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ANSWER KEY

PRELIMINARY EXAMINATION FOR THE Ph.D. DEGREE

Answer four questions (out of five)

Question 1. Complements and substitutes.

A consumer consumes L *consumption goods*, with quantities denoted (x_1, …, x_L), and her preferences are represented by a twice differentiable, strictly quasiconcave and locally nonsatiated utility function $\tilde{u}:\Re^L_+ \to \Re : x \equiv (x_1,...,x_L) \mapsto \tilde{u}(x)$. But these consumption goods are not sold in the market. Instead, consumption good j (j = 1,…, L) is home-produced by using $N[j]$ *marketed goods*, with quantities denoted (z_{1j},…, $z_{N[j],\,j}$), according to the Leontief technology

$$x_j = \min\left\{\frac{z_{1j}}{a_{1j}}, \frac{z_{2j}}{a_{2j}}, ..., \frac{z_{N[j],j}}{a_{N[j],j}}\right\},$$

where all denominators are positive. (The first subscript indicates the marketed good, and the second one the consumption good.) A double index (k, j) labels the kth marketed good in the list of marketed goods used in the production of consumption good j, j = 1,…, L, k = 1,…, $N[j]$. Hence, there are altogether $N \equiv \sum_{j=1}^{L} N[j]$ marketed goods, $N[j]$ of which are used in the home production of consumption good j. Denote by π_{kj} the price of the (k, j) marketed good, j = 1,…, L, k = 1,…, $N[j]$.

1(a). A vector $\pi \equiv (\pi_{11},...,\pi_{N[1],1},\pi_{12},...,\pi_{N[2],2},...,\pi_{1L},...,\pi_{N[L],L})$ of prices of marketed goods induces, via the home production technology, a vector $p \equiv (p_1,…, p_L)$ of (implicit) prices for the consumption goods. For j = 1,…, L, define the function $\tilde{p}$ that expresses the price of consumption good j in terms of the prices of the marketed goods.

ANSWER. For j = 1, …, L, in order to home-produce one unit of consumption good j, the consumer needs to buy a_{kj} units of marketed good (k, j), j = 1,…, L, k = 1,…, $N[j]$. We accordingly define:

$$\tilde{p}_j : \Re^N_{++} \to \Re : \tilde{p}_j(\pi) = a_{1j}\pi_{1j} + a_{2j}\pi_{2j} + \ldots + a_{N[j],j}\pi_{N[j],j} \,. \qquad (1.1)$$

1(b). The *EMIN*[*p*, *u*] *Problem* is defined by $\min_x p \bullet x$ subject to $\tilde{u}(x) \geq u$, with solution function the Hicksian demand function *h* for consumer goods. Show that *S*(*p*, *u*)*p* = 0, where *S*(*p*, *u*) is the Slutsky matrix.

ANSWER. The solution to the EMIN[*p*, *u*] problem is the same as that of the EMIN[*tp*, *u*] problem, for any *t* > 0. Hence, Hicskian demand is homogeneous of degree zero in prices.

Euler's theorem then yields $\begin{bmatrix} \frac{\partial h_j}{\partial p_1} & \ldots & \frac{\partial h_j}{\partial p_L} \end{bmatrix} \begin{bmatrix} p_1 \\ \vdots \\ p_L \end{bmatrix} = 0$, *j* = 1,…, *L*.

1(c). Define the *Hicksian demand for marketed good* (*k*, *j*) by

$$\xi_{kj} : \Re^N_{++} \times U \to \Re : \xi_{kj}(\pi, u) = a_{kj} h_j(\tilde{p}(\pi), u),$$

where *U* is the relevant domain of utility levels.

When can we say that marketed good (*m*, *i*) is a (net) complement of marketed good (*k*, *j*) at (π, *u*)?

When can we say that marketed good (*m*, *i*) is a (net) substitute of marketed good (*k*, *j*) at (π, *u*)?

ANSWER. Adapting the usual definitions, we say that (*m*, *i*) is a (net) complement of (*k*, *j*) at (π, *u*) if $\frac{\partial \xi_{kj}(\pi, u)}{\partial \pi_{mi}} \leq 0$, and that it is a (net) substitute of (*k*, *j*) at (π, *u*) if $\frac{\partial \xi_{kj}(\pi, u)}{\partial \pi_{mi}} \geq 0$.

1(d). Coffee and cream are popular textbook examples of complements. But Paul Samuelson suggested that, when a consumer uses cream both in her coffee and in her tea, cream and coffee may actually behave as substitutes. In order to analyze this somewhat paradoxical result, we specialize the previous model to the case of two home-produced consumer goods: coffee (good 1) and tea (good 2). Coffee requires coffee beans (marketed good (1, 1)) and cream for coffee (marketed good (2, 1)), whereas tea requires tea leaves (marketed good (1, 2)) and cream for tea (marketed good (2, 2)). (The prices of marketed goods (2, 1) and (2, 2) may well be the same, but this does not play any role here.)

1(d)(i). Is marketed good (1, 1) (coffee beans) a complement or a substitute for marketed good (2, 1) (cream for coffee)? Argue your answer.

Answer. In this simplified model, the price functions (1.1) for the consumer goods are as follows.

Good 1 (coffee): $\tilde{p}_1(\pi) = a_{11}\pi_{11} + a_{21}\pi_{21}$, (1.2)

Good 2 (tea): $\tilde{p}_2(\pi) = a_{12}\pi_{12} + a_{22}\pi_{22}$, (1.3)

and the Hicksian demand for marketed good (k, j) can be written

$$\xi_{kj}(\pi,u) = a_{kj}h_j(\tilde{p}_1(\pi), \tilde{p}_2(\pi), u).$$

Applying the chain rule, we compute

$$\frac{\partial \xi_{21}(\pi,u)}{\partial \pi_{11}} = a_{21}\left[\frac{\partial h_1}{\partial p_1}\frac{\partial \tilde{p}_1}{\partial \pi_{11}} + \frac{\partial h_1}{\partial p_2}\frac{\partial \tilde{p}_2}{\partial \pi_{11}}\right]. \quad (1.4)$$

From (1.2), $\frac{\partial \tilde{p}_1}{\partial \pi_{11}} = a_{11}$, and from (1.3) $\frac{\partial \tilde{p}_2}{\partial \pi_{11}} = 0$. Writing $s_{ij} \equiv \frac{\partial h_i(p,u)}{\partial p_j}$ for the (i, j) entry of the Slutsky matrix of 1(b) above, (1.4) becomes

$$\frac{\partial \xi_{21}(\pi,u)}{\partial \pi_{11}} = a_{21}s_{11}a_{11}, \quad (1.5)$$

which must be less than or equal to zero by the negative semidefiniteness of the Slutsky matrix. Hence, as is intuitively plausible, coffee beans are a complement of cream for coffee.

1(d)(ii). Is marketed good (1, 1) a complement or a substitute for marketed good (2, 2) (cream for tea)? Argue your answer.

Answer. Now we compute

$$\begin{aligned}\frac{\partial \xi_{22}(\pi,u)}{\partial \pi_{11}} &= a_{22}\left[\frac{\partial h_2}{\partial p_1}\frac{\partial \tilde{p}_1}{\partial \pi_{11}} + \frac{\partial h_2}{\partial p_2}\frac{\partial \tilde{p}_2}{\partial \pi_{11}}\right] \\ &= a_{22}s_{21}a_{11}.\end{aligned}$$

By 1(b) above,

$$\begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix}\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \quad (1.6)$$

Hence, $s_{21} \geq 0$, and therefore

$$\frac{\partial \xi_{22}(\pi,u)}{\partial \pi_{11}} = a_{22}s_{21}a_{11} \geq 0. \quad (1.7)$$

It follows that coffee beans are a substitute of cream for tea.

1(d)(iii). Define the demand for “cream” as the sum of the demand for marketed goods (2, 1) and (2, 2). What can you say about the complementarity or substitutability of cream and marketed good (1, 1)? Argue and discuss your answer.

ANSWER. The derivative of the Hicksian demand for cream with respect to the price of coffee beans is given by the sum of the nonpositive term (1.5) and the nonnegative term (1.7). Its sign will then depend on the relative strength of these two terms. We can compute the sum as follows.

$$\frac{\partial\xi_{21}(\pi,u)}{\partial\pi_{11}}+\frac{\partial\xi_{22}(\pi,u)}{\partial\pi_{11}}=a_{21}s_{11}a_{11}+a_{22}s_{21}a_{11}. \quad (1.8)$$

By (1.6), $s_{11}\,p_1 + s_{12}\,p_2 = 0$, which given the symmetry of the Slutsky matrix implies that $s_{11}\,p_1 + s_{21}\,p_2 = 0$. Hence (1.8) can be written

$$\frac{\partial\xi_{21}(\pi,u)}{\partial\pi_{11}}+\frac{\partial\xi_{22}(\pi,u)}{\partial\pi_{11}}=a_{21}s_{11}a_{11}-a_{22}\frac{p_1}{p_2}s_{11}a_{11}=\left[\frac{a_{21}}{a_{22}}-\frac{p_1}{p_2}\right]a_{22}a_{11}s_{11}. \quad (1.9)$$

Both definitions in 1(c) are satisfied if $s_{11} = 0$.

So let $s_{11} < 0$. The paradoxical result of coffee beans being a substitute for cream appears when (1.9) is positive, i. e., when $\frac{a_{21}}{a_{22}}-\frac{p_1}{p_2}<0$. This requires that $\frac{a_{21}}{a_{22}}$ be relatively small (the consumer uses a lot more cream in her tea than in her coffee) in relation to $\frac{p_1}{p_2}$, the relative price of coffee. An increase in the price of coffee beans then induces a large substitution of tea for coffee, which may result in an overall increase in the demand for cream.

Question 2. Capping emissions.

A firm that behaves in a perfectly competitive manner in all markets produces one output by using L-1 inputs according to a direct production function

$$f:\Re_{+}^{L-1}\to\Re : z\equiv(z_1,\ldots,z_{L-1})\mapsto f(z).$$

assumed to be differentiable with a strictly positive gradient on $\Re_{++}^{L-1}$ and concave.

Denote by $p > 0$ the price of the output, and by $w \equiv (w_1, \ldots, w_{L-1}) \in \Re^{L-1}_{++}$ the vector of input prices. Assume that the cost function $c: \Re^{L-1}_{++} \times \Re_{+} \to \Re : (w_1, \ldots, w_{L-1}; q) \mapsto c(w_1, \ldots, w_{L-1}; q)$ is differentiable and convex in q, and that at the profit maximizing solution the quantities of the inputs and of output are positive.

2(a). What is the relation between the output price and the marginal cost at a profit-maximizing solution? Prove your answer.

ANSWER. Write the profit maximization problem as:

Given w, choose $q \geq 0$ in order to maximize $p\,q - c(w, q)$.

The first-order condition is: $p \leq \dfrac{\partial c(w,q)}{\partial q}$, with equality if $q > 0$, as assumed, i. e., the output price equals the marginal cost.

2(b). How does an increase in an input price affect the marginal cost at a profit-maximizing solution? Prove your answer.

ANSWER. The PRICE = MARGINAL COST equality is maintained before and after an increase in the price of any input. This necessitates an adjustment in the amount of output, because an increase in an input price increases the marginal cost at any given level of output, as can be seen as follows. The cost-optimization problem can equivalently be written in the minimization or maximization form. Its maximization form is as follows.

Given (w, q) choose $(z_1, \ldots, z_{L-1})$ in order to maximize $-\,w \cdot z$ subject to $q \leq f(z)$. The Lagrangian of the problem is $-\,w \cdot z - \lambda[\,q - f(z)]$. The first-order conditions are:

$$-w_j + \lambda \frac{\partial f}{\partial z_j} \leq 0, \left[-w_j + \lambda \frac{\partial f}{\partial z_j} \right] z_j = 0, \; j = 1, \ldots, L\text{-}1,$$

$$q \leq f(z), \;\; [q - f(z)]\, \lambda = 0.$$

The value function of this problem is $-\,c(w, q)$, and by the envelope theorem,

$\dfrac{\partial(-c(w,q))}{\partial q} = -\lambda \geq 0$. Hence, $\dfrac{\partial c(w,q)}{\partial q} = \lambda \geq 0$, i. e., the marginal cost equals λ.

Implicitly differentiating the FO equality $-w_j + \lambda \dfrac{\partial f}{\partial z_j} = 0$, we obtain $\dfrac{d\lambda}{dw_j} = -\dfrac{-1}{\dfrac{\partial f}{\partial z_j}} > 0$.

Hence, as long as the solution is interior the marginal cost is increasing in every input price.

2(c). In order to limit greenhouse gas emissions, the public authority imposes a fixed cap or quota k on the CO_2 emissions of the firm. All inputs may contribute to emissions: more precisely, the firm's emissions are a convex, differentiable function $\eta : \Re_+^{L-1} \to \Re : (z_1, ..., z_{L-1}) \mapsto \eta(z_1, ..., z_{L-1})$ with nonnegative partial derivatives.

2(c)(i). Is it necessarily true that the production of a larger amount of output requires emissions to increase? Discuss it in the simpler two-dimensional case.

Answer. The answer is in general NO. Assume that, at some point, the slope of the isoquant, $-\frac{\frac{\partial f}{\partial z_1}}{\frac{\partial f}{\partial z_2}}$ is different from the slope of the iso-emissions curve $-\frac{\frac{\partial \eta}{\partial z_1}}{\frac{\partial \eta}{\partial z_2}}$, say $\frac{\frac{\partial f}{\partial z_1}}{\frac{\partial f}{\partial z_2}} > \frac{\frac{\partial \eta}{\partial z_1}}{\frac{\partial \eta}{\partial z_2}}$.

Let $(\varepsilon_1, \varepsilon_2)$ be small and satisfy $\frac{\frac{\partial f}{\partial z_1}}{\frac{\partial f}{\partial z_2}} > \frac{\varepsilon_2}{\varepsilon_1} > \frac{\frac{\partial \eta}{\partial z_1}}{\frac{\partial \eta}{\partial z_2}}$ and consider increasing z_1 by ε_1 while decreasing z_2 by ε_2. The FO approximation to the change in output is $\frac{\partial f}{\partial z_1}\varepsilon_1 - \frac{\partial f}{\partial z_2}\varepsilon_2 > 0$, whereas that of emissions is $\frac{\partial \eta}{\partial z_1}\varepsilon_1 - \frac{\partial \eta}{\partial z_2}\varepsilon_2 < 0$, i. e., output increases, while emissions decrease. See Figure 2.1

If, on the other hand $\frac{\frac{\partial f}{\partial z_1}}{\frac{\partial f}{\partial z_2}} < \frac{\frac{\partial \eta}{\partial z_1}}{\frac{\partial \eta}{\partial z_2}}$, then decrease z_1 by ε_1 while increasing z_2 by ε_2, for $\frac{\frac{\partial f}{\partial z_1}}{\frac{\partial f}{\partial z_2}} < \frac{\varepsilon_2}{\varepsilon_1} < \frac{\frac{\partial \eta}{\partial z_1}}{\frac{\partial \eta}{\partial z_2}}$. In that case the FO approximation to the change in output is $-\frac{\partial f}{\partial z_1}\varepsilon_1 + \frac{\partial f}{\partial z_2}\varepsilon_2 > 0$, whereas that of emissions is $-\frac{\partial \eta}{\partial z_1}\varepsilon_1 + \frac{\partial \eta}{\partial z_2}\varepsilon_2 < 0$. See Figure 2.2.

Hence, it is often technologically possible to increase output while decreasing emissions.

Figure 2.1

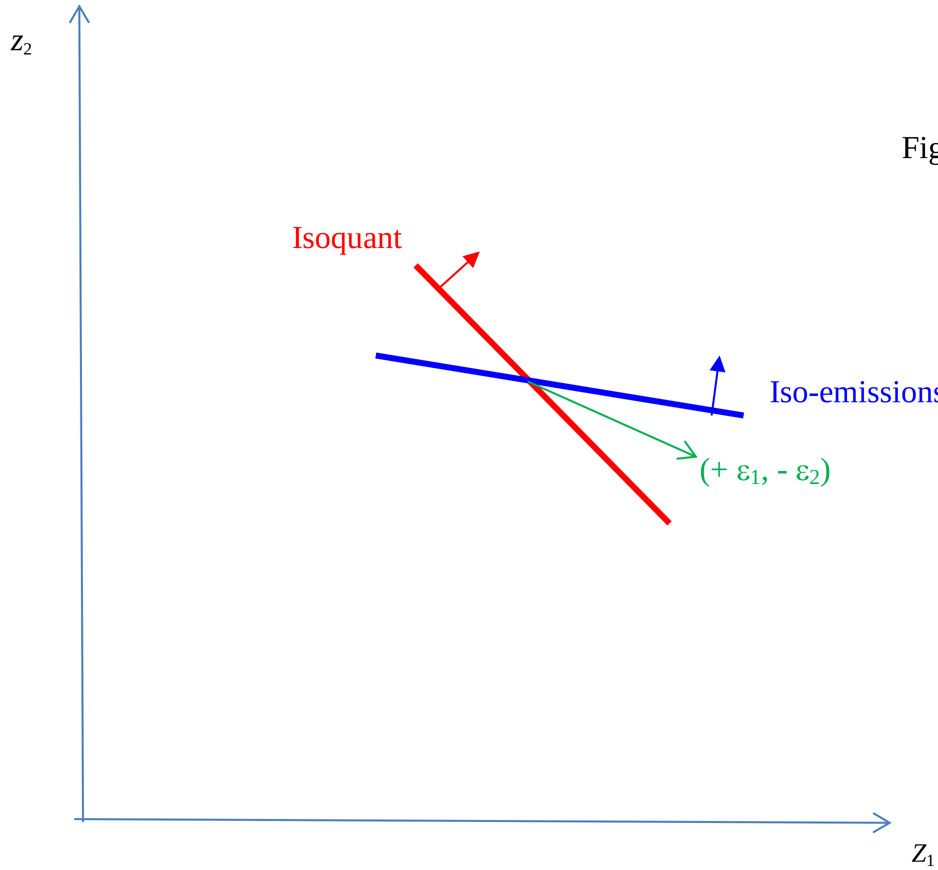
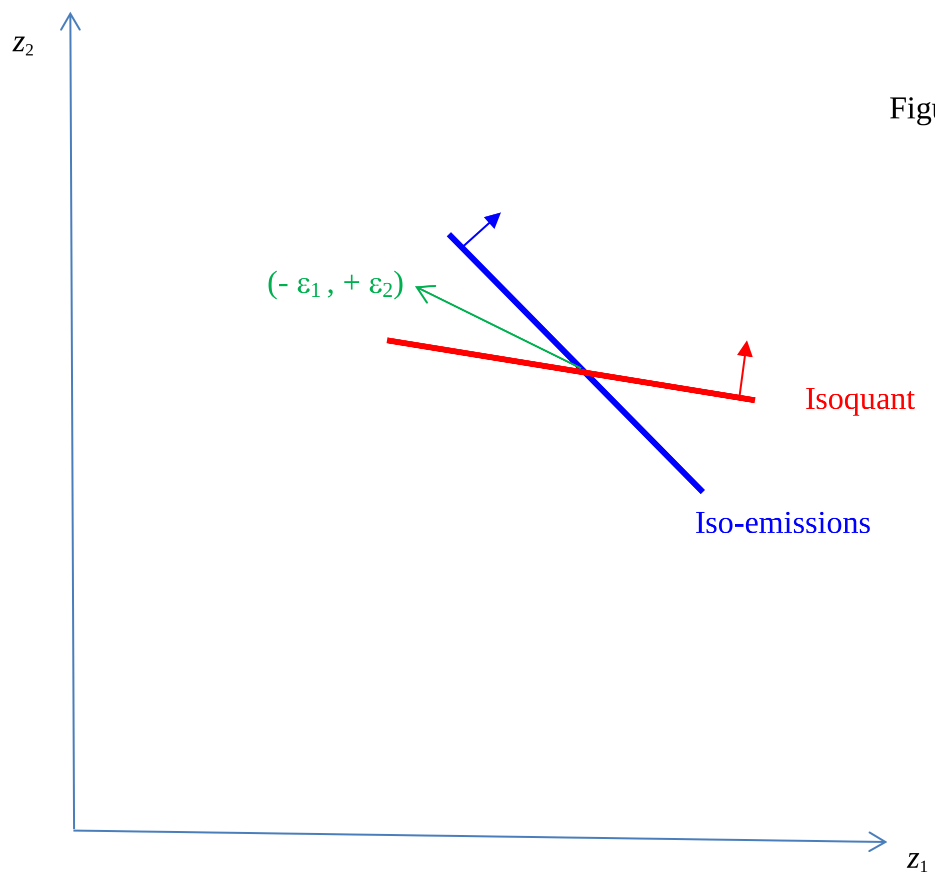


Figure 2.2



2(c)(ii). Write the Kuhn-Tucker conditions of the profit-maximizing problem of the firm that faces an emission cap, and discuss the implications of the size of the emission cap on the Lagrange multiplier

Answer. The problem of the firm when facing the emissions constraint is: Given *p*, *w* and *k*, choose *z* in order to maximize $pf(z) - w.z$ subject to $\eta(z) \leq k$, with Lagrangian $pf(z) - w.z - \mu[\eta(z) - k]$. Its KT conditions are:

$$p\frac{\partial f}{\partial z_j} - w_j - \mu\frac{\partial \eta}{\partial z_j} \leq 0, \left[p\frac{\partial f}{\partial z_j} - w_j - \frac{\partial \eta}{\partial z_j}\right] z_j = 0,\ j = 1,\ldots, L-1,$$

$$\eta(z) - k \leq 0, \mu[\eta(z) - k] = 0.$$

The multiplier μ is nonnegative. If *k* is large enough, the firm just chooses the same input vector as when its emissions are unconstrained and emits an amount less than *k*. The last KT condition then implies than μ is zero. If, on the other hand, the emissions constraint is binding, then μ is typically positive.

2(c)(iii). Compare the Kuhn-Tucker conditions of the profit-maximizing problem of the firm under the emissions constraint with those of the standard profit-maximizing problem (without the emissions constraint).

Answer. Under the assumption that the solution is interior, the first KT conditions can be written:

No emissions constraint: $$p\frac{\partial f}{\partial z_j} = w_j,\ j = 1,\ldots, L-1, \qquad (2.1)$$

Emissions constraint: $$p\frac{\partial f}{\partial z_j} = w_j + \mu\frac{\partial \eta}{\partial z_j},\ j = 1,\ldots, L-1. \qquad (2.2)$$

We observe that the RHS of (2.2) is never lower than that of (2.1), and is higher than (2.1) unless μ or $\frac{\partial \eta}{\partial z_j}$ is zero.

2(c)(iv). Argue that the profit-maximizing output of the firm cannot be higher under the emissions constraint than in the absence of such a constraint.

ANSWER. We see in 2(c)(iii) that the emissions constraint has an effect on the profit maximizing solution of the firm formally comparable to a (weak) increase in input prices. From 2(b) we know that an increase in input prices will induce an increase in the marginal cost, and because marginal costs are nondecreasing in q (by the assumed convexity of the cost function with respect to q), the PRICE = MARGINAL COST equality of 2(a) above cannot be satisfied at a higher level of output. Therefore the profit-maximizing output of the firm cannot be higher under the emissions constraint than in the absence of such a constraint. If the emissions constraint is binding, then the firm will typically choose to produce a smaller amount of output.

Answer Key - Question 3.

(a) $\max \ \alpha \ln(x_i) + \beta \ln(h_i) + \gamma_i \log(z_i + Z^{-i})$

subject to $x_i + r h_i + z_i \leq w_i + r \bar{h}_i$ $\quad \lambda_i$

$z_i \geq 0$ $\quad \rho_i$

FOCs:

$$\frac{\alpha}{x_i} = \lambda_i \quad \frac{\beta}{h_i} = \lambda_i r \quad \frac{\gamma_i}{z_i + Z^{-i}} = \lambda_i - \rho_i \quad \rho_i z_i = 0.$$

suppose $\rho_i = 0$. Then $x_i = \frac{\alpha}{\lambda_i}$ $\quad h_i = \frac{\beta}{\lambda_i r}$ $\quad z_i = \frac{\gamma_i}{\lambda_i} - Z^{-i}$

Inserting in the budget constraint leads to

$$\lambda_i = \frac{1+\gamma_i}{w_i + r\bar{h}_i + Z^{-i}},$$

$$x_i = \frac{\alpha(w_i + r\bar{h}_i + Z^{-i})}{1+\gamma_i}$$

$$h_i = \frac{\beta(w_i + r\bar{h}_0 + Z^{-i})}{(1+\gamma_i) r}$$

$$z_i = \frac{\gamma_i}{1+\gamma_i}(w_i + r\bar{h}_i + Z^{-i}) - Z^{-i}$$ if this is nonnegative

Otherwise $\rho_i > 0$ $z_i = 0$, $\lambda_i = \frac{1}{w_i + r\bar{h}_i}$, $x_i = \alpha(w_i + r\bar{h}_i)$

$h_i = \frac{\beta(w_i + r\bar{h}_i)}{r}$

(b) If $z_i > 0$ $\quad z_i + Z^{-i} = \frac{\gamma_i}{1+\gamma_i}\left(w_i + r\bar{h}_i + Z^{-i}\right)$

$$< \frac{\gamma_i}{1+\gamma_i}\left(w_i + r\bar{h}^i + y\right)$$

$$\Leftrightarrow \quad y\left(1 - \frac{\gamma_i}{1+\gamma_i}\right) < \frac{\gamma_i}{1+\gamma_i}\left(w_i + r\bar{h}^i\right)$$

$$\Leftrightarrow \quad \frac{y}{1+\gamma_i} < \frac{\gamma_i}{1+\gamma_i}\left(w_i + r\bar{h}^i\right) \Leftrightarrow y < \gamma_i\left(w_i + r\bar{h}^i\right)$$

If agent i does not contribute $y = Z^{-i}$ is such that

$$\frac{\gamma_i}{1+\gamma_i}\left(w_i + r\bar{h}^i + y\right) - y \le 0 \Leftrightarrow y \ge \gamma_i\left(w_i + r\bar{h}^i\right)$$

If $i' < i$ contributes $y < \gamma_{i'}\left(w_{i'} + r\bar{h}^{i'}\right) \le \gamma_i\left(w_i + r\bar{h}^i\right)$

since $\bar{h}^{i'}$ is either equal to 0 or equal to $\bar{h}^i$ (if i and i' belong to the highest income group). and $\gamma_{i'} w_{i'} \le \gamma_i w_i$ by assumption. But this contradicts that $y < \gamma_i\left(w_i + r\bar{h}^i\right)$.

(c) If $i \in C$, $Z^{-i} = y - z_i$ and

$$y = z_i + Z^{-i} = \frac{\gamma_i}{1+\gamma_i}\left(w_i + r\bar{h}^i + y - z_i\right)$$

$$\Leftrightarrow \quad (1+\gamma_i)\,y - \gamma_i y = \gamma_i\left(w_i + r\bar{h}^i - z_i\right)$$

$$\Leftrightarrow \quad z_i = \frac{\gamma_i\left(w_i + r\bar{h}^i\right) - y}{\gamma_i} = w_i + r\bar{h}_i - \frac{y}{\gamma_i}$$

Summing over $i \in C$ gives

$$y = \sum_{i \in C} (w_i + r\bar{h}_i) - \frac{1}{\gamma_C} y$$

$$y\left(1 + \frac{1}{\gamma_C}\right) = R_C + rH$$

The equality $H = \sum_{i \in C} \bar{h}_i$ is due to the fact that agents in the highest income group contribute to the city services. Otherwise, no agent would contribute (from the result of (b)) but the utility of all agents would be $-\infty$, which cannot be their optimal choice ~~an optimum~~. Thus

$$\boxed{y = \frac{\gamma_C(R_C + rH)}{1 + \gamma_C}}$$

(d) From (c) $z_i = w_i + r\bar{h}_i - \frac{y}{\gamma_i} = w_i + r\bar{h}_i - \frac{\gamma_C}{\gamma_i}\frac{(R_C + rH)}{1 + \gamma_C}$

and $Z^{-i} = y - z_i = \frac{\gamma_C(R_C + rH)}{1 + \gamma_C} - (w_i + r\bar{h}_i) + \frac{\gamma_C}{\gamma_i}\left(\frac{R_C + rH}{1 + \gamma_C}\right)$

$$= \frac{\gamma_C}{1 + \gamma_C}(R_C + rH)\left(1 + \frac{1}{\gamma_i}\right) - (w_i + r\bar{h}_i)$$

$$\boxed{Z^{-i} = \frac{\gamma_C}{\gamma_i}\frac{1 + \gamma_i}{1 + \gamma_C}(R_C + rH) - (w_i + r\bar{h}_i)}.$$

(e) If $i \in C$, replacing Z^{-i} (in agent i's demand for housing) by its value calculated in

(d) gives $h_i = \frac{\beta\gamma_C}{\gamma_i r}\frac{R_C + rH}{1+\gamma_C}$.

If $i \in N$, then $h_i = \frac{\beta[\omega_i + r\overline{h}_i]}{r} = \frac{\beta\omega_i}{r}$, since $\overline{h}_i = 0$.

“Supply = Demand” gives

$$\left[\sum_{i\in C}\frac{1}{\gamma_i}\right]\beta\frac{\gamma_C}{r}\frac{R_C + rH}{1+\gamma_C} + \frac{\beta}{r}\sum_{i\in N}\omega_i = H,$$

$$\frac{\beta}{r}\left[\frac{R_C}{1+\gamma_C} + R_N\right] = H - \frac{\beta}{1+\gamma_C}H = \frac{[1+\gamma_C - \beta]H}{1+\gamma_C} = \frac{\alpha+\gamma_C}{1+\gamma_C}H.$$

Thus

$$r = \frac{\beta[R_C + (1+\gamma_C)R_N]}{[\alpha+\gamma_C]H}.$$

(f) Substituting *r* in the expression found for *y* gives

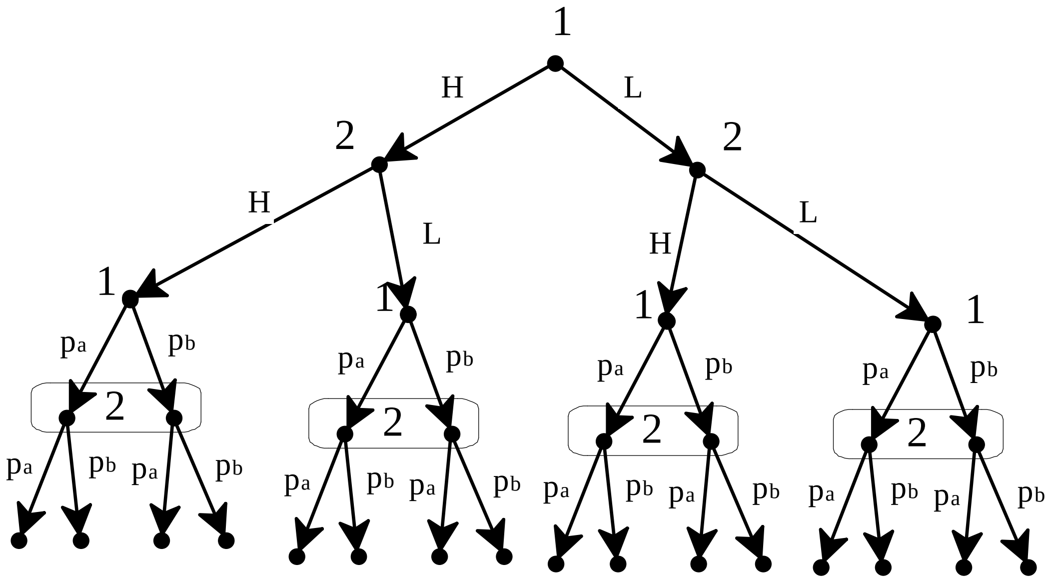
$$y = \frac{\gamma_C}{1+\gamma_C}\left[R_C + \frac{\beta[R_C + (1+\gamma_C)R_N]}{\alpha+\gamma_C}\right] = \frac{\gamma_C}{1+\gamma_C}\left[\frac{[\alpha+\beta+\gamma_C]R_C + \beta(1+\gamma_C)R_N]}{\alpha+\gamma_C}\right],$$

i.e.,
$$y = \frac{\gamma_C}{\alpha+\gamma_C}\left[R_C + \beta R_N\right].$$

Micro Prelim June 24, 2013

Answer Keys for Question 4

4(a) The game is as follows:

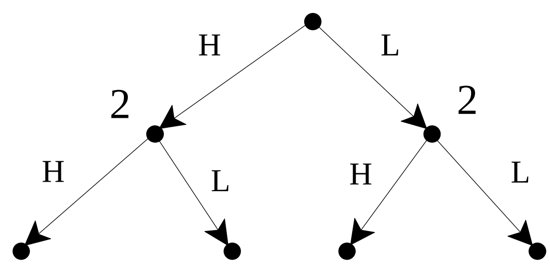


4(b) There are four subgames in the second stage.

– In the subgame where they both choose H, by Bertrand's theorem there is only one Nash equilibrium given by $p_1 = p_2 = 2$ where both firms make zero profits.

– In the subgame where they both choose L, by Bertrand's theorem there is only one Nash equilibrium given by $p_1 = p_2 = 0$ where both firms make zero profits.

Now consider the situation where one firm has chosen H and the other L. Then the profit functions are $\pi_H = (p_H - 2)D_H$ and $\pi_L = p_L D_L$. Solving $\frac{\partial \pi_H}{\partial p_H} = 0$ and $\frac{\partial \pi_L}{\partial p_L} = 0$ we get $p_H = 28,\ p_L = 14$ with corresponding profits of $\pi_H = 169$ and $\pi_L = 49$. Thus the game can be reduced to:



0	169	49	0
0	40	169	0

Hence there is a unique subgame-perfect equilibrium given by

$$\left(\underbrace{\left(H, p_1 = 2, p_1 = 28, p_1 = 14, p_1 = 0\right)}_{\text{strategy of Firm 1}}, \underbrace{\left(L, H, p_2 = 2, p_2 = 14, p_2 = 28, p_2 = 0\right)}_{\text{strategy of Firm 2}} \right)$$

. The subgame-perfect equilibrium play is: Firm 1 chooses H, Firm 2 follows with L and then Firm 1 chooses $p_1 = 28$ and Firm 2 chooses $p_2 = 14$ with corresponding profits of $\pi_H = 169$ and $\pi_L = 49$.

4(c) By Part (b) the problem reduces to finding the Nash equilibria of a simultaneous auction where the winner gets \$(169– her bid) at the loser gets \$49. There is only one Nash equilibrium of this game where both players bid \$120. Let b_A be the bid of Player A and b_B the bid of Player B. Proof: first of all, there cannot be a Nash equilibrium where $b_A > b_B$ because the winner (in this case, Player A) can increase her payoff by reducing her bid slightly. Similarly, there cannot be a Nash equilibrium where $b_B > b_A$. Thus the only candidates for Nash equilibrium are pairs (b_A, b_B) where $b_A = b_B$. Call this common bid *b*. If *b* > 120 then the winner (Player A) gets a payoff of $169 - b < 18$ and she can increase her payoff to 49 by switching to $b_A < b$. It cannot be that $b < 120$, because Player B's payoff is 49 and he can increase it to $(169 - b - \varepsilon)$ by switching to $b_B = b + \varepsilon$ with a sufficiently small $\varepsilon > 0$. Finally we show that (120,120) is indeed a Nash equilibrium. The payoff of each player is 49. Player A is the winner; if she increases her bid to any $b_A > 120$, then her payoff becomes $169 - b_A < 49$ and if she switches to any $b_A < 120$, then her payoff remains 49. If Player B increases his bid to any $b_B > 120$, then his payoff becomes $169 - b_B < 49$ and if he switches to any $b_B < 120$, then his payoff remains 49. Thus there is only one subgame-perfect equilibrium of the entire game given by the bids of 120 together with the strategies determined in Part (b).

4(d) The game can be reduced as in Part (c). In this case there are many Nash equilibria. First note that $b_A = b_B = b$ requires *b* = 120 (if *b* > 120, the winner – Player A – prefers to become the loser and if *b* < 120 then Player B prefers to become the winner).

Secondly, $b_A \geq b_B$ requires $b_A \geq 120$ (otherwise Player B wants to become the winner) and $b_B \leq 120$ (otherwise Player A wants to become the loser).

Finally, $b_B > b_A$ requires $b_B \geq 120$ (otherwise Player A wants to become the winner) and also $b_A \leq 120$ (otherwise Player B wants to become the loser).

The Nash equilibria are:

(1) every pair (b_A, b_B) with $b_A \geq b_B$ and $b_A \geq 120$ and $b_B \leq 120$ (Player B's payoff is 49 and he cannot increase it by changing his bid, while Player A's payoff is at least 49 and she cannot increase it by changing her bid),

(2) every pair (b_A, b_B) with $b_B > b_A$ and $b_B \geq 120$ and $b_A \leq 120$.

4(e) In this case there are no Nash equilibria. Nash equilibrium requires the loser's bid to be zero and the winner's bid to be as low as possible; thus the only candidate would be $b_A = b_B = 0$, but this is not a NE because Player B can increase his payoff by slightly increasing his bid. Here is a more detailed argument:

(1) $b_A \geq b_B > 0$ is not a NE because Player B can increase his payoff by reducing his bid,

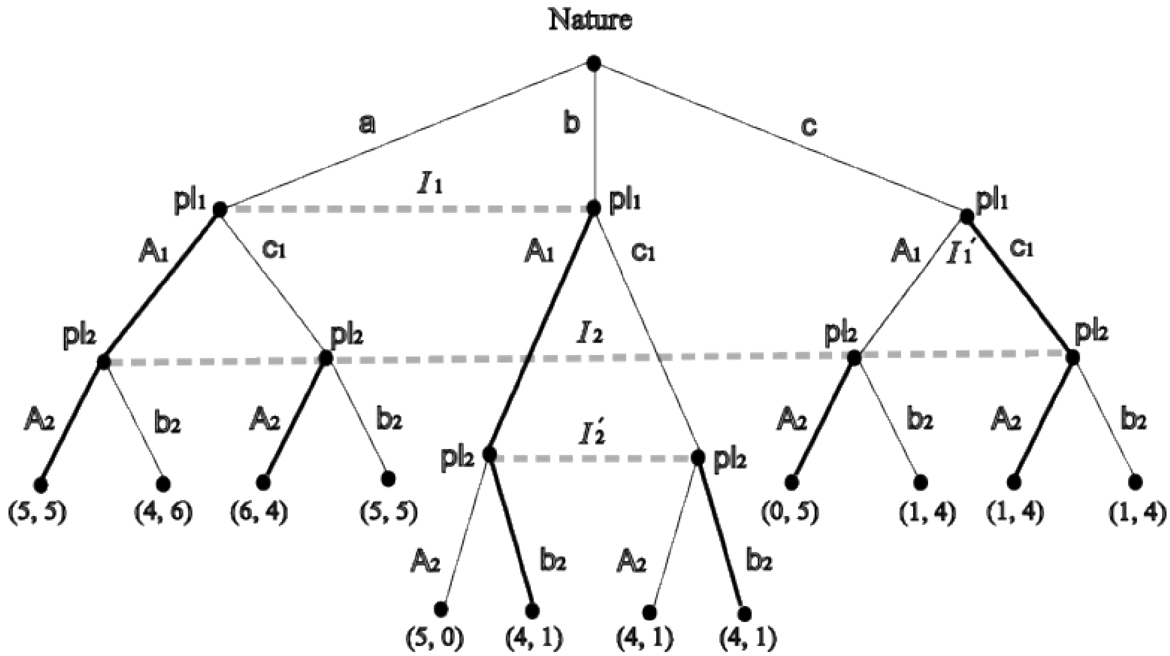
(2) $b_B \geq b_A > 0$ is not a NE because Player A can increase her payoff by reducing her bid,

(3) $b_A > b_B = 0$ is not a NE because Player A can increase her payoff by reducing her bid,

(4) $b_B > b_A = 0$ is not a NE because Player B can increase his payoff by reducing her bid.

Answer Keys for Question 5

5(a) The game is as follows (where $A_1 = \{a,b\}$, $c_1 = \{c\}$, $A_2 = \{a,c\}$, $b_2 = \{b\}$)



5(b.1) The normal form is as follows (where the strategy (x, y) for Player 1 means x if $\{a,b\}$ and y if $\{c\}$ and the strategy (z, w) for Player 2 means z if $\{a,c\}$ and w if $\{b\}$). Inside each cell the corresponding sets of outcomes are given:

		Player 2			
		$(\{a,c\},\{a,c\})$	$(\{a,c\},\{b\})$	$(\{b\},\{a,c\})$	$(\{b\},\{b\})$
	$(\{a,b\},\{a,b\})$	$\{(5,5),(5,0),(0,5)\}$	$\{(5,5),(4,1),(0,5)\}$	$\{(4,6),(5,0),(1,4)\}$	$\{(4,6),(4,1),(1,4)\}$
Pl	$(\{a,b\},\{c\})$	$\{(5,5),(5,0),(1,4)\}$	$\{(5,5),(4,1),(1,4)\}$	$\{(4,6),(5,0),(1,4)\}$	$\{(4,6),(4,1),(1,4)\}$
1	$(\{c\},\{a,b\})$	$\{(6,4),(4,1),(0,5)\}$	$\{(6,4),(4,1),(0,5)\}$	$\{(5,5),(4,1),(1,4)\}$	$\{(5,5),(4,1),(1,4)\}$
	$(\{c\},\{c\})$	$\{(6,4),(4,1),(1,4)\}$	$\{(6,4),(4,1),(1,4)\}$	$\{(5,5),(4,1),(1,4)\}$	$\{(5,5),(4,1),(1,4)\}$

Taking as payoffs the smallest sum of money in each cell (for the corresponding player) the game can be written as follows:

		Player 2			
		$(\{a,c\},\{a,c\})$	$(\{a,c\},\{b\})$	$(\{b\},\{a,c\})$	$(\{b\},\{b\})$
	$(\{a,b\},\{a,b\})$	0 , 0	0 , 1	1 , 0	1 , 1
Pl	$(\{a,b\},\{c\})$	1 , 0	1 , 1	1 , 0	1 , 1
1	$(\{c\},\{a,b\})$	0 , 1	0 , 1	1 , 1	1 , 1
	$(\{c\},\{c\})$	1 , 1	1 , 1	1 , 1	1 , 1

5(b.2) There are 9 Nash equilibria which are highlighted in red.

5(b.3) Truth telling is represented by the strategy profile $((\{a,b\},\{c\}),(\{a,c\},\{b\}))$ and it is one of the Nash equilibria.

5(c.1) No. If the state is b then it is a good idea for Player 2 to report truthfully because $\{a,c\}$ yields her 0 while $\{b\}$ yields her 1. But if the state is either a or c then, by Bayes' rule, Player 2 must assign probability $\frac{1}{2}$ to the left-most node and probability $\frac{1}{2}$ to the right-most node of her information set; thus her expected payoff from reporting $\{a,c\}$ is $\frac{1}{2}5+\frac{1}{2}4=4.5$ while the expected payoff from reporting $\{b\}$ is $\frac{1}{2}6+\frac{1}{2}4=5$.

5(c.2) "Always lie" corresponds to the strategy profile $\left((\{c\},\{a,b\}),(\{b\},\{a,c\})\right)$. By Bayes' rule the corresponding beliefs must be: for Player 1 $\left(\frac{2}{3},\frac{1}{3}\right)$ and for Player 2 $\left(0,\frac{1}{2},\frac{1}{2},0\right)$ at the top information set and (0,1) at the bottom information set. Sequential rationality is then satisfied at every information set: **for Player 1** at the information set on the left $\{c\}$ gives an expected payoff of $\frac{2}{3}5+\frac{1}{3}4=\frac{14}{3}$ while $\{a,b\}$ gives $\frac{2}{3}4+\frac{1}{3}5=\frac{13}{3}$ and at the node on the right $\{a,b\}$ gives 1 and so does $\{c\}$; **for Player 2** at the top information set $\{b\}$ gives an expected payoff of $\frac{1}{2}5+\frac{1}{2}4=4.5$ and $\{a,c\}$ gives $\frac{1}{2}4+\frac{1}{2}5=4.5$ and at the bottom information set both $\{a,c\}$ and $\{b\}$ give 1.