University of California, Davis
Department of Economics
Microeconomic Theory

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PRELIMINARY EXAMINATION FOR THE Ph.D. DEGREE
Please answer four questions (out of five)

ANSWER KEY

Question 1.

Two goods, good 1 and good 2, which is the numeraire good. There are I consumers. For $i = 1,\ldots, I$, Consumer i's utility function is $u^i : \Re_+ \times \Re \to \Re : u^i(x_1^i, x_2^i) = b^i(x_1^i) + x_2^i$, where b^i is differentiable, increasing and strictly concave, with $b^i(0) = 0$. Let $w^i > 0$ be Consumer i's wealth, understood to be sufficiently large. All consumers are price takers (and face the same price). We assume that the price of the numeraire good is equal to 1, and we denote by $p > 0$ the price of good 1. Denote by $\hat{x}_1^i(p)$ Consumer i's Walrasian demand for good 1, and by $X(p) := \sum_{i=1}^{I} \hat{x}_1^i$ the market aggregate demand for good 1.

There is a single firm which produces good 1 by using the numeraire as an input, with a differentiable cost function $C(y)$, where y is the total amount of good 1 produced. We assume in what follows that first order equalities characterize the solution to every optimization problem.

Given a price $p > 0$, we define the *markup* as $\dfrac{p - C'(X(p))}{p}$.

1.1. Given the market price p**,** write the Consumer i's optimization problem that yields her Walrasian demand for good 1. Obtain the first-order equation for this problem.

ANSWER. Given the market price p, Consumer i chooses (x_1^i, x_2^i) in order to maximize $b^i(x_1^i) + x_2^i$ subject to $x_2^i \le w^i - px_1^i$. This problem can be rewritten: choose x_1^i in order to maximize $b^i(x_1^i) + w^i - px_1^i$, with first-order equality

$$\frac{db^i}{dx_1^i} = p\,. \qquad (1.1.a)$$

1.2. Given a price p, what is Consumer i's consumer surplus? What is the aggregate consumer surplus in the economy? If prices are regulated with the objective of maximizing aggregate consumer surplus, what would be the price policy? What can you say about the resulting markup?

ANSWER. Given p, Consumer i's surplus is $b^i(\hat{x}_1^i(p)) - p\hat{x}_1^i(p)$. Its derivative with respect to p is

$$(b^i)'(\hat{x}_1^i(p)) \circ (\hat{x}_1^i)'(p)) - \hat{x}_1^i(p) - p(\hat{x}_1^i)'(p)$$
$$= -\hat{x}_1^i(p),$$

(where 1.1.a has been used), negative as long as demand is positive. (Zero demand cannot solve the consumer surplus maximization problem, since consumer surplus is zero when demand is zero, and positive consumer surplus can typically be achieved at positive demand.) Hence, consumer surplus is decreasing in p: the price policy would be to set a price "as low as possible." Often no maximum will exist, although if the derivative $(b^i)'$ becomes zero at some finite value of x_1^i (as in 1.7 below), then consumer surplus will be maximized at $p = 0$. The markup will then typically be negative.

1.3. Write the profit maximization problem. What can you say about the resulting markup?

ANSWER. The firm's profits are $pX(p) - C(X(p))$; the first-order equality for their maximization is $X(p) + pX'(p) - C'(X(p)) \circ X'(p) = 0$, i. e., $X + [p - C']X' = 0$, which can be written $\frac{p - C'}{p} = \frac{1}{-\frac{pX'}{X}} = \frac{1}{-\eta}$, where $\eta := \frac{pX'}{X} < 0$ is the elasticity of the aggregate (direct) demand function. The equality,

$$\frac{p - C'}{p} = \frac{1}{-\eta} \qquad (1.1.b)$$

often called the Lerner equation, expresses the monopoly markup $\frac{p - C'}{p}$ as the reciprocal of the (absolute value of) the elasticity of demand.

1.4. Suppose now that prices are regulated in order to maximize the sum of total consumer surplus and profits. What would be the price policy? What can you say about the resulting markup?

ANSWER. Because the firm's profits are $pX(p)-C(X(p))$, the sum of the consumer surpluses and profits is

$$\sum_{i=1}^{I} b^i(\hat{x}_1^i(p)) - pX(p) + pX(p) - C(X(p)) = \sum_{i=1}^{I} b^i(\hat{x}_1^i(p)) - C(X(p)),$$

with first order equality $\sum_{i=1}^{I} \frac{db^i}{dx_1^i}\frac{d\hat{x}_1^i}{dp} - C' X' = 0$, which using (1.1.a) can be written $pX' - C'X' = 0$. Dividing through by $X'(p)$, yields

$$p^E = C'$$

i. e., the maximization of the sum of consumer surpluses and profits implies the equality of prices and marginal costs, and, hence, zero markup. As we know, this is a condition for economic efficiency.

1.5. Let Consumer *i* own a share $\theta^i \in (0,1]$ in the profits of the firm. As a consumer, she buys the good in the market, where she is a price taker, but she can vote at the shareholders' meeting on the price that the firm will charge. What is the best price for Consumer-Shareholder *i*?

ANSWER. When voting as a shareholder, Consumer-Shareholder *i* solves the problem $\max_p b^i(\hat{x}_1^i(p)) - p\hat{x}_1^i(p) + w^i + \theta^i[pX(p) - C(X(p))]$, with first-order equality

$$\frac{db^i}{dx_1^i}\frac{d\hat{x}_1^i}{dp} - p\frac{d\hat{x}_1^i}{dp} - \hat{x}_1^i + \theta^i[pX' + X - C' X'] = 0,$$

or, using (1.1.a),

$$\hat{x}_1^i = \theta^i[X + (p - C')X']. \qquad (1.1.c)$$

1.6. Assume now that consumers indexed 1 to *N*, $N \le I$, are the shareholders of the firm (i. e., they own positive shares in the firm's profits, with $\sum_{h=1}^{N} \theta^h = 1$), whereas consumers with indices higher than *N* are not shareholders. Suppose that, at the shareholders' meeting, all shareholders unanimously agree on a price $\bar{p}$. What can you say about the resulting markup? Does it depend on the relative size of the set of shareholders? What can you say about the share θ^h of Consumer-Shareholder *h* when all shareholders agree on the price $\bar{p}$? Interpret.

ANSWER. By assumption, equation (1.1.c) holds at $\bar{p}$ for $h = 1,\ldots, N$, i. e.,

$$\hat{x}_1^h(\bar{p}) = \theta^h[X(\bar{p}) + (\bar{p} - C'(\bar{p}))X'(\bar{p})], h = 1,...,N. \qquad (1.1.d)$$

Write $X^S(\bar{p}) := \sum_{h=1}^{N} \hat{x}_1^h(\bar{p})$ for the aggregate demand by shareholders. Adding up (1.1.d) for $h = 1\ldots,N$, we obtain

$$X^S(\bar{p}) = X(\bar{p}) + [\bar{p} - C'(\bar{p})]X'(\bar{p}), \qquad (1.1.e)$$

or, dividing through by $X(\bar{p})$,

$$\frac{X^S(\bar{p})}{X(\bar{p})} = 1 + \frac{\bar{p} - C'(\bar{p})}{\bar{p}} \frac{\bar{p}X'(\bar{p})}{X(\bar{p})}.$$

Defining $\alpha(\bar{p}) := \dfrac{X^S(\bar{p})}{X(\bar{p})}$ as the share of demand by shareholders in total demand, this can be written:

$$\alpha(\bar{p}) = 1 + \frac{\bar{p} - C'(\bar{p})}{\bar{p}} \eta(\bar{p}),$$

or:

$$\frac{\bar{p} - C'(\bar{p})}{\bar{p}} = \left[\alpha(\bar{p}) - 1\right] \frac{1}{\eta(\bar{p})} \qquad (1.1.f)$$

From (1.1.f) we observe that the markup, or the gap between price and marginal cost, depends on the relative size of the shareholders set, or, more precisely, on the share of the demand by shareholders in total demand. Hence, in a sense, the degree of inefficiency is positively related to the concentration in the ownership of the firm. If ownership is very concentrated and $\alpha(\bar{p})$ is very small, then (1.1.f) looks like the Lerner equation (1.1.b), so that the price and the markup approach their monopoly values. At the other extreme, when every consumer is a shareholder, then $\alpha(\bar{p})$= 1, and the price equals the marginal cost, achieving efficiency, as in 1.4 above.

Let $N \leq I$. What can we say about the share θ^h of Consumer-Shareholder h when all shareholders agree on the price $\bar{p}$? From (1.1.c) and (1.1.e) we have that

$$\hat{x}_1^h(\bar{p}) = \theta^h X^S(\bar{p}), h = 1,..., N,$$

i. e.,

$$\theta^h = \frac{\hat{x}_1^h(\bar{p})}{X^S(\bar{p})}.$$

In other words, for unanimity to obtain, a shareholder's profit share must equal the share of her demand in the aggregate demand by all shareholders.

Intuitively, the price has two effects on a consumer-shareholder's welfare. As a consumer, she will prefer a very low price (zero, as in 1.2 above), but as a shareholder she would prefer the monopoly price. Hence, a big consumer who is a small shareholder prefers low prices, whereas a big shareholder who is a small consumer prefers high prices. When everybody is a shareholder and the shares in profits equal the shares in consumption, this two effects balance out in a way that all shareholders prefer the surplus maximizing price.

1.7. We now relax the assumption that b^i is increasing and differentiable on $\Re_+$, and specialize the model to a very simple case, where for $i = 1,\ldots, I$,

$$b^i(x_1^i) = \begin{cases} ax_1^i - \frac{1}{2}[x_1^i]^2, \text{ if } x_1^i \leq a, \\ \\ \frac{1}{2}a^2, \ \text{ if } x_1^i > a, \end{cases} ,$$

and $C(y) = cy$, with $a > c$. We assume that only a fraction $\sigma = N/I$ of the population of I consumers are shareholders in the firm, each owning a share $\theta = \frac{1}{\sigma I}$ in the firm's profits.

1.7.1 Compute the profit-maximizing price p^M.

Answer. For $p > a$, demand is zero. For $p \leq a$, individual demand is given by $p = a - x_1^i$, i. e., $\hat{x}_1^i(p) = a - p$, and aggregate demand by $X(p) = I[a-p]$. The (monopoly) profit maximizing problem is

$$\max{}_p [p-c]I[a-p]$$

with FOC: $(a-p)-(p-c)=0$, i. e., $a-2p+c=0$, or: $p^M = \frac{a+c}{2}$.

1.7.2. Show that all shareholders agree on a price $p^*(\sigma)$, and compute it. How does $p^*(\sigma)$ vary with σ? What are the limits of $p^*(\sigma)$ as $\sigma \to 0$? As $\sigma \to 1$? Comment.

Answer. Applying (1.1.c), we can write the FOC for a shareholder as

$$a - p = \frac{1}{\sigma I}[I[a-p]+[p-c][-I]].$$

Aggregating over the σI shareholders, we obtain

$$\sigma I[a-p] = I[a-p]+[p-c][-I]],$$

i. e., $\sigma a - \sigma p = a - p - p + c$,

or: $p[2-\sigma] = [1-\sigma]a + c$,

i. e.,

$$p^*(\sigma) = \frac{[1-\sigma]a + c}{2-\sigma}.$$

We compute: $\frac{dp^*(\sigma)}{d\sigma} = \frac{-a[2-\sigma]+[1-\sigma]a+c}{[2-\sigma]^2} = \frac{-2a + a\sigma + a - \sigma a + c}{[2-\sigma]^2} = \frac{-a+c}{[2-\sigma]^2} < 0$,

$\lim_{\sigma\to 0} p^*(\sigma) = \frac{a+c}{2} = p^M$, the profit maximizing price,

$\lim_{\sigma\to 1} p^*(\sigma) = c.$

Question 2.

2.1. We consider the behavior of agents that can invest in a risky asset. Agents 1 and 2 have identical (nonrandom) initial wealth of ω units of a safe asset, which asset yields one unit of the single consumption good in any state of the world. There is also a risky asset, which yields different amounts of consumption good in the various states of the world: an agent can *ex ante* exchange safe and risky asset on a one-to-one basis in the financial markets. An action by Agent i ($i = 1, 2$) is now a *portfolio*, defined by amount γ^i of the risky asset, which leaves her with amount $\omega - \gamma^i$ of the safe asset.

There are two states concerning the returns of the risky asset, the bad one and the good one. The bad state occurs with probability $\pi \in (0,1)$: in it the risky asset (gross) return rate is $v_1 < 1$ (i. e., the net return rate is then $v_1 - 1 < 0$). The good state occurs with probability $(1 - \pi)$: in it the risky asset (gross) return rate is $v_2 > 1$ (i. e., the net return rate is then $v_2 - 1 > 0$). We assume in this section that $\frac{v_2 - 1}{1 - v_1} > \frac{\pi}{1 - \pi}$.

Agents 1 and 2 are expected utility maximizers, and their vNMB utility functions, defined on an outcome space X which is an interval of the real line, are denoted u^1 and u^2, respectively. For $i = 1, 2$, u^i is twice continuously differentiable, with positive first-order derivative, denoted $(u^i)'$, and negative second order derivative, denoted $(u^i)''$.

Here and in the remaining of Question 2 we assume that the solution to any agent's optimization problem is interior to its constraint set. We also assume that there are no constraints on short sales of any asset.

2.1.1. Write Agent i's optimization problem and obtain its first-order equality.

ANSWER. A portfolio γ^i induces the following contingent consumptions for Agent i:

* $x_1^i = [\omega - \gamma^i] + \gamma^i v_1 = \omega + \gamma^i [v_1 - 1]$ in the bad state for the risky asset,
* $x_2^i = [\omega - \gamma^i] + \gamma^i v_2 = \omega + \gamma^i [v_2 - 1]$ in the good state for the risky asset.

The budget set in contingent-consumption space is therefore

$$\{(x_1^i, x_2^i) \in X: x_1^i \leq \omega + \gamma^i [v_1 - 1], x_2^i \leq \omega + \gamma^i [v_2 - 1], \text{ for some } \gamma^i \in \Re\}.$$

If $x_1{}^i = \omega + \gamma^i [v_1 - 1]$, and $x_2{}^i = \omega + \gamma^i [v_2 - 1]$, then $\gamma^i = \frac{x_1^i - \omega}{v_1 - 1} = \frac{\omega - x_1^i}{1 - v_1}$, and thus the equation of the budget line can be written $x_2^i = \omega + \frac{\omega - x_1^i}{1 - v_1}[v_2 - 1] = -\frac{v_2 - 1}{1 - v_1}x_1^i + \frac{v_2 - 1}{1 - v_1}\omega + \omega$, or

$$x_2^i = -\frac{v_2 - 1}{1 - v_1}x_1^i + \frac{v_2 - 1}{1 - v_1}\omega + \omega. \qquad (2.1.a)$$

The maximization of expected utility $\pi u^i(x_1^i) + [1 - \pi]u^i(x_2^i)$ subject to the budget constraint yields the first-order equality

$$\frac{\pi(u^i)'(x_1^i)}{[1 - \pi](u^i)'(x_2^i)} = \frac{v_2 - 1}{1 - v_1}, \qquad (2.1.b)$$

where $x_1{}^i = \omega + \gamma^i [v_1 - 1]$, and $x_2{}^i = \omega + \gamma^i [v_2 - 1]$.

2.1.2. Let $u^2(x) = \psi(u^1(x)), \forall x \in X$, for some twice continuously differentiable function ψ with positive first-order derivative ($\psi' > 0$). Recalling that $(u^i)' > 0$ and $(u^i)'' < 0$, what is the relation between the sign of its second-order derivative ψ'' and the quotient of the coefficients of absolute aversion of Agents 1 and 2, denoted $r_A(x, u^1)$ and $r_A(x, u^2)$, respectively? Prove your answer.

ANSWER. We compute, using the chain rule, $(u^2)'(x) = \psi' \cdot (u^1)'(x)$, and $(u^2)''(x) = \psi''(u^1(x)) \cdot (u^1)'(x) \cdot (u^1)'(x) + \psi'(u^1(x) \cdot (u^1)''(x)$. Hence,

$$\frac{r_A(x,u^2)}{r_A(x,u^1)} = \frac{(u^2)'' \cdot (u^1)'}{(u^2)' \cdot (u^1)''} = \frac{[\psi'' \cdot (u^1)' \cdot (u^1)' + \psi' \cdot (u^1)''] \cdot (u^1)'}{\psi' \cdot (u^1)' \cdot (u^1)''} = \frac{\psi'' \cdot (u^1)' \cdot (u^1)'}{\psi' \cdot (u^1)''} + 1.$$

Because $(u^1)'' < 0$, the sign of $\frac{\psi'' \cdot (u^1)' \cdot (u^1)'}{\psi' \cdot (u^1)''}$ is that of $-\psi''$. Hence,

$$\frac{r_A(x,u^2)}{r_A(x,u^1)} \geq 1 \Leftrightarrow \psi''(u^1(x)) \leq 0, \text{ and } \frac{r_A(x,u^2)}{r_A(x,u^1)} = 1 \Leftrightarrow \psi''(u^1(x)) = 0.$$

2.1.3. Postulate that $u^2(x) = \psi(u^1(x)), \forall x \in X$, for a twice-continuously differentiable function ψ with $\psi' > 0$ and $\psi'' < 0$. How do the investments γ^1 and γ^2 in the risky asset chosen by Agents 1 and 2, respectively, compare? Prove your answer.

ANSWER. Each agent satisfies the first-order equality (2.1.b). Hence,

$$\frac{\pi(u^1)'(x_1^1)}{[1-\pi](u^1)'(x_2^1)}=\frac{\pi(u^2)'(x_1^2)}{[1-\pi](u^2)'(x_2^2)}=\frac{\pi\psi'(u^1(x_1^2))\bullet(u^1)'(x_1^2)}{[1-\pi]\psi'(u^1(x_2^2))\bullet(u^1)'(x_2^2)},$$

where the chain rule has been used in the last equality, i. e.,

$$\frac{\pi(u^1)'(x_1^1)}{[1-\pi](u^1)'(x_2^1)}=\frac{\psi'(u^1(x_1^2))}{\psi'(u^1(x_2^2))}\bullet\frac{\pi(u^1)'(x_1^2)}{[1-\pi](u^1)'(x_2^2)}. \qquad (2.1.c)$$

Because $x_1^i = \omega + \gamma^i\,[v_1 - 1] < \omega + \gamma^i\,[v_2 - 1] = x_2^i$, we have that $u^1(x_1^2) < u^1(x_2^2)$, and hence, $\psi'(u^1(x_1^2)) > \psi'(u^1(x_2^2))$ (because $\psi'' < 0$). Hence, from (2.1.c),

$$\frac{(u^1)'(x_1^1)}{(u^1)'(x_2^1)} > \frac{(u^1)'(x_1^2)}{(u^1)'(x_2^2)}. \qquad (2.1.d)$$

If $\gamma^2 \geq \gamma^1$, then $x_2^2 = \omega + \gamma^2\,[v_2 - 1] \geq \omega + \gamma^1\,[v_2 - 1] = x_2^1$ (because $v_2 - 1 > 0$), while $x_1^2 = \omega + \gamma^2[v_1 - 1] \leq \omega + \gamma^1[v_1 - 1] = x_1^1$. Because $(u^1)'' < 0$, this would imply that

$$(u^1)'(x_2^2) \leq (u^1)'(x_2^1), \text{and } (u^1)'(x_1^2) \geq (u^1)'(x_1^1),$$

$$\text{i. e., } \frac{(u^1)'(x_1^1)}{(u^1)'(x_2^1)} \leq \frac{(u^1)'(x_1^2)}{(u^1)'(x_2^2)},$$

contradicting (2.1.d). Hence, $\gamma^2 < \gamma^1$. ■

2.2. Assume now that the initial wealth ω is random, and can take the values $\varpi+\delta_1$ or $\varpi+\delta_2$, where $\delta_1 < \delta_2$. No market exists for contracting insurance to cover the loss $\delta_2 - \delta_1$. But the market for the risky asset is open, as in Section 2.1 above: one unit of the risky asset yields the gross return of $v_1 < 1$ with probability π, and the gross return of $v_2 > 1$ with probability $1-\pi$.

We postulate two agents with preferences as in Section 2.1.3 above, i. e., $u^2(x) = \psi(u^1(x)), \forall x \in X$, for a twice-continuously differentiable function ψ with $\psi' > 0$ and $\psi'' < 0$. Agent i chooses *ex ante* (i. e., before any uncertainty is resolved) her investment γ^i in the risky asset.

In this section we assume that $\frac{v_2 - 1}{1 - v_1} < \frac{\pi}{1-\pi}$. We postulate that the (random) initial wealth and the (random) returns to the risky asset are perfectly *negatively* correlated: with probability π, the return to the risky asset is v_1 *and* the wealth of the agent is $\varpi + \delta_2$, whereas with probability $1-\pi$ the return to the risky asset is v_2 *and* the wealth of the agent is $\varpi + \delta_1$.

How do the investments γ^1 and γ^2 in the risky asset chosen by Agents 1 and 2, respectively, now compare? Argue your answer and give an intuitive explanation.

ANSWER. Again, (2.1.b) and (2.1.c) hold. But now, because $\frac{v_2 - 1}{1 - v_1} < \frac{\pi}{1-\pi}$, (2.1.b) implies that $\frac{(u^i)'(x_1^i)}{(u^i)'(x_2^i)} < 1$, which, because $(u^i)'' < 0$, implies that $x_1^i > x_2^i$, and hence that $\psi'(u^1(x_1^2)) < \psi'(u^1(x_2^2))$, i. e.,

$$\frac{(u^1)'(x_1^1)}{(u^1)'(x_2^1)} < \frac{(u^1)'(x_1^2)}{(u^1)'(x_2^2)}. \tag{2.2.a}$$

If $\gamma^2 \leq \gamma^1$, then

$$\begin{aligned} x_2^2 &= \varpi + \delta_1 + \gamma^2[v_2 - 1] \\ &\leq \varpi + \delta_1 + \gamma^1[v_2 - 1] \\ &= x_2^1, \end{aligned}$$

(because $v_2 > 1$), and

$$\begin{aligned} x_1^2 &= \varpi + \delta_2 + \gamma^2[v_1 - 1] \\ &\geq \varpi + \delta_2 + \gamma^1[v_1 - 1] \\ &= x_1^1, \end{aligned}$$

(because $v_1 < 1$). Because $(u^1)'' < 0$, this would imply that $(u^1)'(x_1^2) \leq (u^1)'(x_1^1)$ and $(u^1)'(x_2^2) \geq (u^1)'(x_2^1)$, i. e. , $\frac{(u^1)'(x_1^1)}{(u^1)'(x_2^1)} \geq \frac{(u^1)'(x_1^2)}{(u^1)'(x_2^2)}$, contradicting (2.2.a). Hence, $\gamma^2 > \gamma^1$. ■

Hence, the more risk-averse agent buys a higher amount of the risky asset! The intuition is that, because of the negative correlation between wealth and the returns to the risky asset, purchases of the risky asset offer an indirect insurance against the risk in initial wealth.

2.3. Assume now that the (random) returns to the risky asset are distributed independently from the (random) initial wealth. In particular, assume that the initial wealth is low ($\varpi + \delta_1$) with probability $\rho \in (0,1)$ and high ($\varpi + \delta_2$) with probability $1-\rho$. These probabilities are independent from those of the returns of the risky asset, which are as before, $\pi \in (0,1)$ for the low return ($v_1 < 1$) and $1-\pi$ for the high return ($v_2 > 1$). We assume in this section that $\frac{v_2 - 1}{1 - v_1} > \frac{\pi}{1-\pi}$. Again, Agent *i* (*i* = 1, 2) chooses *ex ante* (i. e., before any uncertainty is resolved) her investment γ^i in the risky asset.

2.3.1. Write Agent *i*'s optimization problem.

ANSWER. Agent *i*, with vNMB utility function u^i, chooses γ^i in order to maximize her expected utility. Note that now there are four states of the world, described, together by their probabilities, by the following table.

	Low Initial Wealth $\varpi + \delta_1$	High Initial Wealth $\varpi + \delta_2$
Low Return to Risk Asset $v_1 < 1$	$\pi\rho$	$\pi[1-\rho]$
High Return to Risk Asset $v_2 > 1$	$[1-\pi]\rho$	$[1-\pi][1-\rho]$

Agent *i* chooses γ^i in order to maximize

$$\begin{aligned}&\pi\rho u^i(\varpi + \delta_1 + \gamma^i[v_1 - 1])\\ &\quad + \pi[1-\rho]u^i(\varpi + \delta_2 + \gamma^i[v_1 - 1])\\ &\quad + [1-\pi]\rho u^i(\varpi + \delta_1 + \gamma^i[v_2 - 1])\\ &\quad + [1-\pi][1-\rho]u^i(\varpi + \delta_2 + \gamma^i[v_2 - 1]).\end{aligned} \tag{2.3.a}$$

2.3.2. Consider now a Fictional Agent who can choose the variable $\gamma \in \Re$ which defines the uncertain prospect

$$z = \left\langle \begin{array}{l} \gamma[v_1 - 1] \text{ with probability } \pi, \\ \gamma[v_2 - 1] \text{ with probability } 1\text{-}\ \pi, \end{array}\right.$$

and has expected utility preferences with vNMB utility function *V* (also denoted *V*(*z*) when making explicit its argument $z \in \Re$). What is the optimization problem of this fictional agent?

ANSWER. Fictional Agent chooses γ in order to maximize

$$\pi V(\gamma[v_1 - 1]) + [1-\pi]V(\gamma[v_2 - 1]). \tag{2.3.b}$$

2.3.3. We consider now two such fictional agents, named Fictional Agent 1 and Fictional Agent 2, with vNMB utility functions $V^1(z)$ and $V^2(z)$. Assume that the coefficient of absolute risk aversion of $V^1(z)$, $r_A(z, V^1)$, is higher than that of $V^2(z)$, $r_A(z, V^2)$. How would their investments γ^1 and γ^2 in the risky asset compare?

ANSWER. As in Section 2.1 above, the agent with the highest coefficient of absolute risk aversion chooses the smallest holdings of the risky asset, i. e., $\gamma^1 < \gamma^2$.

2.3.4. Let the vNMB utility functions of Fictional Agent 1 and Fictional Agent 2, be defined by:

$$V^i(z) = \rho u^i(z+\varpi+\delta_1) + [1-\rho]u^i(z+\varpi+\delta_2).$$

where for $i = 1, 2$, the utility function u^i is the one of 2.3.1

What is the relationship between the optimization problem of Agent i of 2.3.1 above and Fictional Agent i of this section?

ANSWER. They are identical! From (2.3.b), Fictional Agent i chooses γ^i in order to maximize $\pi V^i(\gamma^i[v_1-1]) + [1-\pi]V^i(\gamma^i[v_2-1])$, which in view of the definition of $V^i(z)$ in this section can be written:

$$\begin{aligned}
&\pi V^i(\gamma^i[v_1-1]) + [1-\pi]V^i(\gamma^i[v_2-1]) = \\
&\quad \pi\left[\rho u^i(\varpi+\delta_1+\gamma^i[v_1-1]) + [1-\rho]u^i(\varpi+\delta_2+\gamma^i[v_1-1])\right] \\
&\qquad + [1-\pi]\left[\rho u^i(\varpi+\delta_1+\gamma^i[v_2-1]) + [1-\rho]u^i(\varpi+\delta_2+\gamma^i[v_2-1])\right] \\
&\quad = \pi\rho u^i(\varpi+\delta_1+\gamma^i[v_1-1]) \\
&\quad + \pi[1-\rho]u^i(\varpi+\delta_2+\gamma^i[v_1-1]) \\
&\quad + [1-\pi]\rho u^i(\varpi+\delta_1+\gamma^i[v_2-1]) \\
&\quad + [1-\pi][1-\rho]u^i(\varpi+\delta_2+\gamma^i[v_2-1]),
\end{aligned}$$

same as in (2.3.a).

2.3.5. It turns out that examples can be constructed (involving the vNMB utility functions u^1 and u^2 of two agents, as well as the parameters $\pi, v_1, v_2, \omega, \rho, \varpi, \delta_1, \delta_2$) where $r_A(x,u^1) < r_A(x,u^2)$, yet $r_A(z,V^1) > r_A(z,V^2)$, for V^i ($i = 1, 2$) defined as in 2.3.4. In such an example:

(i) Which agent invests a higher amount of the risky asset when initial wealth is certain?

(ii) Which agent invests a higher amount of the risky asset when initial wealth is uncertain?

Explain.

ANSWER. As in the case discussed in Section 2.2, such an example displays a kind of reversal. When wealth is certain, then the analysis of Section 2.1 applies, and the more risk averse Agent 2 has a lower investment in the risky asset, i. e., $\gamma^1 > \gamma^2$. But if wealth is uncertain, the fact that $r_A(z,V^1) > r_A(z,V^2)$ implies, as analyzed in Section 2.3, that $\gamma^1 < \gamma^2$. When wealth is uncertain, what matters is the degree of risk aversion displayed by the induced function V^i, rather than that of the original vNMB utility function u^i.

Answer Key Question 3

(3a) Maximizing $K^\alpha L^{1-\alpha} - rK - wL$ implies that capital and labor inputs must satisfy

$$r = \alpha \left(\frac{L}{K}\right)^{1-\alpha}, \qquad w = (1-\alpha)\left(\frac{K}{L}\right)^{\alpha}$$

(3b) Since for typical rich agent $x_0 = e - k, x_1 = rk - m$, maximizing $\log(e-k) + \gamma \log(rk - m) + \delta \log(m)$ leads to the FOCs

$$-\frac{1}{e-k} + \frac{\gamma r}{rk - m} = 0, \qquad -\frac{\gamma}{rk-m} + \frac{\delta}{m} = 0$$

After re-arranging (k, m) solve the linear equations

$$rk(1+\gamma) - m = \gamma re, \qquad \delta rk - m(\delta + \gamma) = 0$$

with solution

$$k = \frac{e(\delta+\gamma)}{1+\gamma+\delta}, \qquad m = \frac{\delta re}{1+\gamma+\delta}$$

Both k and m increase with the endowment e, k increase with γ (utility of date 1 consumption) and δ (utility of gift which is made possible by transferring capital at date 1); m is increasing in δ but decreasing in γ since utility for date 1 consumption competes with utility for gift. The supply of capital is independent of the interest rate and equal to

$$K = \frac{n_r\, e(\delta+\gamma)}{1+\gamma+\delta}$$

(3c) The solution of the worker problem is

$$x_1 = \beta w, \qquad \ell = (1-\beta), \qquad 1 - \ell = \beta$$

The supply of labor is

$$L = n_w \beta$$

(3d) From the previous questions we deduce that the equilibrium prices for capital and labor are

$$\bar{r} = \alpha \left(\frac{n_w \beta(1+\gamma+\delta)}{n_r e(\gamma+\delta)}\right)^{1-\alpha}, \qquad \bar{w} = (1-\alpha)\left(\frac{n_r e(\gamma+\delta)}{n_w\beta(1+\gamma+\delta)}\right)^{\alpha}$$

The interest rate is increasing in α (productivity of capital), decreasing in n_r, γ and δ (supply of capital) and increasing in n_w and β (supply of labor).

The gift of the typical rich agent is

$$\bar{m} = \alpha e^\alpha \beta^{1-\alpha} \left(\frac{n_w}{n_r}\right)^{1-\alpha} \frac{\delta}{(\gamma+\delta)^{1-\alpha}(1+\gamma+\delta)^{\alpha}}$$

It is increasing in δ (the utility for gift), decreasing in γ (the competing utility for date 1 consumption) and increasing with the parameters which increase the interest rate, which provides the income from which m is drawn. The consumption of the agents are

$$\bar{x}_{0r} = \frac{e}{1+\gamma+\delta}, \qquad \bar{x}_{1r} = \alpha e^{\alpha}\beta^{1-\alpha}\left(\frac{n_w}{n_r}\right)^{1-\alpha}\frac{\gamma}{(\gamma+\delta)^{1-\alpha}(1+\gamma+\delta)^{\alpha}}$$

$$\bar{x}_w = (1-\alpha)\beta^{1-\alpha}\left(\frac{n_r}{n_w}\right)^{\alpha}\left(\frac{\gamma+\delta}{1+\gamma+\delta}\right)^{\alpha}, \quad \bar{x}_p = \frac{n_r^{\alpha}n_w^{1-\alpha}}{n_p}\frac{\alpha e^{\alpha}\beta^{1-\alpha}\delta}{(\gamma+\delta)^{1-\alpha}(1+\gamma+\delta)^{\alpha}}$$

The consumption of a rich agent increases with the endowment e. Date 0 consumption decreases with the date 1 utility coefficients γ and δ. Date 1 consumption increases with the parameter γ and decreases with the parameter δ which increases the utility of gift. It increases with the parameters which increase the interest rate (number of workers, their supply of labor, the productivity of capital) and decreases with n_r which decreases the interest rate. The consumption of the workers increase with the parameters which increase the wage ($(1-\alpha)$ which increases the productivity of labor, n^r, γ, δ which increase the amount of capital) and it increases with β, which increases the utility for consumption but decreases the wage, the first effect dominating the other. It decreases with n_w, which increases the supply of labor and thus decreases the wage. As for x_p it is proportional to m, increases with the number of rich agents who give (although this tends to decrease the interest rate and the per-capita gift) and of course decreases with the number of poor agents.

(3e) Inserting the values for $\bar{r}, \bar{k}, \bar{m}$ leads to

$$\bar{\tau} = \frac{\delta}{\gamma+\delta}$$

Part 2. In this part, a rich agent chooses k to maximize $\log(e-k) + \gamma\log(r(1-\bar{\tau})k)$, which gives the FOC

$$-\frac{1}{e-k} + \frac{\gamma}{k} = 0 \iff k = \frac{e\gamma}{1+\gamma}$$

Since the function $x \to \dfrac{\gamma+x}{1+\gamma+x}$ is increasing (take the derivative)

$$\frac{e\,\gamma}{1+\gamma} < \frac{e\,(\gamma+\delta)}{1+\gamma+\delta}$$

Thus the supply of capital is lower with taxes for 2 reasons: the rich agents have less of a reason for transferring income at date 1 since they do not have the "warm-glow" effect of giving, and the price at which they can sell capital is lower, making consumption at date 0 more attractive. Since the supply of labor is the same

- the interest rate is higher in the economy with tax (less capital)
- the wage is lower (the productivity of labor decreases if capital decreases) so that the consumption of each worker is lower.
- as for the poor it depends whether $\bar{\tau}rk$ is higher or lower than $\bar{m}$. Without calculation it is not completely obvious since r increases but k decreases. From 3(a)

$$rk = r = \alpha\left(\frac{L}{K}\right)^{1-\alpha}k = \alpha\left(\frac{L}{n_r\,k}\right)^{1-\alpha}k$$

so that rk is proportional to k^α which is increasing in k. Thus the amount taxed from each rich agent is less than the contribution $\bar{m}$ in the previous part and the consumption of the poor agents decline. This seems to suggest that it is better to leave the care of the poor to 'charities' rather than taxing capital. However this exercise does not say anything about other form of redistributive taxes which may create less distorsion than taxes on capital.

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(a) No, for FQ choosing $\ell = L$ is not a dominant strategy. For example, if $\pi > L$ then choosing $\ell = L$ yields FQ a payoff equal to *R*, while choosing an ℓ such that $L < \ell \le \pi$ yields a payoff equal to $R + \ell - L > R$.

(b) For MS choosing $\pi = P$ is a weakly dominant strategy. Proof. Fix an arbitrary ℓ. We must show that $\pi = P$ gives at least as high a payoff against ℓ as any other π. Three cases are possible. **Case 1:** $\ell < P$. In this case $\pi = P$ or any other π such that $\pi \ge \ell$ yields MS a payoff of $P - \ell > 0$, while any $\pi < \ell$ yields a payoff of 0. **Case 2:** $\ell = P$. In this case MS's payoff is zero no matter what π he chooses. **Case 3:** $\ell > P$. In this case $\pi = P$ or any other π such that $\pi < \ell$ yields a payoff of 0, while $\pi \ge \ell$ yields a payoff of $P - \ell < 0$.

(c) **Suppose that** $P > L$. If (π, ℓ) is a Nash equilibrium with $\pi \ge \ell$ then it must be that $\ell \le P$ (otherwise MS could increase its payoff by reducing π below ℓ) and it must be that $\ell \ge L$ (otherwise FQ would be better off by increasing ℓ above π). Thus it must be $L \le \ell \le P$, which is possible, given our assumption. However, it cannot be that $\pi > \ell$, because FQ would be getting a higher payoff by increasing ℓ to π. Thus it must be $\pi \le \ell$, which implies that $\pi = \ell$. Thus the following are Nash equilibria:

all the pairs (π, ℓ) with $L \le \ell \le P$ and $\pi = \ell$.

Now consider pairs (π, ℓ) with $\pi < \ell$. Then it cannot be that $\ell < P$, because MS could increase its payoff by increasing π to ℓ. Thus it must be $\ell \ge P$ (hence – by our supposition – $\ell > L$). Furthermore, it must be that $\pi \le L$ (otherwise FQ could increase its profits by reducing ℓ to, or below, π). Thus

(π, ℓ) with $\pi < \ell$ is a Nash equilibrium if and only if $\pi \le L$ and $\ell \ge P$.

(d) **Suppose that** $P < L$. For the same reasons given above, an equilibrium with $\pi \ge \ell$ requires $L \le \ell \le P$. However, this is not possible given that $P < L$. Hence,

there is no Nash equilibrium (π, ℓ) with $\pi \ge \ell$.

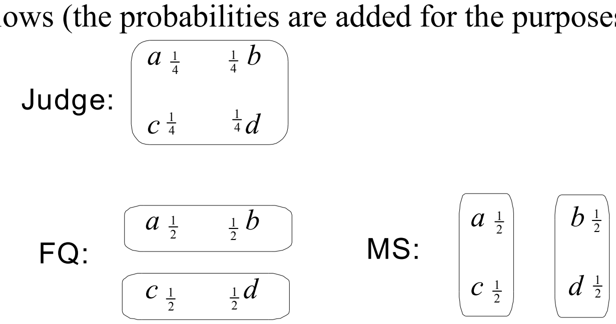
Thus we must restrict attention to pairs (π, ℓ) with $\pi < \ell$. As explained before, it must be that $\ell \ge P$ and $\pi \le L$. Thus,

(π, ℓ) with $\pi < \ell$ is a Nash equilibrium if and only if $P \le \ell$ and $\pi \le L$.

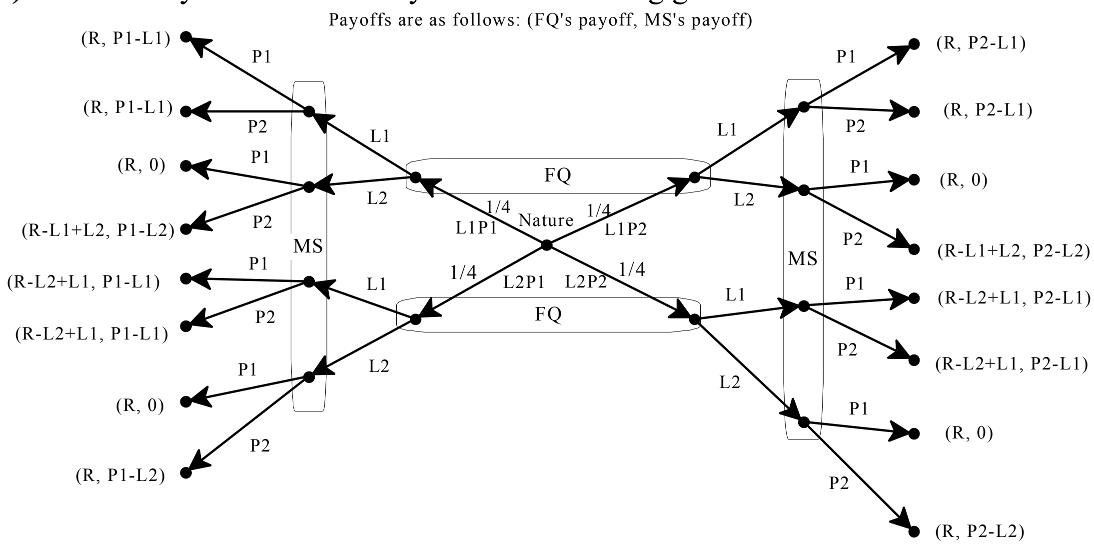
(e) Pareto efficiency requires that the discotheque be shut down if $P < L$ and that it remain open if $P > L$. Now, when $P < L$ all the equilibria have $\pi < \ell$ which leads to shut-down, hence a Pareto efficient outcome. When $P > L$, there are two types of equilibria: one where $\pi = \ell$ and the discotheque remains open (a Pareto efficient outcome) and the other where $\pi < \ell$ in which case the discotheque shuts down, yielding a Pareto inefficient outcome.

(f) **(f.1)** Let $a = (L_1, P_1)$, $b = (L_1, P_2)$, $c = (L_2, P_1)$, $d = (L_2, P_2)$. These are the states.

The partitions are as follows (the probabilities are added for the purposes of part f.2):



(f.2) The Harsanyi transformation yields the following game:



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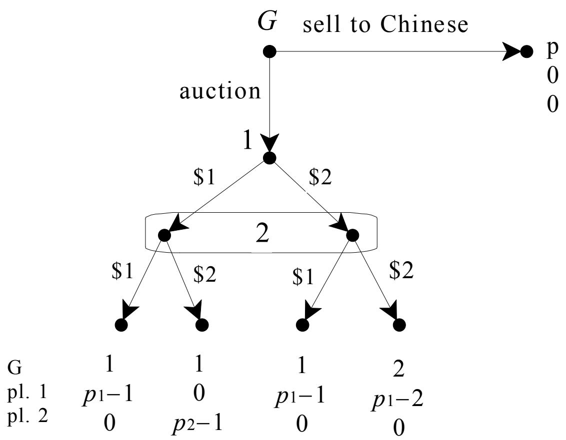
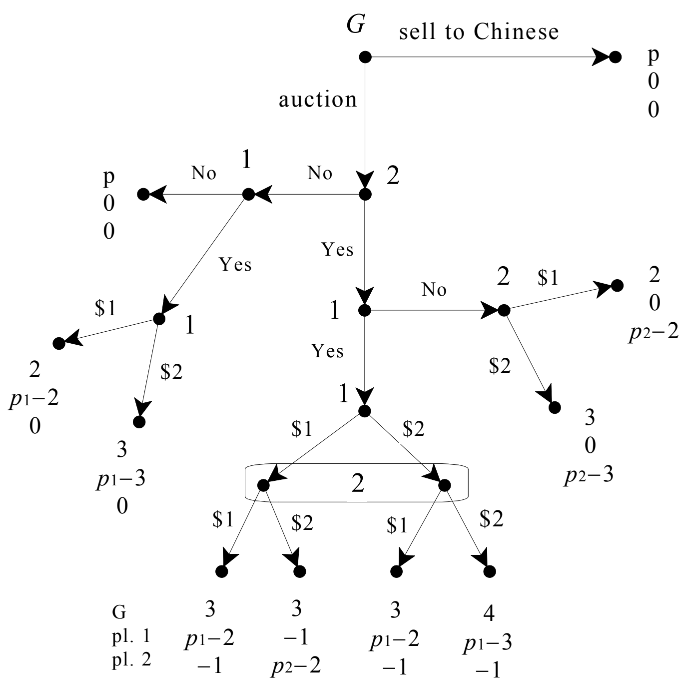
(a) The extensive form is as follows:



(b) In the auction subgame for every player it is a weakly dominant strategy to bid his own value. Let $p_j = \max_{i=1,\ldots,n}\{p_i\}$ be the highest value and $p_k = \max_{\substack{i=1,\ldots,n \\ i\neq j}}\{p_i\}$ be the second highest value. Then the auction, if it takes place, will be won by Player j and he will pay p_k. Hence there are three cases. **Case 1**: $p > p_k$. In this case Player G will sell to the Chinese (and the strategy of player i in the subgame is to bid p_i), G's payoff is p and the payoff of player i is 0. **Case 2:** $p < p_k$. In this case Player G announces the auction, the strategy of player i in the subgame is to bid p_i, the winner is player j and he pays p_k, so that the payoff of G is p_k, the payoff of player j is $p_j - p_k$ and the payoff of every other player is 0. **Case 3:** $p = p_k$. In this case there are two subgame-perfect equilibria: one as in Case 1 and the other as in Case 2 and G is indifferent between the two.

(c) The game is as follows:



(d) In the simultaneous subgame after both players have said Yes, the participation fee paid is a sunk cost and for every player bidding the true value is a weakly dominant strategy. Thus the outcome there is as follows: player 1 bids p_1, gets the palace by paying p_2, G's payoff is $(p_2 + 2)$, 1's payoff is $(p_1 - p_2 - 1)$ and 2's payoff is -1. In the subgames where one player said No and the other said Yes the optimal choice is obviously $x = 1$, with payoffs of 2 for Player G, 0 for the player who said No and $p_i - 2$ for the player who said Yes. Thus the game reduces to the one shown in Figure A:

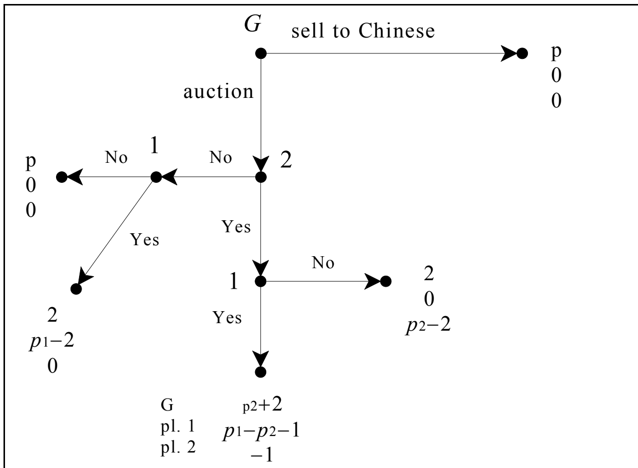


Figure A

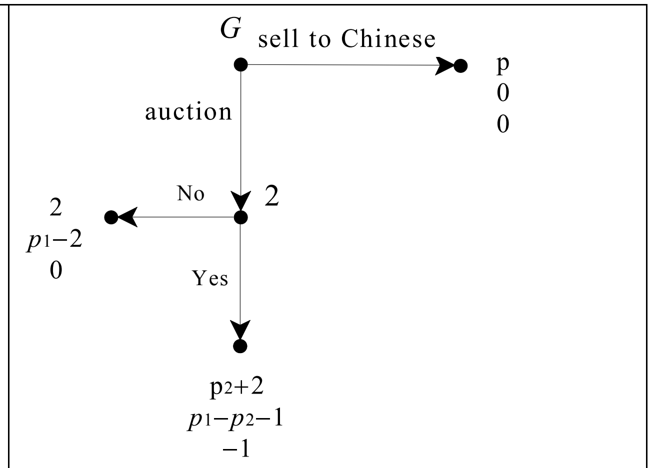


Figure B

By assumption, $p_1 > p_2 + 1 > 2$, so that $p_1 - p_2 - 1 > 0$ and $p_1 - 2 > 0$. Thus at the bottom node and at the left node player 1 prefers Yes to No. Thus the game reduces to the one shown in Figure B.
Hence player 2 will say No. Thus the subgame-perfect equilibrium is as follows:
(1) if $p > 2$ then player G will sell to the dealer (and the choices off the equilibrium path are as explained above) and the payoffs are $(p, 0, 0)$;
(2) if $p < 2$ then G chooses to auction, 2 says No, 1 says Yes and then offers \$1 and the payoffs are $(2, p_1 - 2, 0)$ (and the choices off the equilibrium path are as explained above);
(3) if $p = 2$ then there are two equilibria: one as in (1) and the other as in (2).

(e) When the loser is given the fraction a of the amount paid by the winner (that is, the loser is given the fraction a of his own bid), **it is no longer true that bidding one's true value is a dominant strategy**. In fact, (p_1, p_2) is not even a Nash equilibrium any more. To see this, imagine that 1's true value is 10 and 2's true value is 6 and $a = 50\%$. Then if player 1 bids 10 and 2 bids 6, player 2 ends up losing the auction but being given \$3, while if he increased his bid to 8 then he would still lose the auction but receive \$4. This shows that there cannot be a Nash equilibrium where player 2 bids less than player 1. Now there are several Nash equilibria of the auction, for example, all pairs (b_1, b_2) with $b_1 = b_2 = b$ and $p_2 \le b < p_1$ provided that $p_1 - b \ge a(b-1)$, that is, $b \le \dfrac{p_1 + a}{1 + a}$ (but there are more: for example all pairs (b_1, b_2) with $b_1 = b_2 = b$ and $b < p_2$ provided that $p_1 - b \ge a(b-1)$ and $ab \ge p_2 - b$). Thus to find a subgame-perfect equilibrium of the game one first has to select a Nash equilibrium of the auction game and then apply backward induction to see if the players would want to say Yes or No to the auction, etc.