/8}{p_1 + p_2} + \frac{1/8}{p_1 + p_3}$;

Good 2. $\frac{6/8}{p_1 + p_2} + \frac{1/8}{p_2 + p_3}$;

Good 3. $\frac{1/8}{p_1 + p_3} + \frac{1/8}{p_2 + p_3}$.

If all three prices are positive at equilibrium, then aggregate demand must equal aggregate supply in all three markets, i. e.,

Good 1. $\frac{6/8}{p_1 + p_2} + \frac{1/8}{p_1 + p_3} = 1$, (1.3)

Good 2. $\frac{6/8}{p_1 + p_2} + \frac{1/8}{p_2 + p_3} = 1$, (1.4)

Good 3. $\frac{1/8}{p_1 + p_3} + \frac{1/8}{p_2 + p_3} = 1$. (1.5)

But (1.3) and (1.4) imply that $p_1 = p_2 := p$, which substituted into (1.5) yields $\frac{1/8}{p+1-2p} + \frac{1/8}{p+1-2p} = \frac{2/8}{1-p} = 1$, i. e., $1 = 4 - 4p$ or $p = ¾$, which makes $p_3 = 1 - 2p$ negative, so this does not work.

Note that if only one price is zero the expressions used above still give a solution to the utility maximization problem (although there are others with larger amounts of the good whose price is zero). But we cannot have two zero prices, because then one of the demands would be not defined (infinite). Accordingly, we try only one zero price, with the corresponding good possibly in excess supply at equilibrium. Setting $p_1 = 0$ would not work, because then the condition of "no excess demand for good 1" would read (adapting (1.3)) $\frac{6/8}{p_2} + \frac{1/8}{p_3} \leq 1$, which together with (1.4) would imply $\frac{6/8}{p_2}+\frac{1/8}{p_3}\le\frac{6/8}{p_2}+\frac{1/8}{p_2+p_3}$, or $p_3 \geq p_2 + p_3$, i. e., p_2 would also be zero, in addition to p_1. Similarly for trying $p_2 = 0$.

But setting $p_3 = 0$ does work. As before, (1.3) and (1.4) imply $p_1 = p_2$, i. e., $p_1 = p_2 = ½$, which satisfies equalities (1.3) and (1.4) and yields $\frac{1/8}{1/2}+\frac{1/8}{1/2}=\frac{1}{2}<1$ for the LHS of (1.5), i. e., good 3 is in excess supply, at zero price, in equilibrium.

1(f). Consider the equilibrium allocation obtained in 1(e). Is it Pareto efficient (or Pareto optimal)? Does it involve any waste? Argue and discuss your answer.

ANSWER. The equilibrium allocation obtained in 1(e) is indeed Pareto efficient. The first fundamental theorem of welfare economics does apply. More directly, improving the welfare level of Consumer 2 or Consumer 3 requires an extra amount of good 1 or of good 2, which would decrease the utility of Consumer 1.

Is there waste? Certainly not in the sense of Pareto inefficiency. Yet it is true that ½ units of good 3 go unused.

Question 2. Variations on a theme by Joan Robinson

We consider a profit-maximizing firm that produces one output (named *output*) by using one input (named *labor*), according to the continuous, differentiable and concave production function $f:\Re_+ \to \Re : L \mapsto f(L)$, where L denotes the amount of labor used by the firm, and where we assume that $f(0) = 0$. The firm has no market power in the output market, where it faces a price equal to one ($p = 1$), but in the input market it faces a differentiable (inverse) supply-of-labor function $\tilde{w}:\Re_+ \to \Re_+ : L \mapsto \tilde{w}(L)$, where $\tilde{w}(L)$ is the wage per unit of labor that the firm has to pay in order to hire L units of labor, and where we assume that $\tilde{w}'(L) \geq 0$.

2(a). What do we mean when we say that the firm has market power in the input market?

ANSWER. We say that the firm does not have market power in the input market if it faces a flat supply-of-labor curve, i. e., if $\tilde{w}'(L) = 0$, and that it has market power there if it faces an upwards-sloping supply-of-labor curve, i. e., if $\tilde{w}'(L) > 0$ (at relevant labor levels).

2(b). Write the profit maximizing program of the firm. Obtain its first-order condition and verbally interpret it. Represent the profit maximizing solution in a graph, labeled Figure 2.1, which depicts the relevant marginal magnitudes.

Answer. The maximization problem is $\max_L \; f(L) - \tilde{w}(L)L$ subject to $L \geq 0$. Its FOC is $f'(L) - [\tilde{w}(L) + \tilde{w}'(L)L] \leq 0$, with equality if $L > 0$.

When $L > 0$, the condition is $f'(L) = \tilde{w}(L) + \tilde{w}'(L)L$, interpreted as:

MARGINAL PRODUCT OF LABOR = MARGINAL HIRING COST.

The profit-maximizing solution is graphically represented in Figure 2.1 for the case where $\tilde{w}'(L) > 0$ and, accordingly, the firm enjoys market power in the input market. The intersection between the marginal product and marginal hiring cost curves gives the profit-maximizing demand for labor L^M by the firm, which the firm buys at the wage rate w^M. The wage rate is lower than the marginal product of labor when the firm has market power.

2(c). Rewrite the first-order condition of 2(b) à la Lerner equation, i. e., in such a way that on one side of the equation only the labor elasticity of supply appears. Interpret, separately considering the cases $\tilde{w}'(L) > 0$ and $\tilde{w}'(L) = 0$.

Answer. From $f'(L) = \tilde{w}(L) + \tilde{w}'(L)L$, we write $f'(L) - \tilde{w}(L) = \tilde{w}'(L)L$, or dividing through by $\tilde{w}(L)$,

$$\frac{f'(L) - \tilde{w}(L)}{\tilde{w}(L)} = \tilde{w}'(L)\frac{L}{\tilde{w}(L)}. \qquad (2.1)$$

The RHS of (2.1) is the elasticity of the inverse supply-of-labor function, which in a manner parallel to the Lerner equation can be called the *degree of monopsony*. The LFH is the relative excess of the marginal product of labor over the wage rate.

If $\tilde{w}'(L) > 0$, then the degree of monopsony is positive and the marginal product of labor exceeds the wage rate: Joan Robinson would say that there is "neoclassical exploitation of labor." If, on the contrary, $\tilde{w}'(L) = 0$, then the degree of monopsony is zero and the marginal product of labor equals the wage rate.

Writing $\varepsilon(w) = \frac{d\tilde{L}}{dw}\frac{w}{L}$ for the elasticity of the *direct* supply-of-labor function, equation (2.1) can be written

$$\frac{f' - w}{w} = \frac{1}{\varepsilon}. \qquad (2.2)$$

2(d). Assume now and in what follows that the supply-of-labor function $\tilde{w}(L)$

is generated by the UMAX problem of a (representative) consumer who consumes leisure (with amounts denoted x_1) and output (with amounts denoted x_2), with preferences represented by the utility function $u: \Re_+ \times \Re : u(x_1, x_2) = \varphi(x_1) + x_2$, where φ is a differentiable function satisfying φ(0) = 0, and with nonnegative initial endowments ω_1 and ω_2 of leisure and output, respectively. We denote by L^S the consumer's supply of labor, i. e., $x_1 = \omega_1 - L^S$, and define the *disutility-of-labor* function

$$\gamma(L^S) := \varphi(\omega_1) - \varphi(\omega_1 - L^S).$$

The consumer is a price and wage taker, facing the wage rate w in the labor market and the price $p = 1$ in the output market.

Obtain the supply-of-labor function of the consumer under the assumption that the solution to her maximization problem is interior.

ANSWER. The UMAX problem of the consumer is as follows: given w, choose L^S in order to maximize

$$\varphi(\omega_1 - L^S) + wL^S \text{ or } \varphi(\omega_1) - \gamma(L^S) + wL^S$$

subject to the constraints $L^S \geq 0$, $\omega_1 - L^S \geq 0$, where $\varphi(\omega_1)$ is a constant.

The FOC for an interior maximum is $w = \gamma'(L^S)$, which gives the supply-of-labor function in inverse form $\tilde{w}(L^S)$.

2(e). Given an amount of labor L, define the *social surplus* associated with L as $S(L) := f(L) - \gamma(L)$. What amount of labor maximizes social surplus? Graphically represent the surplus-maximizing solution, denoted L^*, and the maximal social surplus, denoted S^*, in a copy of Figure 2.1, to be labeled Figure 2.2.

ANSWER. The maximization of $f(L) - \gamma(L)$, if achieved at a point interior to the interval $[0, \omega_1]$, requires $f'(L) = \gamma'(L)$, satisfied at point L^* in Figure 2.2.

By appealing to the Fundamental Theorem of Calculus and noting that $f(0) = \gamma(0) = 0$, The level of social surplus S^*achieved at L^* can be written as $S^* = \int_0^{L^*} [f'(L) - \gamma'(L)]dL$ and graphically represented as the area below the curve $f'(L)$ and above the curve $\gamma'(L)$ between 0 and L^*, i. e., *AREA* S^M +*AREA DL*.

2(f). Define the *deadweight loss* associated with an amount L of labor as $S^* - S(L)$, i. e., the amount by which the level of social surplus achieved at L falls short of the maximal social surplus.

Graphically represent in Figure 2.2 the level of social surplus and the deadweight loss associated with the profit-maximizing solution.

ANSWER. As in 2(e), $S(L^M) = \int_0^{L^M} [f'(L) - \gamma'(L)]dL$, which is *AREA* S^M in Figure 2.2. Accordingly, the deadweight loss is represented by *AREA DL*.

2(g). In addition to the assumptions introduced in 2(a)-2(f), postulate now and in what follows that the production function is linear, i. e., $f(L) = \frac{L}{c}$, where c is a positive constant. Express the firm's *profit percentage*, defined as the ratio PROFITS/COSTS, in terms of the elasticity of labor supply, and interpret. (Note that "COSTS" are defined by the outlays incurred by the firm in order to purchase inputs, and "PROFITS" as the difference between revenues and costs.)

ANSWER. We now have that $f'(L) = \frac{1}{c}$, which substituted into (2.2) above, yields:

$$\frac{\frac{1}{c} - w}{w} = \frac{1}{\varepsilon}. \qquad (2.3)$$

Multiplying the numerator at the denominator of the LHS of (2.3) by L yields $\frac{\frac{L}{c} - wL}{wL} = \frac{1}{\varepsilon}$. But $\frac{L}{c}$ is now the amount of output produced by the firm, equal to its revenue under the assumption that p = 1, and wL is the input cost for the firm. Thus, $\frac{L}{c} - wL$ is the profit of the firm. The equation

$$\frac{\frac{L}{c} - wL}{wL} = \frac{1}{\varepsilon} \qquad (2.4)$$

can then be read:

PROFIT PERCENTAGE = INVERSE SUPPLY-OF-LABOR ELASTICITY,

or:

PROFIT PERCENTAGE = DEGREE OF MONOPSONY.

2(h). In addition to the assumptions introduced in 2(a)-2(g), postulate now that the leisure valuation function φ is quadratic, i. e., $\varphi(x_1) = qx_1 - \frac{1}{2}k[x_1]^2$, with $q > k\omega_1 > 0$.

Provide a graphical illustration in a figure, labeled Figure 2.3, which specializes Figure 2.2 to the assumptions adopted here.

What is the relation between the surplus-maximizing amount of labor and the profit-maximizing amount of labor?

At the profit maximizing solution, what is the relation between the level of profits and the deadweight loss?

Answer. We first compute the disutility-of-labor function

$\gamma(L) = \varphi(\omega_1) - \varphi(\omega_1 - L)$

$$= q\omega_1 - \frac{1}{2}k[\omega_1]^2 - \left[q[\omega_1 - L] - \frac{1}{2}k[\omega_1 - L]^2\right]$$

$$= q\omega_1 - \frac{1}{2}k[\omega_1]^2 - q\omega_1 + qL + \frac{1}{2}k[[\omega_1]^2 - 2\omega_1 L + L^2]$$

$$= [q - k\omega_1]L + \frac{1}{2}kL^2$$

$$= gL + \frac{1}{2}kL^2,$$

where $g \equiv q - k\omega_1$, positive by assumption.

The marginal disutility of labor is then $g + kL$, which if equated to the marginal product $\frac{1}{c}$ yields, by solving for L the equation $g + kL = \frac{1}{c}$, the social-surplus-maximizing amount of labor $L^* = \frac{\frac{1}{c} - g}{k}$. Instead, the profit maximizing firm equates the marginal, and average, product $\frac{1}{c}$ to the marginal hiring cost $g + kL + kL = g + 2kL$, resulting in an amount of labor $L^M = \frac{L^*}{2}$, i. e., the profit-maximizing firm demands one-half the surplus-maximizing amount of labor. The wage paid by the firm is the marginal disutility of labor at $\frac{L^*}{2}$, i. e., $g + k\frac{L^*}{2}$. Hence the profit of the firm equals

$$\left[\frac{1}{c} - g - k\frac{L^*}{2}\right]\frac{L^*}{2} \qquad (2.5)$$

which can be written $\left[\frac{1}{c} - g - k\frac{\frac{1}{c} - g}{2k}\right]\frac{\frac{1}{c} - g}{2k} = \left[\frac{1}{2c} - \frac{g}{2}\right]\frac{1}{k}\left[\frac{1}{2c} - \frac{g}{2}\right] = \frac{1}{4k}\left[\frac{1}{c} - g\right]^2$.

Figure 2.3 illustrates.

The deadweight loss is given by the area of the triangle of base L^* - L^M = $= \frac{L^*}{2}$ and height $\left[\frac{1}{c} - g - k\frac{L^*}{2}\right]$ in Figure 2.3, i. e., $DL = \frac{1}{2}\left[\frac{1}{c} - g - k\frac{L^*}{2}\right]\frac{L^*}{2}$, or using (2.5)

$$DEADWEIGHT\ LOSS = \frac{PROFITS}{2}. \qquad (2.6)$$

Summarizing, in the special case where the production function is linear and the disutility of labor quadratic, the profit maximizing firm demands an amount of labor which is one-half the surplus-maximizing amount, and generates a deadweight loss equal to one half of its profits.

2(i). Under all the assumptions introduced in 2(a)-2(h), express the ratio

$$\frac{DEADWEIGHT\ LOSS}{COSTS}$$

in terms of the elasticity of labor supply.

Answer. Under the stated assumptions there is a simple relationship between the DEADWEIGHT-TO-COSTS ratio and the elasticity of labor supply. From (2.4), the ratio $\frac{PROFITS}{COSTS} = \frac{1}{\varepsilon}$, and from (2.6) *PROFITS* = 2×[*DEADWEIGHT LOSS*]. Hence,

$\frac{2DL}{COSTS} = \frac{1}{\varepsilon}$, i. e., $\frac{DL}{COSTS} = \frac{1}{2\varepsilon}$, i. e., the *Deadweight-loss*-to-*Costs* ratio is one half of the elasticity of the inverse supply-of-labor function, or:

$$\frac{DL}{COSTS} = \frac{DEGREE\ OF\ MONOPSONY}{2}.$$

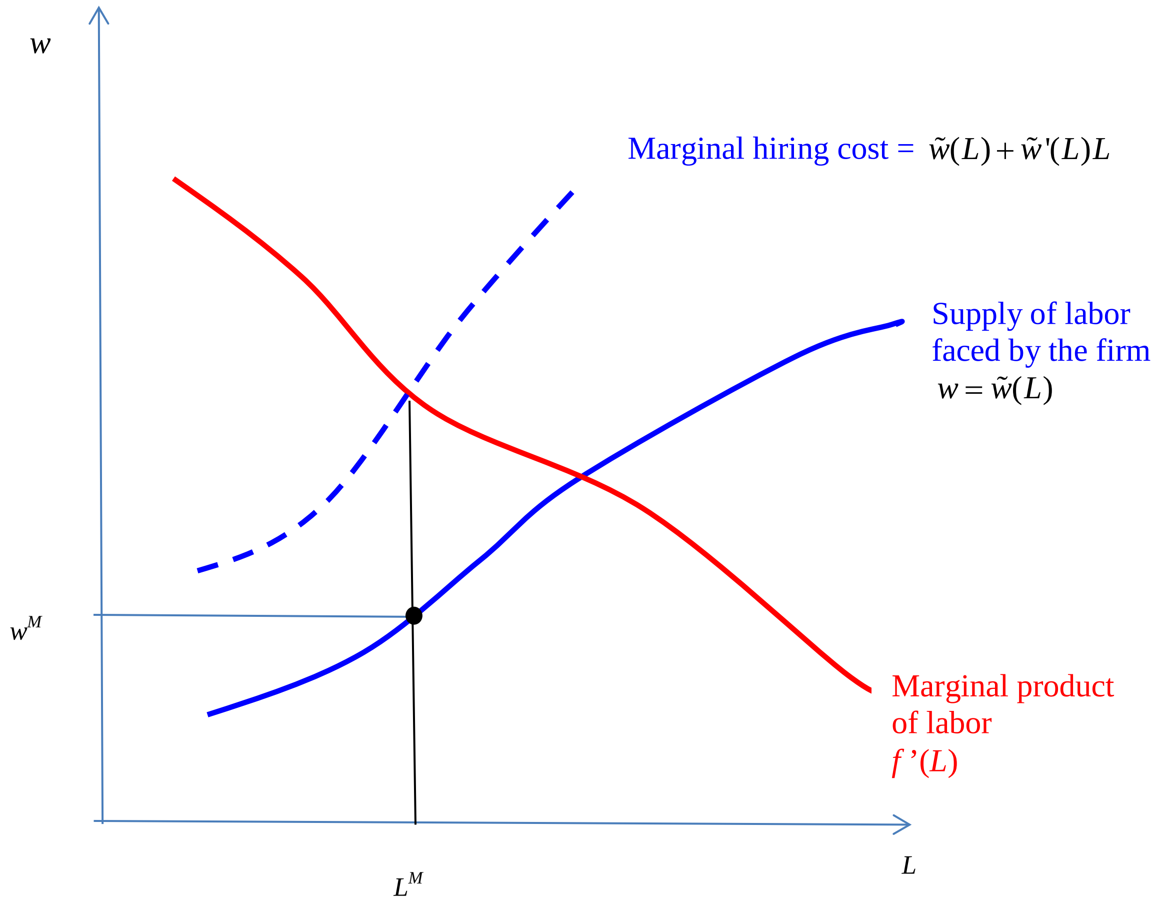


Figure 2.1
The profit-maximizing solution

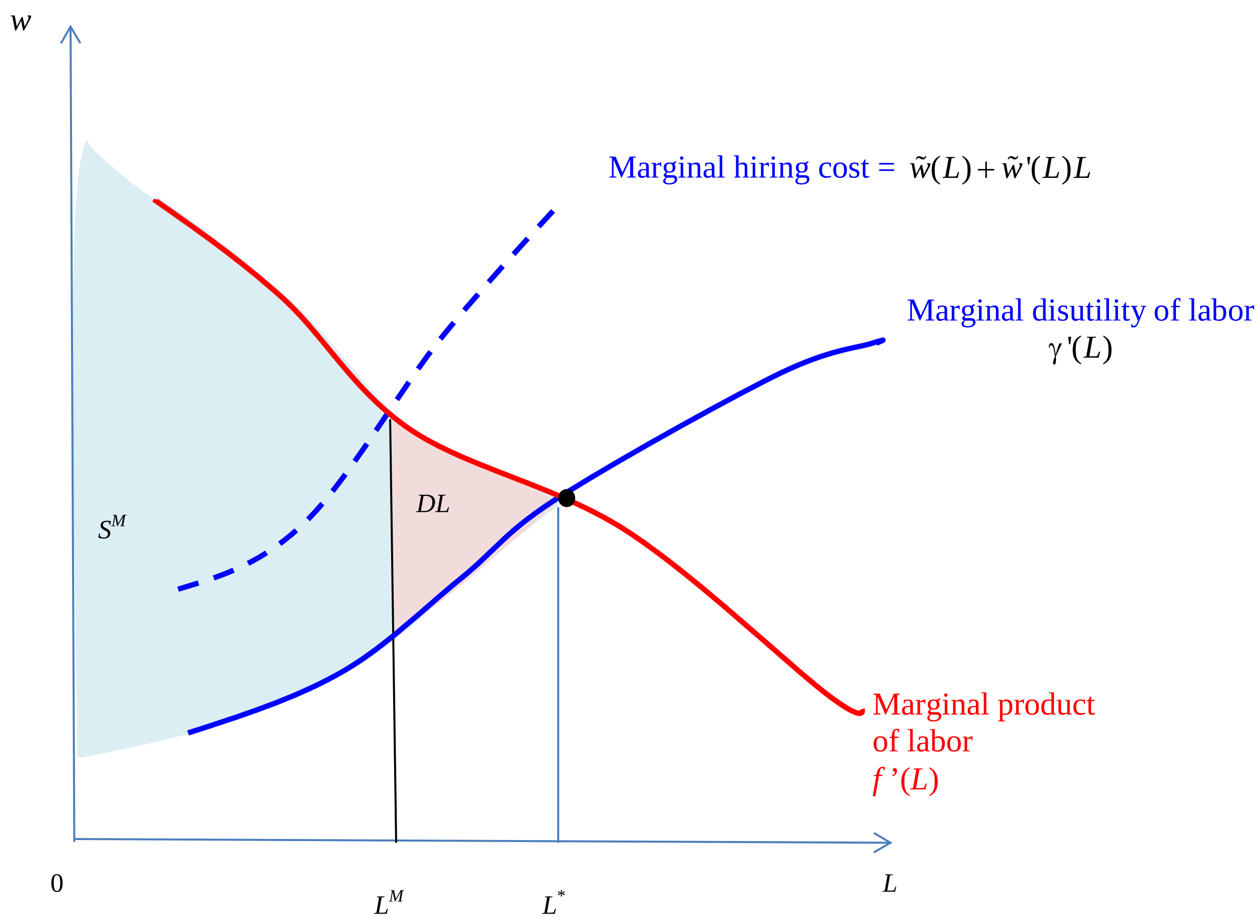


Figure 2.2
Social surplus and deadweight loss

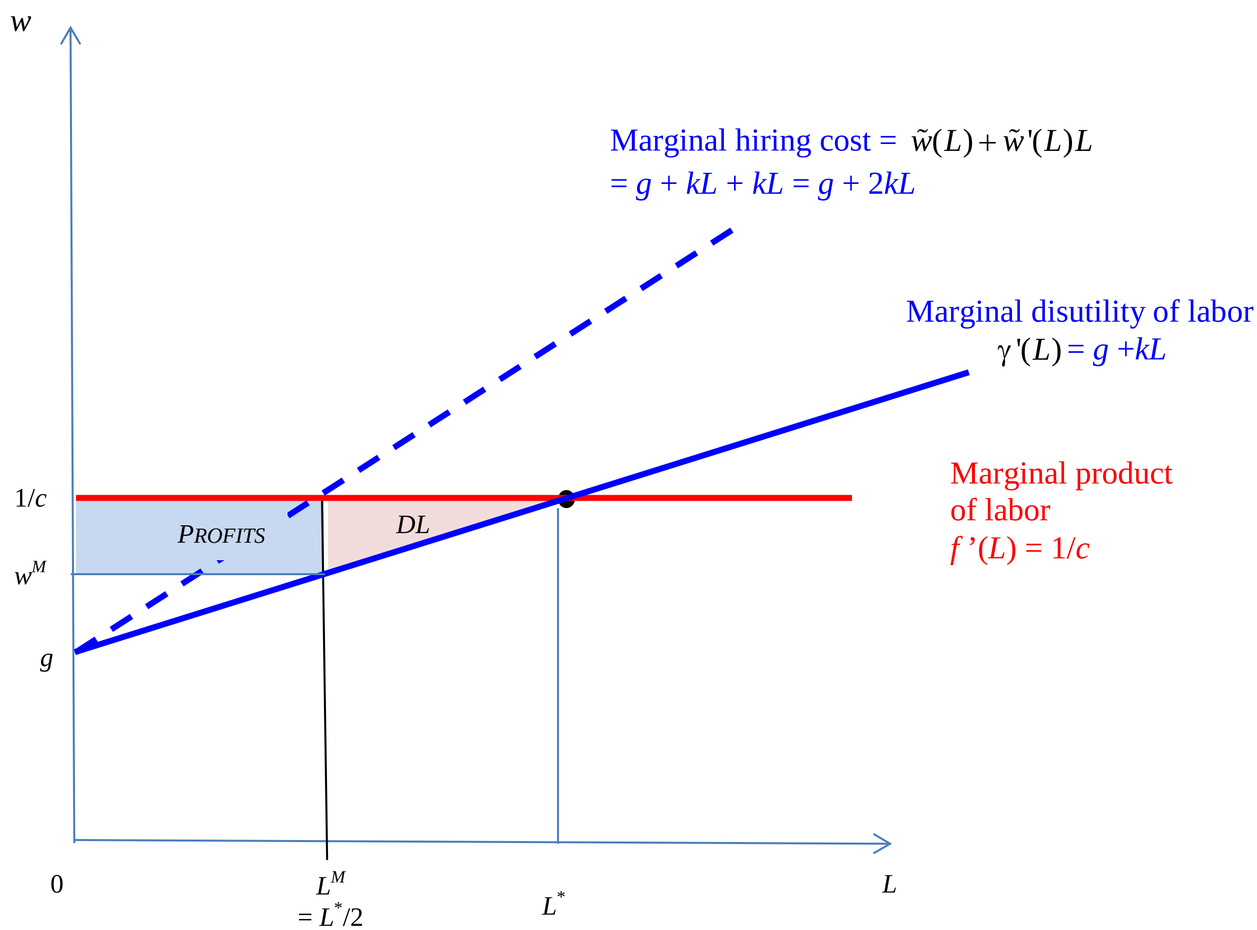


Figure 2.3
The linear-quadratic case

Answer Key Question 3

3. (a) $(\bar{x}, \bar{s}, \bar{r})$ is a competitive equilibrium is

- $\forall i = 1, \ldots, I \quad (\bar{x}^i, \bar{s}^i) \in \arg\max \left\{ u^i(x^i) \;\middle|\; \begin{array}{l} x^i_1 = w^i_1 - s^i \\ x^i_2 = w^i_2 + s^i(1+\bar{r}) \end{array}, \begin{array}{l} s^i \in \mathbb{R} \\ x^i \geq 0 \end{array} \right\}$

- $\Sigma \bar{s}^i = 0$.

Note that the budget set of agent i is the same as the budget set

$$\left\{ x^i \in \mathbb{R}^2_+ \;\middle|\; x^i_1 + \frac{1}{1+\bar{r}} x^i_2 = w^i_1 + \frac{1}{1+\bar{r}} w^i_2 \right\} \qquad (1)$$

or

$$\left\{ x^i \in \mathbb{R}^2_+ \;\middle|\; \bar{p}_1 x^i_1 + \bar{p}_2 x^i_2 = \bar{p}_1 w^i_1 + \bar{p}_2 w^i_2 \right\}$$

with $\bar{p}_1 = 1 \quad \bar{p}_2 = \frac{1}{1+\bar{r}}$

Also note that $\Sigma \bar{s}^i = 0$ is equivalent to $\Sigma \bar{x}^i = \Sigma w^i$. Thus $(\bar{x}, \bar{s}, \bar{r})$ is an equilibrium of the economy with borrowing and lending if and only if $(\bar{x}, \bar{p})$ with $\bar{p}_1 = 1$ $\bar{p}_2 = 1/_{1+\bar{r}}$ is an equilibrium of the 2-good economy $\mathcal{E}((u^i, w^i)^I_{i=1})$. By the First Theorem of Welfare economy, an equilibrium of the two-good economy $\mathcal{E}((u^i, w^i)^I_{i=1})$ is such that $\bar{x}$ is Pareto optimal. Thus the equilibria of the economy with borrowing and lending are Pareto optimal.

(b) $$MRS_{x_1^i, x_2^i}(x^i) = \frac{\partial u^i/\partial x_1^i (x^i)}{\partial u^i/\partial x_2^i (x^i)} = \frac{v_i'(x_1^i)}{\delta v_i'(x_2^i)}$$

Since the function v_i is concave, the marginal utility v_i' is decreasing. If $x_1^i < x_2^i$, $v_i'(x_1^i) > v_i'(x_2^i)$ and $MRS_{x_1^i, x_2^i}(x^i) > \frac{1}{\delta}$. The inequality is reversed if $x_1^i > x_2^i$.

In a Pareto optimal allocation x the marginal rates of substitution are equalized. Since $\sum_i w_1^i = \sum_i w_2^i$, if one agent i has a consumption stream such that $x_1^i < x_2^i$, some other agent k must have a consumption stream such that $x_1^k > x_2^k$. But the MRS of agent i is larger than $1/\delta$ and the MRS of agent k is smaller than $1/\delta$, which contradicts Pareto optimality. Thus all agents must have equal consumption in both periods.

(c) If $(\bar{x}, \bar{s}, \bar{r})$ is an equilibrium, $\bar{x}$ is Pareto optimal, and we know from question (b) that $\bar{x}_1^i = \bar{x}_2^i$, which implies that
$$\frac{v_i'(\bar{x}_1^i)}{\delta v_i'(\bar{x}_2^i)} = \frac{1}{\delta}$$

The first-order conditions for maximizing utility under the budget constraint (1) imply
$$\frac{v_i'(\bar{x}_1^i)}{\delta v_i'(\bar{x}_2^i)} = 1 + \bar{r}$$

Thus $\frac{1}{\delta} = 1+\bar{r}$, i.e. the equilibrium interest rate in an economy without growth is such that $\delta = \frac{1}{1+\bar{r}}$

(d) Suppose that $\sum_i \omega_2^i > \sum_i \omega_1^i$. If there is an agent who has in equilibrium a consumption $\bar{x}_2^i \leq \bar{x}_1^i$, it must be that there is an agent k who has a consumption stream such that $\bar{x}_2^k > \bar{x}_1^k$. But then the marginal rates of substitution cannot be equalized, which contradicts Pareto optimality.
Thus all agents must have consumption streams such that $\bar{x}_2^i > \bar{x}_1^i$, which implies that $\frac{v_i'(\bar{x}_1^i)}{\delta v_i'(\bar{x}_2^i)} > \frac{1}{\delta}$

Thus $1+\bar{r} > \frac{1}{\delta} \Leftrightarrow \delta > \frac{1}{1+\bar{r}}$

In an equilibrium of an economy with growth the interest rate is higher than in an economy without growth: agents must be induced by the price of borrowing and lending to want to consume less at date 1 than at date 2: a high interest rate prevents agents to want to borrow too much against date 2 wealth.

Answer Key - Question 4

(a) $\left((x^i)_{i=1}^{I}, y\right) \gg 0$

$$\sum_{i=1}^{I} \mu_i \, u^i(x^i; y)$$

s.t $\quad \Sigma x^i + ky \leq \Sigma w^i \qquad \lambda$

for some $\mu_i \gg 0$ (strictly positive consumption for all agents).

FOCs: $\quad \mu_i \dfrac{\partial u^i}{\partial x^i}(\bar{x}_i, \bar{y}) = \lambda$

$$\Sigma \mu_i \frac{\partial u^i}{\partial y}(\bar{x}_i, \bar{y}) = \lambda k$$

for some $\lambda > 0$, which is equivalent to the Lindahl-Samuelson condition

$$\sum_{i=1}^{I} \frac{\frac{\partial u^i}{\partial y}(\bar{x}^i; \bar{y})}{\frac{\partial u^i}{\partial x^i}(\bar{x}^i; \bar{y})} = k$$

(b) Lindahl equilibrium $\left[\left((\bar{x}^i)_{i=1}^{I}, \bar{y}\right), \left(1, (\bar{p}^i)_{i=1}^{I}\right)\right]$ such that

(i) $\bar{y} \in \arg\max_{y \geq 0} \left(\sum_{i=1}^{I} \bar{p}_i\right) y - ky$

(ii) $(\bar{x}^i; \bar{y}) \in \arg\max_{(x_i, y) \in \mathbb{R}^2_+} \left\{ u^i(x^i, y) \mid x^i + \bar{p}^i y \leq w^i \right\}$

(no profit to distribute because of constant returns)

(iii) $\Sigma \bar{x}^i + k\bar{y} = \Sigma w^i$

Answer Key - Question 4

(a) $\left((x^i)_{i=1}^{I}, y\right) \gg 0$

$$\sum_{i=1}^{I} \mu_i \, u^i(x^i, y)$$

s.t $\quad \sum x^i + k y \leq \sum w^i \qquad \lambda$

for some $\mu_i \gg 0$ (strictly positive consumption for all agents).

FOCs: $\quad \mu_i \dfrac{\partial u^i}{\partial x^i}(\bar{x}^i, \bar{y}) = \lambda$

$$\sum \mu_i \frac{\partial u^i}{\partial y}(\bar{x}^i, \bar{y}) = \lambda k$$

for some $\lambda > 0$, which is equivalent to the Lindahl-Samuelson condition

$$\sum_{i=1}^{I} \frac{\frac{\partial u^i}{\partial y}(\bar{x}^i, \bar{y})}{\frac{\partial u^i}{\partial x^i}(\bar{x}^i, \bar{y})} = k$$

(b) Lindahl equilibrium $\left[\left((\bar{x}^i)_{i=1}^{I}, \bar{y}\right), \left(1, (\bar{p}^i)_{i=1}^{I}\right)\right]$ such that

(i) $\bar{y} \in \arg\max_{y \geq 0} \left(\sum_{i=1}^{I} \bar{p}_i\right) y - k y$

(ii) $(\bar{x}^i, \bar{y}) \in \arg\max_{(x_i, y) \in \mathbb{R}^2_+} \left\{ u^i(x^i, y) \,\middle|\, x^i + \bar{p}^i y \leq w^i \right\}$

(no profit to distribute because of constant returns)

(iii) $\sum \bar{x}^i + k \bar{y} = \sum w^i$

(c) Under the assumptions on preferences it is sufficient to show that the FOCs for Pareto optimality are satisfied at a Lindahl equilibrium

FOCs for profit max: $\sum \bar{p}_i = k$

FOCs for u^i max $\quad \frac{\partial u^i}{\partial x^i}(\bar{x}^i, \bar{y}) = \lambda^i, \quad \frac{\partial u^i}{\partial y}(\bar{x}^i, \bar{y}) = \lambda^i \bar{p}^i$

$$\Rightarrow \bar{p}^i = \frac{\frac{\partial u^i}{\partial y}(\bar{x}^i, \bar{y})}{\frac{\partial u^i}{\partial x^i}(\bar{x}^i, \bar{y})} \Rightarrow \text{Samuelson condition.}$$

(d) subscription equilibrium $\left((\tilde{z}^i)_{i=1}^{I}, \tilde{y}\right)$ such that

$$\text{(I)} \quad \tilde{z}^i \in \arg\max_{0 \le z^i \le w^i} u^i\left(w^i - z^i, \frac{z^i + \tilde{Z}^{-i}}{k}\right) \quad \left(\tilde{Z}^{-i} = \sum_{i' \ne i} \tilde{z}^{i'}\right), \forall i$$

$$\text{(II)} \quad k\tilde{y} = \sum_{i=1}^{I} \tilde{z}^i$$

FOCs at interior equilibrium

$$-\frac{\partial u^i}{\partial x_i}(\tilde{x}^i, \tilde{y}) + \frac{1}{k}\frac{\partial u^i}{\partial y}(\tilde{x}^i, \tilde{y}) = 0$$

$$\Leftrightarrow \frac{\frac{\partial u^i}{\partial y}(\tilde{x}^i, \tilde{y})}{\frac{\partial u^i}{\partial x^i}(\tilde{x}^i, \tilde{y})} = k \qquad \forall i = 1, \ldots I$$

(e) Since the Lindahl equilibrium is symmetric $\bar{p}_i$ is the same for all agents and given $b(i)$, $\bar{p}_i = \frac{k}{I}$.

$(\bar{x}, \bar{y})$ maximizes $u(x,y)$ on the budget line

$$x + \frac{k}{I} y = w$$

Since the subscription equilibrium is symmetric

$$\tilde{z}^i = \check{z} = w - \tilde{x} \quad \text{for all } i$$

and

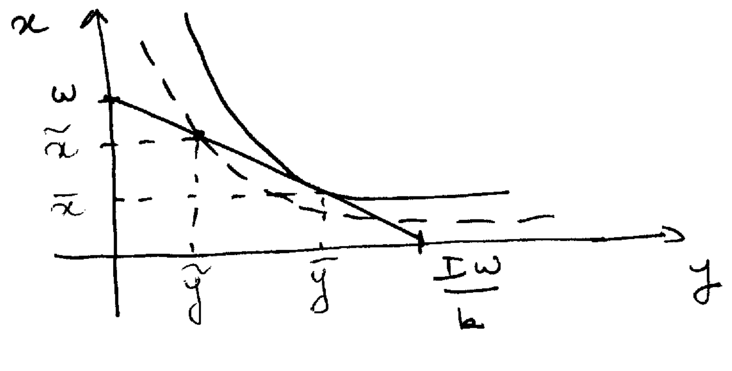
$$I(w - \tilde{x}) = k\tilde{y} \iff I\tilde{x} + k\tilde{y} = Iw$$

$$\iff \tilde{x} + \frac{k}{I}\tilde{y} = w$$

Thus both $(\bar{x}, \bar{y})$ and $(\tilde{x}, \tilde{y})$ are on the budget line $x + \frac{k}{I} y = w$. The indifference curve through $(\bar{x}, \bar{y})$ is tangent to the budget line, while the marginal rate of substitution

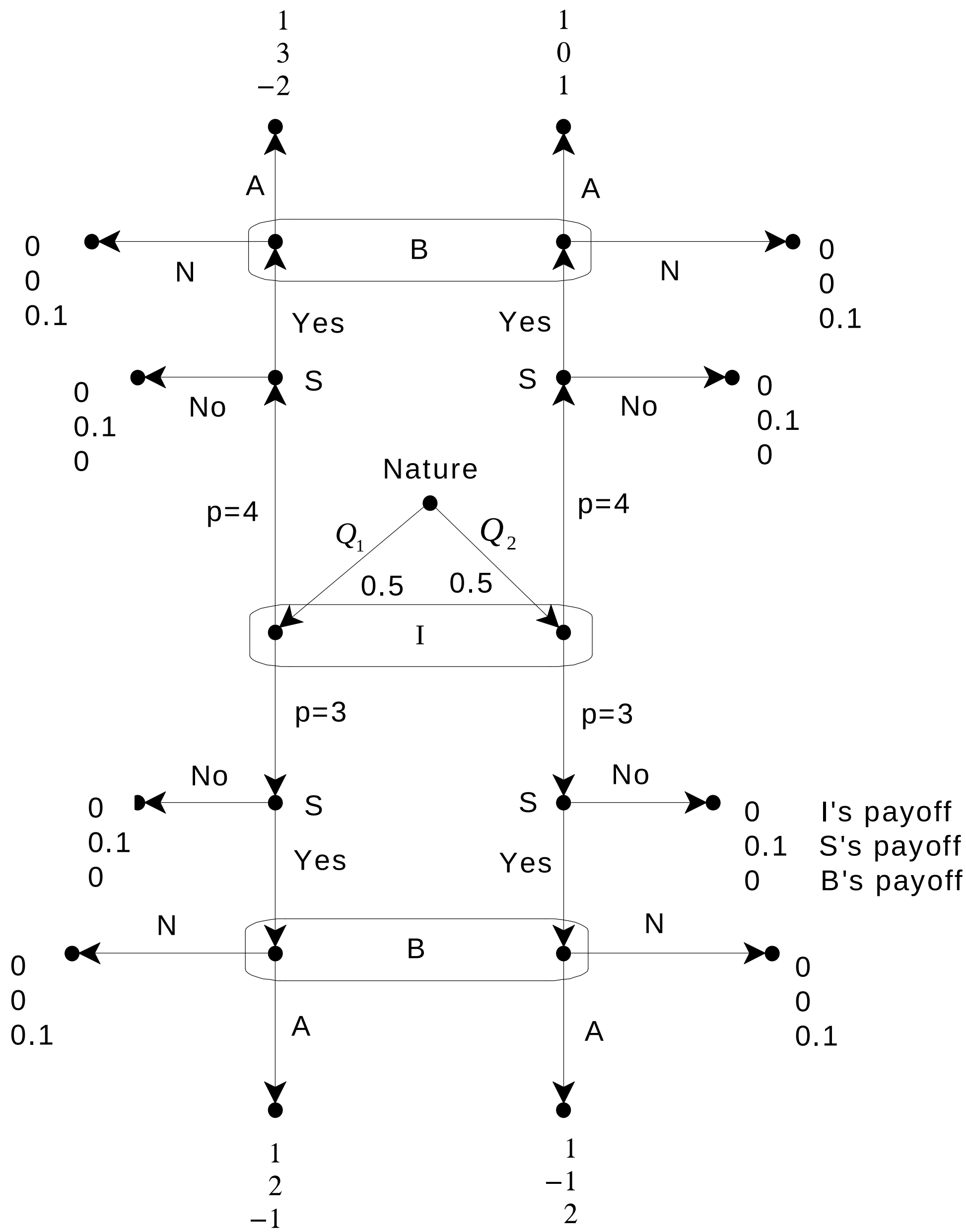
$$\frac{\frac{\partial u}{\partial y}(\tilde{x}, \tilde{y})}{\frac{\partial u}{\partial x}(\tilde{x}, \tilde{y})} = k > \frac{k}{I}$$

In axes (Oy, Ox) the indifference curve is steeper at $(\tilde{x}, \tilde{y})$ than at $(\bar{x}, \bar{y})$, which implies that $\tilde{y} < \bar{y}$ and $\tilde{x} > \bar{x}$



Answer keys for Question 5, Micro Prelim August 2013

(a) The game is shown below:



(b) At S's nodes that follow Nature's choice of Q_2, No gives a payoff of 0.1, while Yes a payoff of at most 0. Thus sequential rationality requires that S choose No at those two nodes. Now consider S's node following Nature's choice of Q_1 and I's choice of $p = 3$. Suppose that S's strategy involves choosing Yes at that node. Then at B's information set that follows, by Bayes' rule B must assign probability 1 to the left node (that is, to Q_1), in which case sequential rationality dictates the choice of Not Accept. But then Yes cannot be optimal for S at the previous node. Hence it cannot be that S's equilibrium strategy involves Yes at S's node following Q_1 and $p = 3$. Similar reasoning applies to the node following Q_1 and $p = 4$. Hence S's strategy must be to say No at each of her 4 nodes. Given this, I's payoff is 0 whatever price he suggests. Finally, B's strategy must be to say Not Accept at both information sets (otherwise

S would want to say Yes if her car's quality is Q_1, but then this would lead to a contradiction, as explained above). In order for Not Accept to be optimal, B must assign sufficiently high probability to the left node of that information set. Denote that probability by *r*; then it must be that $0.1 \geq -r + 2(1-r)$, that is, $r \geq \frac{1.9}{3}$. Since B's information sets are not reached, any beliefs are allowed there by the notion of WSE. Thus we have the following sets of pure-strategy WSE (r_1 denotes the probability assigned to the left node of B's *bottom* information set and r_2 denotes the probability assigned to the left node of B's *top* information set): First set:

$\sigma = \left(\underbrace{p=3}_{I's\,strategy}, \underbrace{No, No, No, No}_{S's\ strategy}, \underbrace{NotAccept, NotAccept}_{B's\ strategy} \right)$, $\mu = (r_1, r_2)$ with $r_i \geq \frac{1.9}{3}$ (i = 1,2). Second set : $\sigma = \left(\underbrace{p=4}_{I's\,strategy}, \underbrace{No, No, No, No}_{S's\ strategy}, \underbrace{NotAccept, NotAccept}_{B's\ strategy} \right)$ and μ as before.

(c) In order for I's expected payoff to be 1, it must be that the type Q_3 of S says Yes (in which case also the other types say Yes). A necessary condition for this is that I suggests *p* = 5 (if *p* = 4, then the Q_3 type by saying Yes will get $p - U(Q_3) = 4 - 4 = 0$ if B says Accept and 0 if B says Not Accept, while by saying No she gets 0.1; similarly, if *p* < 4). If *p* = 5 and all the types of S say Yes, then B's expected payoff from Accept is $2q_1 + 6q_2 + 8(1 - q_1 - q_2) - 5$. This needs to be greater than 0.1, which is the payoff from saying Not Accept. If B says Accept then the payoff of the Q_3 type of S is $5 - U(Q_4) = 5 - 4 = 1 > 0.1$ and thus by saying Yes she is better off than by saying No (and for the other types the payoff from saying Yes is even greater). Finally, suggesting *p* = 5 gives I a payoff of 1, while suggesting a lower price gives him a lower payoff. Thus a necessary and sufficient condition for

$$\left(\underbrace{p=5}_{I's\ strategy}, \underbrace{Yes\,if\ Q_1\ and\ p=5, Yes\,if\ Q_2\ and\ p=5, Yes\,if\ Q_3\ and\ p=5}_{part\,of\ S's\ strategy}, \underbrace{Accept\,if\ \ p=5\,and\,Yes}_{part\,of\ B's\ strategy} \right)$$

to be part of a WSE is $2q_1 + 6q_2 + 8(1 - q_1 - q_2) - 5 > 0.1$, that is, $6q_1 + 2q_2 < 2.9$. For example, $q_1 = q_2 = \frac{1}{6}$ satisfy this inequality.

(d) I's expected payoff is q_1 if and only if he suggests a price that induces both the Q_2 and the Q_3 types of S to say No and the Q_1 type to say Yes. If it is optimal for the Q_2 type to say No, then it is also optimal for the Q_3 type to say No. The Q_2 type will say No if $p \leq 3$ and the Q_1 type will say Yes if $p > \frac{1}{2}$ (that is, at any price in P). Thus we are looking for a WSE where I suggests either a price of 1 or of 3. In either case only the Q_1 type will say Yes and thus, by Bayes' rule, B's beliefs must assign probability 1 to the Q_1 type, in which case his expected payoff from saying Accept is 2 − *p*, which is greater than 0.1 only if *p* = 1. Hence we are looking for a WSE where the relevant parts of the strategies are

$\left(\underbrace{p=1}_{I's\ strategy}, \underbrace{Yes\,if\ Q_1\ and\ p=1, No\,if\ Q_2, No\,if\ Q_3}_{S's\ strategy}, \underbrace{Accept\,if\ \ p=1}_{part\,of\ B's\ strategy} \right)$. This gives I an expected payoff of $1q_1 + 0q_2 + 0q_3 = q_1$. In order for this to be part of a WSE, it must be the case that I cannot get a higher expected payoff from suggesting a higher price. We have already shown that p = 3 gives I a lower payoff. From part (c) we know that if $6q_1 + 2q_2 \geq 2.9$ then p = 5 cannot give a payoff of 1. The other possible expected payoff for I would be $q_1 + q_2$, which can only be achieved by suggesting p = 4 if the Q_2 type says Yes (so that also the Q_1 type says Yes, while we know that the Q_3 type says No) and B says Accept. Then, using Bayes' rule, B's expected payoff from Accept is $\frac{q_1}{q_1+q_2}2 + \frac{q_2}{q_1+q_2}6 - 4$ and this needs to be greater than 0.1 for B to want to say Accept. Thus a necessary and sufficient condition for B to say Not Accept when p = 4 is $\frac{q_1}{q_1+q_2}2 + \frac{q_2}{q_1+q_2}6 - 4 \leq 0.1$, that is, $1.9q_2 \leq 2.1q_1$. Thus the necessary and sufficient conditions that we are seeking are $6q_1 + 2q_2 < 2.9$ and $1.9q_2 \leq 2.1q_1$. For example, $q_1 = q_2 = \frac{1}{6}$ satisfy these two inequalities.

(e) Pareto efficiency requires that the car be sold with probability 1, since B values each type of car more than S does; furthermore, if the car is sold, I gets the largest possible payoff, namely 1. Thus we are looking for prices that, if suggested by I, induce the Q_3 type of S (and thus all the types) to say Yes and subsequently B to Accept. Anticipating B to accept, the Q_3 type of S will say Yes if $p - 4 \geq 0.1$, that is, is if $p \geq 4.1$. If all the types say Yes, then B's expected payoff from saying Accept is $2\frac{1}{2} + 6\frac{1}{6} + 8\frac{1}{3} - p = \frac{14}{3} - p$ and this needs to be greater than or equal to 0.1 in order for Accept to be at least as good as Not Accept. Thus we need, $\frac{14}{3} - p \geq 0.1$ or $p \leq \frac{137}{30} = 4.57$. Thus the required range for p is [4.1 , 4.57].