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Microeconomics

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PRELIMINARY EXAMINATION FOR THE Ph.D. DEGREE

Please answer any four questions

PART A. Question 1. Ambiguity aversion and Insurance

Assume that nature randomly chooses one of two "states or the world," numbered 1 and 2, where state 1 is interpreted as the bad state, involving a loss. Two experts, named *G* and *R*, have estimated the probabilities of the two states. Expert *G* (for "Gloomy") estimates the probability of state 1 as $\pi^G \in (0, 1)$ (and that of state 2 as $1 - \pi^G$), whereas Expert *R* (for "Rosy") estimates the probability of state 1 as
$\pi^R \in (0, 1)$ (and that of state 2 as $1 - \pi^R$). We postulate that

$$\pi^G > \pi^R \ .$$

We also postulate that Expert *G* is correct with probability $\mu \in (0, 1)$, and Expert *R* with probability $1 - \mu$.

The outcome set is $\Re_+$, i. e., there is a single good, interpreted as "consumption." We will alternatively consider decision maker (DM) Leonard or Frank. Leonard and Frank will be endowed with the same utility function over outcomes $u: \Re_+ \to \Re$, concave, strictly increasing and twice differentiable. Denote by x_i the DM's consumption contingent to state i ($i = 1, 2$).

In the absence of insurance, the DM ends up with the level of consumption ω_1 in state 1 and ω_2 in state 2, where $\omega_2 - \omega_1 > 0$ is the amount of the loss. The insurance market offers contracts with variable *coverage* $\alpha \in [0, \omega_2 - \omega_1]$, and linear *premia*, i. e., the premium per unit of coverage is q, which the DM takes as given while choosing α (hence the total premium is $q\alpha$). Leonard and Frank will face the same q and will have the same endowments.

Given q, an amount of coverage α yields the contingent consumption vector
$(x_1, x_2) = (\omega_1 + \alpha(1 - q), \omega_2 - q\alpha)$. The premium rate q and the endowments (ω_1, ω_2) induce a budget equality of the form $P_1x_1 + P_2x_2 = W$, where $W = P_1\omega_1 + P_2\omega_2$.

Leonard and Frank differ in the following respect. Leonard combines the probability estimates of the two experts, and his objective function is

$$U^L(x_1, x_2) \equiv \beta u(x_1) + (1 - \beta)u(x_2), \tag{1}$$

where $$\beta \equiv \mu\pi^G + (1 - \mu)\pi^R, \tag{2}$$

and, accordingly, $$1 - \beta = \mu(1 - \pi^G) + (1 - \mu)(1 - \pi^R).$$

Frank's objective function is, instead

$$U^F(x_1, x_2) \equiv \mu\varphi(\pi^G u(x_1) + (1 - \pi^G)u(x_2)) + (1 - \mu)\,\varphi(\pi^R u(x_1) + (1 - \pi^R)u(x_2)), \tag{3}$$

where $\varphi: \Re \rightarrow \Re$ is a strictly increasing, twice differentiable, concave function. Accordingly, U^F is a concave function.

1(a). Consider a given vector $(x_1, x_2) \in \Re^2_{++}$. Find, by implicit differentiation, expressions for the marginal rates of substitution of Leonard and of Frank at (x_1, x_2). Show that Leonard and Frank have the same marginal rates of substitution at any point (x_1, x_2) satisfying $x_1 = x_2$ (i. e., any point on the *certainty line.*)

1(b). Assume that φ is affine, but that u is strictly concave. Do Leonard and Frank always make the same choice? If NO, who buys more insurance when they make a different choice? Prove your answer.

1(c). Assume now that φ is strictly concave. Do Leonard and Frank always make the same choice? If NO, who buys more insurance when they make different choices? Does the answer depend on whether u is affine or strictly concave? Prove your answer.

We assume, for the remainder of this question, that both φ and u are strictly concave. The strict concavity of φ has been interpreted in the literature as (strict) *ambiguity aversion*, in a manner parallel to the interpretation of the strict concavity of u as (strict) *risk aversion*. Similarly, DM is *ambiguity neutral* when φ is affine.

1(d). Let the insurance premium rate be $q = \beta$ (β as defined by (2)). Do Leonard and Frank always make the same choice? If NO, who buys more insurance when they make different choices? Prove and graphically illustrate your answer.

1(e). Assume now that $q > \beta$. Do Leonard and Frank always make the same choice? If NO, who buys more insurance when they make different choices? Prove and graphically illustrate your answer.

1(f). Comment on your results.

Part B. Question 2

2. Consider an exchange economy with I agents and L goods. Each agent ($i = 1, \dots, I$) has an endowment $\omega^i \in \mathbb{R}^L_{++}$ and a utility function $u^i : \mathbb{R}^L_+ \to \mathbb{R}$ which is strictly quasi-concave, increasing, differentiable in $\mathbb{R}^L_{++}$ and which satisfies the Inada conditions: $\partial u^i(x^i)/\partial x^i_\ell \to \infty$ if $x^i_\ell \to 0$. Let $\bar{\omega} = \sum_{i=1}^I \omega^i \gg 0$ be the aggregate endowment.

(a) Use the Kuhn-Tucker theorem to show that if $\bar{x} = (\bar{x}^1, \dots, \bar{x}^I)$ is a Pareto optimal allocation then there is a vector of prices $\bar{p}$ which supports the preferred set of each agent i at $\bar{x}^i$ ($i = 1, \dots, I$).

(b) Suppose that in addition to the assumptions above the agents have identical homothetic preferences, i.e. the utility functions u^i are identical ($u^i = u$) and homogeneous of degree 1. Let $p \in \mathbb{R}^L_{++}$ be a price vector, $(v_1, \dots, v_I) \in \mathbb{R}^I$ be a family of attainable utility levels for the agents ($v_i > u^i(0)$), and let $(\bar{x}^1, \dots, \bar{x}^I) \in \mathbb{R}^{LI}_{++}$ denote the family of solutions of the expenditure minimization problems

$$\min\{px^i \mid u(x^i) \geq v_i\}, \qquad i = 1, \dots, I$$

Show that for all $i, j = 1, \dots, I$, $\bar{x}^i = \dfrac{v_i}{v_j}\bar{x}^j$.

(c) Deduce that under the assumptions of question (b) all Pareto optimal allocations are supported by the same price vector. Express this vector as a function of the common utility function u and the aggregate endowment $\bar{\omega}$.

(d) Application: suppose there are 3 agents and 3 goods. The agents have the same preferences represented by the utility function

$$u(x) = \Big(\sum_{\ell=1}^{3}(x_\ell)^{-2}\Big)^{-\frac{1}{2}}$$

and endowments

$$\omega^1 = (1, 0, 1), \quad \omega^2 = (2, 1, 1), \quad \omega^3 = (1, 1, 2)$$

Normalize prices by $p_1 = 1$. Find the vector of prices and the transfers that would be needed to generate the equilibrium with transfers which gives the same consumption bundle to each agent.

Part B. Question 3

3. Consider an economy with a private good and a public good. The economy consists of two consumers whose utility functions are given by

$$u_1(x_1, y) = \frac{1}{2}\log(x_1) + \frac{1}{2}\log(y), \qquad u_2(x_2, y) = \frac{1}{3}\log(x_1) + \frac{2}{3}\log(y)$$

Consumer 1 is endowed with 4 units of private good and consumer 2 with 2 units of private good. The public good can be produced from the private good according to the linear technology $y = z$ where $z \geq 0$ is the number of units of private goods used as inputs. The public good that we are considering is such that it is technologically possible to exclude agents from consuming it so that, if desired, the public good can be sold as an ordinary good (think of TV broadcast sold by subscription or tolls on roads). To (partially) solve the problem of financing, a benevolent planner decides to take over the production of the public good and finance its production by making agents pay for its consumption. The planner has not found a way of knowing the agents' preferences, so he/she charges the same price p to both agents for each unit of public good consumed (the price of the private good is normalized to 1). Whatever the price, an agent cannot consume more than the total quantity of public good produced, which is decided by the planner.

(a) Derive the constrained demand function $(x_i(p, \bar{y}), y_i(p, \bar{y}))$ of agent i $(i = 1, 2)$ when charged a price p per unit of public good consumed and constrained to a maximum consumption $\bar{y}$. Carefully indicate the prices for which the constraint $y_i \leq \bar{y}$ is, or is not, binding.

(b) An equilibrium with exclusion is a pair $\left((\bar{x}_i, \bar{y}_i)_{i=1,2}, (\bar{p}, \bar{z}, \bar{y})\right)$ for the agents and the planner such that the agents optimize as in (a), the firm producing the amount $\bar{y}$ of public good (known by all agents) and selling it at price $\bar{p}$ breaks even, and the allocation is feasible. Show that there are potentially 3 types of equilibria with exclusion: (i) the constraint $y \leq \bar{y}$ is binding for both agents; (ii) the constraint $y \leq \bar{y}$ is binding for agent 1 and not for agent 2; (iii) the constraint $y \leq \bar{y}$ is not binding for any agent; Write all the equalities/inequalities which have to be satisfied in each case.

(c) Study the equilibria of type (i). Show that they all have the same price $\bar{p}$ for the public good. Show that the one with the highest production of public good Pareto dominates the others.

(d) Study the equilibria of type (iii). Show that they all have the same level of public good $\bar{y}$. Show that the equilibrium with the smallest price Pareto dominates the others. Calculate this price.

(e) Study the equilibria of type (ii). Show that the possible equilibrium prices are in an interval $[p_{min}, p^{max}]$ and compare p_{min} and p^{max} with the results of questions (c) and (d). Show that the equilibrium with the cheapest price for the public good dominates in the Pareto sense all the other equilibria (if you are short of time only check it dominates the equilibrium with the maximum price).

(f) Deduce from question (c)-(e) what is the best choice $(\bar{p}, \bar{y})$ for the benevolent planner. Explain intuitively why it is the best solution to the public good problem that can be reached by charging a uniform price for the public good.

PART C. Question 4.

Consider the problem of allocating an indivisible object. We shall restrict attention to the case where there are only two agents: *A* and *B*. The von Neumann-Morgenstern utility function of a player with valuation *v* for the object is as follows:

$$U(m,p)=\begin{cases} v+m-p & \text{if she gets the object and receives \$}m\text{ and pays \$}p \\ m-p & \text{if she does not get the object, receives \$}m\text{ and pays \$}p \end{cases}$$

(a) We start with a simple case where it is common knowledge that Agent *A* has valuation *a* and agent *B* has valuation *b*, with $0 < a < b$. The two agents play the following game. Each agent can say either "give it to me" or "give it to her". If both say "to me" then each gets nothing. If one says "to me" and the other says "to her" then the object is given for free to the one who said "to me", while the other gets nothing. If both say "to her" then nobody gets the object, but each receives a sum of money equal to $\frac{a+b}{2}$.

- **(a.1)** Does any player have a weakly/strictly dominant strategy?
- **(a.2)** What are the pure-strategy Nash equilibria of this game?

(b) Let us now change the above game slightly as follows: when both say "to me", each has to pay \$*d* (to a third party), where $d > 0$.

- **(b.1)** For each player state whether she has a weakly/strictly dominant strategy.
- **(b.2)** What are the Nash equilibria of this game, including the mixed strategy equilibria (if any)?

(c) Now consider the following incomplete-information situation. It is common knowledge between *A* and *B* that (1) there are three possible valuations: 10, 20 and 25, (2) the two agents have different valuations, (3) each agent knows her own valuation but is uncertain about the valuation of the other agent and (4) when uncertain among several possibilities, each agent assigns equal probability to each possibility she has in mind. *Please note that we are no longer assuming that player A has a lower valuation than player B* (it could be higher or lower).

- **(c.1)** Use a set of states and information partitions to represent the possible states of knowledge of the two agents.
- **(c.2)** Let *E* be the event that agent *A* has a higher valuation than agent *B*. Find the events (1) $K_A E$, (2) $K_B E$, (3) $K_A K_B E$ and (4) $K_B K_A E$.

A and *B* play the following game. Each agent can say either "give it to me" or "give it to her". If both say "to me" then each agent gets nothing. If one says "to me" and the other says "to her" then the object is given for free to the one who said "to me" and the other agent gets nothing. If both say "to her", then the two valuations are revealed to both agents (they become common knowledge) and the agents participate in a *second-price* sealed-bid auction for the object (if the two bids are equal, then agent *A* is the winner). *The only bids that are allowed in the auction are 10, 20 and 25* and the payment goes to a third party.

- **(c.3)** For the case where the valuation of *A* is 25 and the valuation of *B* is 20, write the normal (or strategic) form of the auction and find the dominant-strategy equilibria.
- **(c.4)** Apply the Harsanyi transformation and draw an extensive-form game that represents this incomplete-information game. To simplify the representation, replace each auction with the payoffs associated with the "truthful" bids.
- **(c.5)** **(c.5.1)** In the game of part (c.4) find a weakly dominant strategy for player *B*.
  - **(c.5.2)** Find all the pure-strategy Nash equilibria at which player *B* chooses the weakly dominant strategy that you have just found in part (c.5.1).
  - **(c.5.3)** How many pure-strategy Nash equilibria are there where player *B* chooses the weakly dominant strategy that you have just found in part (c.5.1)?

PART C. Question 5. There are two types of individuals: type 1 and type 2. They all derive utility from *m* (money) and *t* (training). Assume throughout that $t \in [0,12]$. The utility functions are:

$$\text{for type 1:} \quad u_1(m,t) = 100 + 10m - 4t^2$$
$$\text{for type 2:} \quad u_2(m,t) = 100 + 10m - 2t^2$$

Training affects productivity (denoted by μ) as follows.

$$\text{for type 1:} \quad \mu_1(t) = 8 + 2t$$
$$\text{for type 2:} \quad \mu_2(t) = 8 + 2.5t$$

(a) Derive the equations of the indifference curves of the two types that go through the point $m = 10$, $t = 2$.

(b) Suppose first that employers are able tell the two types apart and offer to pay each individual an amount of money *m* equal to the person's productivity. For each type calculate the amount of training that the individual chooses, the amount of money she is offered and her level of utility.

(c) From now on assume that employers **cannot** tell whether job applicants are of type 1 or type 2. However, they can observe their level of training. Suppose that employers offer to pay new hires according to the following rule: if your level of training is *t* then I will pay you *m(t)* where $m(t) = \begin{cases} 8+2t & \text{if } t < t^* \\ 8+2.5t & \text{if } t \geq t^* \end{cases}$. For each of the following cases either calculate a separating equilibrium (and prove that it is indeed an equilibrium) or show that a separating equilibrium does not exist. *As part of the definition of separating equilibrium we require that each worker is paid an amount that is equal to his actual productivity*.

(c.1) $t^* = 4$**;** **(c.2)** $t^* = 9$**;** **(c.3)** $t^* = 11$**.**

(d) Suppose now that employers offer to pay new hires according to the following rule: "if your level of training is *t* then I will pay you *m(t)* as follows: if $t < 4$, *m(t)* is the productivity of a type 1 with training *t* and if $t \geq 4$ then $m(t)$ is the *average* productivity of the two types with training *t* computed using the proportion of type 1 (q) and of type 2 ($1-q$) in the population [where $0 < q < 1$]."

(d.1) For every value of $q \in (0,1)$ calculate the optimal choice of *t* for both types.

(d.2) Is there a value of *q* at which there is a pooling equilibrium in this case? [Note: we no longer require that each type be paid his true productivity: at a pooling equilibrium they are all paid the same amount.]