

PRELIMINARY EXAMINATION FOR THE Ph.D. DEGREE
Please answer four questions (out of five)

Question 1.

Throughout this question we assume that all the optimization problems considered have unique solutions which are interior and characterized by the first-order equalities. We also assume that all decision makers take all prices as given, and that the Lagrange multipliers are nonzero.

1.1. This part covers standard price-taking production theory, and we minimally adapt the notation used in class and in Chapter 5 of Mas-Colell *et al.* We consider a price-taking firm that produces one output by using N inputs, according to a direct production function

$$f : \mathfrak{R}_+^N \rightarrow \mathfrak{R} : (z_1, \dots, z_N) \mapsto f(z_1, \dots, z_N),$$

where $(z_1, \dots, z_N) \equiv z$ denotes the input vector. We denote by q the amount of output, and in order to avoid notational confusion with Parts 1.2 below we use capital letters for prices, i. e., $P \in \mathfrak{R}_{++}$ denotes the price of the output, and $W \equiv (W_1, \dots, W_N) \in \mathfrak{R}_{++}^N$ the vector of input prices.

We consider the following optimization problems.

Problem INPUTPROFITMAX $[W, P]$.

Given (W, P) , choose (z_1, \dots, z_N) in order to maximize $Pf(z) - W \cdot z$.

Denote by $\tilde{z}(W, P)$ and by $\Pi(W, P)$ the solution and the value, respectively, of this problem.

Problem COSTMIN $[W, q]$.

Given (W, q) , choose (z_1, \dots, z_N) in order to minimize $W \cdot z$ subject to $f(z) \geq q$.

Denote by $\hat{z}(W, q)$ and by $c(W, q)$ the solution and the value, respectively, of this problem, and by $\beta(W, q)$ its Lagrange multiplier.

Problem OUTPUTPROFITMAX $[W, P]$.

Given (W, P) , choose q in order to maximize $Pq - c(W, q)$.

1.1(a). Prove that z^* solves INPUTPROFITMAX[W, P] if and only if, defining $q^* = f(z^*)$:

(i) z^* solves COSTMIN[W, q^*], and

(ii) q^* solves OUTPUTPROFITMAX[W, P].

Remark. It is here preferable that you write direct proofs by contradiction rather than manipulating first order conditions

1.1(b). Interpret $\beta(W, q)$.

1.1(c). Argue that, if z^* solves INPUTPROFITMAX[W, P], and $q^* = f(z^*)$, then $P = \beta(W, q^*)$.

1.2. We now consider a consumer with utility function $\tilde{u} : \mathfrak{R}_+^N \rightarrow \mathfrak{R} : (x_1, \dots, x_N) \mapsto \tilde{u}(x_1, \dots, x_N)$, which is in particular assumed to be locally nonsatiated, where $(x_1, \dots, x_N) \equiv x$ denotes her consumption vector. We denote by $p \equiv (p_1, \dots, p_N) \in \mathfrak{R}_{++}^N$ the price vector for the consumption goods, and by $w > 0$ the wealth of the consumer. We consider the following optimization problems.

Problem UMAX[p, w].

Given (p, w) , choose $x \equiv (x_1, \dots, x_N)$ in order to maximize $\tilde{u}(x)$ subject to $p \cdot x \leq w$.

Denote by $\tilde{x}(p, w)$ and by $v(p, w)$ the solution and the value, respectively, of this problem, and by $\lambda(p, w)$ its Lagrange multiplier.

Problem EMIN[p, u].

Given (p, u) , choose $x \equiv (x_1, \dots, x_N)$ in order to minimize $p \cdot x$ subject to $\tilde{u}(x) \geq u$.

Denote by $h(p, u)$ and by $e(p, u)$ the solution and the value, respectively, of this problem, and by $\mu(p, u)$ its Lagrange multiplier.

Now a new one:

Problem FRISCH[p, r].

Given $p \in \mathfrak{R}_{++}^N$ and $r \in \mathfrak{R}_{++}$, choose $x \equiv (x_1, \dots, x_N)$ in order to maximize $r\tilde{u}(x) - p \cdot x$.

Denote by $\varphi(p, r)$ and by $\Omega(p, r)$ the solution and the value, respectively, of this problem.

1.2(a). Interpret Problem FRISCH[p, r] and the parameter r .

1.2(b). What can you say about $\frac{\partial \Omega}{\partial p_j}(p, r)$? Justify your answer.

1.2(c). Some of the optimization problems in this part (Part 1.2) are formally (mathematically) identical to some of the ones in Part 1.1 above. Which ones are those? Explain.

1.2(d). Interpret the Lagrange multipliers $\lambda(p, w)$ and $\mu(p, u)$.

1.2(e). Show that if $u^* = v(p, w^*)$, then

$$\lambda(p, w^*) = \frac{1}{\mu(p, u^*)}.$$

Interpret. (Hint. Use duality.)

1.2(f). Let $x^* = \varphi(p, r)$ and $u^* = \tilde{u}(x^*)$. Show that $r = \mu(p, u^*)$ and interpret.

1.2(g). Let $x^* = \varphi(p, r)$, $u^* = \tilde{u}(x^*)$ and $w^* = e(p, u^*)$. Show that $r = \frac{1}{\lambda(p, w^*)}$ and interpret.

1.3. Comment on the results of parts 1.1-1.2 above.

Question 2

2. An economy is composed of two sectors of production and one (representative) consumer. The first sector of production (represented by firm 1) produces good 1 from the two factors of production, capital and labor, with the Leontief production function $y^1 = \min(3K^1, L^1)$. The second sector of production (represented by firm 2) produces good 2 from capital and labor with the Leontief production function $y^2 = \min(K^2, L^2)$. The representative consumer, who owns the total supply of capital $\bar{K} > 0$, the total supply of labor time $\bar{L} > 0$ and the firms, has the utility function

$$u(x_1, x_2, \ell) = x_1 x_2 \ell$$

where x_1 denotes the consumption of good 1, x_2 the consumption of good 2 and ℓ the consumption of leisure time.

- (a) Justify the terminology: “the production of good 2 is relatively more capital intensive than the production of good 1”.
- (b) The goal of the exercise is to compute the competitive equilibrium of this economy for different values of the parameters (\bar{K}, \bar{L}) and study what happens when the economy becomes relatively richer in capital. Without loss of generality let $\bar{L} = 1$. Let p_1 denote the price of good 1, p_2 the price of good 2, r the price of capital and w the price of labor. Argue that, in a competitive equilibrium, $w > 0$ and $r \geq 0$. Intuitively when do you expect r to be zero? We use the property $w > 0$ to normalize the wage to 1.
- (c) Find the relation between the price of each produced good and the prices of the factors that must hold in equilibrium.
- (d) Show there exists κ such that if $\bar{K} \geq \kappa$, there is an equilibrium such that $r = 0$. Describe this equilibrium.
- (e) Assume $\bar{K} < \kappa$. Use the market-clearing equations for the capital and goods 1 and 2 to find the equation that $r > 0$ must satisfy to be an equilibrium price. Check that if $\bar{K} < \kappa$, this equation does have one positive root, and if $\bar{K} \geq \kappa$ it does not have a positive root— so that in this case the equilibrium found in (d) is the unique equilibrium.
- (f) Compute the equilibrium price of capital in equilibrium when $\bar{K} < \kappa$. Show that if \bar{K} increases from $\varepsilon > 0$ to κ , the interest rate decreases and the equilibrium ratio y_2/y_1 increases. Explain these results.

Question 3

3. Let us take an aggregated view of the economy to show that, with perfect knowledge of the characteristics of the economy, an optimal cap (and trade) system can restore Pareto optimality in an economy with externalities. Suppose that the economy is composed of a productive sector which produces a (composite) consumption good y from a factor of production z with the linear technology $y = z$. The production process releases a polluting substance e in quantity $e = \phi(y, z_e)$, where y is the output of the production sector and z_e is the amount of factor of production used to reduce pollution. The function $\phi : \mathbb{R}_+^2 \rightarrow \mathbb{R}$ is a differentiable and strictly convex function, increasing in y and decreasing in z_e . Also $\phi(0, z_e) = 0, \forall z_e \geq 0$. There is a representative consumer with an endowment ω of the factor of production, no endowment of the consumption good, and utility U such that

$$U(y, e) = u(y) - \psi(e)$$

where $u : \mathbb{R}_+ \rightarrow \mathbb{R}$ is differentiable, strictly concave, increasing, with $u'(y) \rightarrow \infty$ when $y \rightarrow 0$, and $\psi : \mathbb{R}_+ \rightarrow \mathbb{R}$ is differentiable, convex, increasing, with $\psi'(e) \rightarrow 0$ when $e \rightarrow 0$.

- (a) Write the maximization problem whose solution is the Pareto optimum of this economy. Explain why there is a unique solution that is characterized by the first-order conditions.
- (b) In what follows assume that the Pareto optimal allocation is such that some capital is devoted to pollution reduction: $z_e^* > 0$. Give the equation that characterizes the optimal division (z^*, z_e^*) of ω between the input in production, z^* , and the input in the pollution-reduction technology, z_e^* . Let (y^*, e^*) denote the associated levels of production and pollution.
- (c) Derive the equation that you found in (b) by marginal economic reasoning without using the formalism of maximization under constraints that you used in (b). (This gives you the occasion to check that your answer to (b) is correct).
- (d) Suppose that the (Robinson Crusoe) economy described above is decentralized, with the firm (production sector) maximizing profit taking prices as given, and the consumer maximizing utility under her budget constraint. Suppose that a benevolent planner can calculate e^* and impose the limit e^* on the firm's emission of the polluting substance. Write the definition of a competitive equilibrium with cap e^* on pollution.
- (e) Show that there exists a price p^* for good y such that $(y^*, z^*, z_e^*), (p^*, 1)$ is a competitive equilibrium with cap e^* on pollution. Comment on the expression for p^* .
- (f) Show that there cannot be any other competitive equilibrium with cap e^* .
- (g) Conclude that imposing the cap e^* on pollution and letting markets work leads to Pareto optimality. How does this generalize to an economy with several polluting firms?

QUESTION 4

A new start-up firm, I-Start, wants to hire 50 workers. There are two types of workers that would have the required skills: H and L . A worker of type H would generate a monthly revenue of $\$R_H$ for the firm and a type L worker would generate a monthly revenue of $\$R_L$ for the firm. All workers are currently employed at a monthly salary of $\$S_0$. We assume throughout that $R_L < S_0 < R_H$. Every worker who decides to quit her current job and go to work for I-Start will always be able to immediately return to her current job (at the current monthly salary of $\$S_0$) if laid off by I-Start. The proportion of H types is q_H and the proportion of L types is $(1 - q_H)$ with $0 < q_H < 1$. I-Start would hire workers for a period of n months. Each worker knows her own type, while I-Start would not be able to tell if an applicant is of type H or of type L , that is, there is asymmetric information. The firm and the workers are risk neutral. Assume that there is no discounting. *All of the expected amounts to be computed below are from the initial point of view, before any hiring is done.* I-Start has no other costs besides the labor costs. The firm has the following options.

OPTION 1: offer a monthly salary of $\$w$ and hire 50 of those who apply.

- (a) For every $w \in [0, \infty)$, $w \neq S_0$, calculate the firm's expected monthly profit per worker if it chooses this option.
- (b) Write an inequality in terms of the parameters listed above that expresses the fact that Option 1 is not feasible for the firm because it cannot yield positive profits.
- (c) Calculate the firm's maximum profits when $R_H = 6,000$, $R_L = 4,000$, $S_0 = 4,800$, $q_H = 25\%$ and $n = 36$.

OPTION 2: offer to hire workers on a probationary contract that lasts m months, with $m < n$; at the end of the m -month period the firm will audit the fraction p of the workers ($0 \leq p \leq 1$) and will be able to tell with 100% accuracy whether each audited worker is of type L or of type H . The workers who are not audited and the workers who are audited and determined to be of type H will be retained at a monthly salary of $\$c$; the workers who are audited and determined to be of type L will be laid off (these workers can then go back to their old job that pays $\$S_0$ per month). The monthly salary during the probation period is $\$d$, with $d < c$.

Auditing involves a cost of $\$k > 0$ per audited worker. Note that *no new workers are hired after the m -month probationary period.*

- (d) Write an expression (in terms of the parameters) that gives I-Start's expected **total** profits over the entire n -month period under Option 2. [Hint: you need to consider two cases.]

From now on assume that $R_H = 6,000$, $R_L = 4,000$, $S_0 = 4,800$, $q_H = 25\%$, $n = 36$, $c = 5,500$ and $d = 3,000$. Thus the only remaining parameters are m , p and k .

- (e) Assume for this question that $k = 0$. Rewrite the expressions of part (d) for the values given above and explain the significance of the inequality $2,500m + 40,500p - 1,125mp < 36,000$.

Now add the following additional assumption: if indifferent between applying to the new firm and staying at the current job, L -type workers choose to stay where they are and H -type workers choose to apply to the new firm.

- (f) Assume that $k \geq 0$. Write a set of inequalities (in terms of m , p and k) that – under Option 2 – guarantee that (1) only the H workers apply and (2) I-Start will make positive profits.
- (g) Assume that $k \geq 0$. Write the firm's maximization problem if it decides to implement Option 2 in such a way that only H workers apply.
- (h) Solve the maximization problem of part (f).
- (i) Explain why the firm would want to publicly commit to auditing an appropriate fraction of workers at the end of the probation period (e.g. by signing a contract with a third party – the auditor – and making that contract public).

QUESTION 5

The US customs know that a drug shipment is being smuggled into the country in the baggage of an unknown passenger on one of two flights, one from Bogota and one from Cartagena. Both flights arrive at the same time. There are 15 customs officers available, and each officer can inspect 10 passengers. The flight from Bogota is carrying 200 passengers and the flight from Cartagena 100 passengers. Nothing is known about the characteristics of the suspected smuggler, but it is known that, *if the smuggler is coming from Bogota, he/she is carrying a larger shipment than if he/she is coming from Cartagena*. All of this is common knowledge between the US customs and the drug kingpins. This situation can be seen as a simultaneous game, where the US customs must decide how many of the 15 officers should be assigned to the Cartagena flight (possibly none, possibly all; the remaining officers will be assigned to the Bogota flight), the drug kingpins must decide which flight to use to send the shipment and there are four possible outcomes: a **large** shipment is **intercepted** by the US customs (I_L), a **small** shipment is **intercepted** by the US customs (I_S), a **large** shipment **passes** undetected through US customs (P_L) and a **small** shipment **passes** undetected through US customs (P_S). Both the US customs and the drug kingpins have von Neumann-Morgenstern preferences. For the US customs the utility of I_L is twice the utility of I_S , while the utility of P_L is equal to the utility of P_S . For the drug kingpins the utility of P_L is twice the utility of P_S , while the utility of I_L is equal to the utility of I_S .

- (a) Write the normalized von Neumann-Morgenstern utility functions of the two players.
- (b) If the US customs office assigns 8 officers to the Bogota flight and 7 to the Cartagena flight, what is the probability that the drug shipment will be intercepted if it is on the Bogota flight? What is the probability that the drug shipment will be intercepted if it is on the Cartagena flight?
- (c) Does the US customs office have any strategies that are dominated? If your answer is No then prove it and if your answer is Yes then list the dominated strategies and state whether they are weakly or strictly dominated.
- (d) Prove that there are no pure strategy Nash equilibria.
- (e) Prove that there is no mixed-strategy Nash equilibrium where the US customs uses a pure strategy and the drug kingpins use a completely mixed strategy.
- (f) Using the insight you got from the calculations for part (e), find a Nash equilibrium of the game and prove that it is a Nash equilibrium. [Note: remember that necessary conditions are not always sufficient.]
- (g) Now let us change the game as follows. It is a game of incomplete information where the size of the drug shipment was chosen earlier on and cannot be changed; the drug kingpins know the size of the shipment while the US customs do not know the shipment size and believe that it is large with probability p and small with probability $1 - p$; these beliefs are common knowledge between the drug kingpins and the US customs. The drug kingpins choose on which flight to send the shipment and the US customs choose the number of officers to assign to the Cartagena flight (and the remaining are assigned to the Bogota flight); these choices are made “simultaneously” (that is, in ignorance of the other player’s choice). The payoffs are the same as above.
 - (g.1) Draw an extensive game that represents this situation and indicate what the payoffs are (you can do so as a function of n , where n is the choice of the US customs).
 - (g.2) Suppose that $p = \frac{1}{2}$ and the US customs use the mixed strategy found in part (f). Is there a pure strategy of the drug kingpins that is a best reply to that?
 - (g.3) Does the drug kingpins’ pure strategy of part (g.2) together with the US customs’ mixed strategy of part (f) constitute a Bayesian Nash equilibrium?