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Department of Economics
Microeconomics

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Reading Time: 20 minutes

PRELIMINARY EXAMINATION FOR THE Ph.D. DEGREE

Please answer any four questions (out of five)

Question 1.

Throughout this question, we consider two alternative (L-1)-input production functions (where, as usual, the jth argument is interpreted as the amount of input j used):

$$f^C : \Re^{L-1}_{++} \to \Re : f^C(z_1^C,\ldots,z_{L-1}^C) = c\prod_{j=1}^{L-1}\left[z_j^C\right]^{\alpha_j},$$

where the parameters c, $\alpha_1,\ldots,\alpha_{L-1}$ are strictly positive and satisfy $\sum_{j=1}^{L-1}\alpha_j = 1$;

$$f^A : \Re^{L-1}_{++} \to \Re : f^A(z_1^A,\ldots,z_{L-1}^A) = a\left[\sum_{j=1}^{L-1}\gamma_j\left[z_j^A\right]^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}},$$

where the parameters a, σ and γ_j , $j = 1,\ldots, L-1$, are strictly positive, with $\sigma \in (0,1)\cup(1,\infty)$.

1(a). What is the interpretation of the parameter σ in the function f^A? In which sense are the functions f^A and f^C related?

1(b). Consider the following cost minimization problem for a generic production function f^J. Interpret its solution function and its value function in words. Does the interpretation entail perfect competition (price taking) in all markets?

Cost Minimization Problem. Given an input price vector $w := (w_1,\ldots, w_{L-1}) >> 0$, and an amount of output $q > 0$, choose $(z_1,\ldots, z_{L-1}) \in \Re^{L-1}_{+}$ in order to minimize $\sum_{j=1}^{L-1} w_j z_j$ subject to $f^J(z_1,\ldots, z_{L-1}) \geq q$.

Solve the cost minimization problem separately for the production functions f^C and f^A (i. e., f^J for $J = C$ and for $J = A$). You may assume that the solution vector is strictly positive. For the production function f^C (resp. f^A), denote the solution function by $(\tilde{z}_1^C(w,q),\ldots,\tilde{z}_{L-1}^C(w,q))$ (resp. $(\tilde{z}_1^A(w,q),\ldots,\tilde{z}_{L-1}^A(w,q))$, and the value function by $c^C(w,q)$ (resp. $c^A(w,q)$.)

For the remainder, assume that all markets are perfectly competitive (price taking) and that the input and output amounts are all positive. We name input 1 "capital." Define the *share of capital income in total income* (or "factor share of capital") *according to the production function* f^J, $J = C$ or A, as the quotient $\dfrac{w_1\tilde{z}_1^J(w,q)}{pq}$, and denote it by the symbol ψ^J .

1(c). For the production function f^J, $J = C, A$, compute the share of capital income in total income as a function of the input price vector w, i. e., as an expression $\tilde{\psi}^J(w)$ involving only w and parameters.

1(d). Define the *capital/output ratio according to the production function* f^J, $J = C, A$, as $\dfrac{\tilde{z}_1^J(w;q)}{q}$, and denote it by the symbol θ^J. For $J = C, A$, compute the share of capital income in total income as a function of θ^J, i. e., as an expression $\hat{\psi}^J(\theta^J)$ involving only θ^J and parameters.

1(e). For $J = A, C$, what happens to the share of capital income in total income as the price w_1 of capital increases? Justify and comment.

1(f). Address this question and question 1(g) below by using your answer to 1(d) above. For $J = A, C$, what happens to the share of capital income in total income as the capital-output ratio θ^J increases? Justify and explain in detail.

1(g). In macroeconomic growth theory the functions f^C or f^A are often interpreted as depicting the production possibilities of the economy, with output q interpreted as GDP (or NDP).
It is sometimes postulated that the savings rate s and the growth rate g are exogenous, and that the capital-output ratio is a function $\widehat{\theta}(s, g)$ of s and g with $\dfrac{\partial\widehat{\theta}}{\partial s} > 0$ and $\dfrac{\partial\widehat{\theta}}{\partial g} < 0$. Under these assumptions, for $J = C, A$, how does the share ψ^J of capital income in total income vary with s? With g? Explain and discuss.

Question 2

1. Consider an exchange economy $\mathcal{E}$ with L goods and I agents. Agent i, $i = 1, \ldots, I$ has an initial endowment $\omega^i \in \mathbb{R}^L_+$ and preferences represented by a utility function $u^i : \mathbb{R}^L_+ \to \mathbb{R}$ which is continuous, strictly quasi concave and strongly monotone. We assume $\sum_{i=1}^I \omega^i \gg 0$ and that the excess demand $Z(p)$ of this economy satisfies the Gross Substitute (GS) property: if the price of a good (let us say good k) increases, the excess demand in all the other goods increases (i.e. agents substitute the other goods for the good whose price has increased). Formally the excess demand Z is such that if p and p' are two price vectors in $\mathbb{R}^L_{++}$ such that $p'_k > p_k$ and $p'_\ell = p_\ell$ for $\ell \neq k$, then $Z_\ell(p') > Z_\ell(p)$ for any $\ell \neq k$. We want to prove that an economy with the GS property has a unique competitive equilibrium.

(a) Suppose that p and $\tilde{p}$ are 2 price vectors in $\mathbb{R}^L_{++}$. Show that there exist p' collinear to $\tilde{p}$ such that $p'_\ell = p_\ell$ for at least one good ℓ and $p'_k \geq p_k$ for $k \neq \ell$. [Hint: Rank the the ratios of prices $\frac{\tilde{p}_k}{p_k}$.]

(b) Suppose that p and p' are two equilibrium price vectors for the economy $\mathcal{E}$. Deduce from (a) that we can assume without loss of generality that $p'_\ell = p_\ell$ for at least one good ℓ and $p'_k \geq p_k$ for $k \neq \ell$.

(c) Let p and p' be two equilibrium price vectors satisfying the property in (b). Show that if $p \neq p'$, then $Z_\ell(p') > Z_\ell(p)$ (Hint: Note that you can go from p to p' in a series of steps).

(d) Deduce from the previous questions that the economy $\mathcal{E}$ can not have more that one competitive equilibrium.

(e) An important application: Consider an exchange economy in which all agents have Cobb-Douglas preferences

$$u^i(x^i) = (x^i_1)^{\alpha^i_1} \ldots (x^i_L)^{\alpha^i_L}, \qquad \alpha^i_1 > 0, \ldots, \alpha^i_L > 0, \quad \alpha^i_1 + \ldots + \alpha^i_L = 1$$

Show that the economy satisfies the GS property and thus has a unique equilibrium.

Question 3

Consider a community of I agents in which agent' i's utility function is given by

$$u_i(x_i, h_i, y) = \alpha \ln(x_i) + \beta \ln(h_i) + \gamma_i \ln(y), \qquad \alpha + \beta = 1, \qquad \gamma^i < \beta, \qquad i = 1, \dots I$$

where x_i denote the amount of the consumption good, h_i the number of units of living space and y the services provided to the community. The coefficients $\alpha > 0$ and $\beta > 0$ are the same for all agents but the coefficients $\gamma_i = \gamma(\omega_i) > 0$ are different since typically rich agents need public services less than poorer agents. Each agent has an endowment $(\omega_i, \bar{h}_i)$ in consumption good and housing. City services can be produced from the consumption good with constant returns, one unit of consumption good producing one unit of services.

(a) Find the Pareto optimal allocations for this economy (you can parameterize them by the agents' weights in the social welfare function which is maximized).

(b) Cities are not allowed to tax income but they can tax goods and services sold in their jurisdictions. The city council thus decides to finance the public services by taxing housing at rate τ, i.e. when an inhabitant buys h units of living space at price p, he/she pays $p\tau h$ in taxes to the city. The tax rate will be chosen by majority voting.

  (i) Normalize the price of consumption good to 1 and solve the utility maximum problem of agent i who takes the price of housing p and the tax rate as given.

  (ii) Define agent i 's indirect utility function $V_i(\tau)$ for tax rates, under the assumptions that agent i takes the price p of housing as given and knows the total supply of houses in the community. Show that V^i is single peaked.

  (iii) Find the tax rate τ^* chosen by majority voting.

  (iv) Calculate the competitive equilibrium price p^* of living space in the city, the consumption bundle (x^{i*}, h^{i*}) of each agent in the community and the amount of public services y^* produced. [It simplifies the calculations and expressions at this stage to keep the notation τ^* without substituting the value found in (ii).]

(c) Show that there exist parameters $((\gamma^i, \omega^i, \bar{h}^i)$ such that the Samuelson condition is satisfied at $\left((x^{i*}, h^{i*})_{i=1}^{I}, y^*\right)$ and the allocation is Pareto optimal. Interpret the condition found as a comparison between median and average utility for public services.

Question 4.

A seller (S) owns an item that a buyer (B) would like to purchase. The seller's reservation price is s (that is, she is willing to sell if and only if the price paid by the buyer is at least s) and the buyer's reservation price is b (that is, he is willing to buy if and only if the price is less than or equal to b). It is common knowledge between the two that (1) both b and s belong to the set $\{1,2,\ldots,n\}$, (2) the buyer knows the value of b and the seller knows the value of s, (3) both the buyer and the seller attach equal probability to all the possibilities among which they are uncertain.

Buyer and Seller play the following game. First the buyer makes an offer of a price $p \in \{1,\ldots,n\}$ to the seller. If $p = n$ the game ends and the object is exchanged for \$$p$. If $p < n$ then the seller either accepts (in which case the game ends and the object is exchanged for \$$p$) or makes a counter-offer of $p' > p$, in which case either the buyer accepts (and the game ends and the object is exchanged for \$ p') or the buyer rejects, in which case the game ends without an exchange. Payoffs are as follows

$$\pi_{seller} = \begin{cases} 0 & \text{if there is no exchange} \\ x - s & \text{if exchange takes place at price \$}x \end{cases} \quad \text{and}$$

$$\pi_{buyer} = \begin{cases} 0 & \text{if there is no exchange} \\ b - p & \text{if exchange takes place at price \$}p \text{ (the initial offer)} \\ b - p' - \varepsilon & \text{if exchange takes place at price \$}p' \text{ (the counter-offer)} \end{cases}$$

where $\varepsilon > 0$ is a measure of the buyer's "hurt feelings" for seeing his initial offer rejected. These are von Neumann-Morgenstern payoffs.

4(a) For the case where $n = 2$ use states and information partitions to represent this situation of incomplete information. Make sure that (1) you specify the players' beliefs and (2) you associate with every state the game that is played in that state, with payoffs appropriate to that state.

4(b) For the case where $n = 2$ apply the Harsanyi transformation to transform the situation represented in part (a) into a game.

4(c) Find all the pure-strategy weak sequential equilibria of the game of part (b) and calculate the corresponding payoffs for both players.

4(d) Now consider the case where $n = 100$. Describe a pure-strategy Nash equilibrium of the game that is obtained by applying the Harsanyi transformation to the corresponding situation of incomplete information.

4(e) Now go back to the game of part (b). Suppose that we add an extra option for the buyer at (and only at) the beginning: besides offering a price p, the buyer can "pass", in which case the game ends with a payoff of 0 for both players. Explain whether adding this extra option increases or shrinks the set of pure-strategy weak sequential equilibria (relative to the set you found in part (c)).

Question 5.

Consider an economy with one good, three individuals (1, 2 and 3) and four states (B_1B_2, B_1G_2, G_1B_2 and G_1G_2: the interpretation of B_iG_j is that the state is Bad for Individual i and Good for Individual j, with $i, j \in \{1,2\}$, $i \neq j$ and similarly for the other three states). Individuals 1 and 2 have the same von Neumann-Morgenstern utility function $U : \mathbb{R}^+ \to \mathbb{R}$ (whose argument is the consumption of the only good in the economy) with $U' > 0$ and $U'' < 0$, while Individual 3 is risk neutral. We assume that (1) the endowment of Individual $i \in \{1,2\}$ is ω_B^i in both states B_iB_j and B_iG_j and is ω_G^i in both states G_iB_j and G_iG_j (that is, the endowment of each individual depends only on whether the state is Bad or Good for her) and (2) the two individuals have the same endowment in the sense that $\omega_B^1 = \omega_B^2 = \omega_B$ and $\omega_G^1 = \omega_G^2 = \omega_G$, with $0 < \omega_B < \omega_G$. Individual 3's endowment is w in every state, with $w > 2(\omega_G - \omega_B)$. Individuals 1 and 2 have different characteristics which affect the probability that a bad state will occur for them. For each $i \in \{1,2\}$, let p_i^B be the probability that the bad state will occur for Individual i and $\left(1 - p_i^B\right)$ be the probability that the good state will occur for Individual i. The states are independent, in the sense that B_1B_2 will occur with probability $p_1^B p_2^B$; B_1G_2 will occur with probability $p_1^B\left(1 - p_2^B\right)$, etc. We assume that $0 < p_1^B < p_2^B < 1$. Individual 3 proposes a trade $x_i = \left(x_i^B, x_i^G\right) \in \mathbb{R}^2$ to individual $i \in \{1,2\}$ with $0 \le x_i^B \le \omega_G - \omega_B$ and $x_i^G \le 0$, where x_i^B is a payment from Individual 3 to Individual i if the Bad state for i realizes and x_i^G is a payment from Individual i to Individual 3 if the Good state for i realizes . Let $\omega = (\omega_B, \omega_G)$; then Individual i will end up consuming $\omega + x_i$ if she accepts the trade offered by 3 and ω if she does not accept the trade.

5(a) Suppose that Individual 3 can tell who Individual 1 is. Give the details of a trade that 3 would be willing to offer to 1 and 1 would be better off accepting than rejecting.

For (b)-(e) below assume that Individual 3 has all the bargaining power and that Individuals 1 and 2 – if indifferent between accepting and not accepting – will accept 3's offer and, in general, will break indifference in favor of Individual 3.

5(b) Benchmark case: Individual 3 can tell who is Individual 1 and who is Individual 2 (he can tell them apart). What trade(s) will Individual 3 offer to 1 and 2 in this case? Prove your claim and write down the expected utility of each individual.

For (c) and (d) below assume that Individual 3 **cannot** tell 1 and 2 apart (that is, when dealing with somebody, he does not know if he is dealing with 1 or with 2). We will consider several cases.

5(c) **Case 1:** Individual 3 is required by law to offer only one type of trade, that is, he must offer the same deal to anybody who is willing to trade with him (but there is no requirement that it be a deal which is attractive to everybody). What trade will Individual 3 offer? Justify your answer and explain if both 1 and 2 trade with 3 or only one of them.

5(d) **Case 2:** Individual 3 is allowed to offer a menu of trades (and then Individuals 1 and 2 will pick whichever they like better). Write down the maximization problem for Individual 3 and illustrate the optimal solution graphically in a two-dimensional diagram, where consumption in the bad state is measured on the horizontal axis.

5(e) Now consider the following special case: $\begin{pmatrix} \text{state} & B_1B_2 & B_1G_2 & G_1B_2 & G_1G_2 \\ \text{probability} & \frac{1}{128} & \frac{7}{128} & \frac{15}{128} & \frac{105}{128} \end{pmatrix}$,

$U(x) = \sqrt{x}$, $\omega_B = 1{,}600$, $\omega_G = 3{,}600$ and $w = 4{,}500$. **(e.1)** Calculate the solution for case (b) and compute the expected utility of each individual. **(e.2)** Write down the maximization problem for part (d) [give precise values for the range of the choice variable(s)].