# PRELIMINARY EXAMINATION FOR THE Ph.D. DEGREE

## Answer FOUR questions

### Question 1.

A consumer with wealth $w$ buys goods 1 and 2 in the market, at prices $p_1$ and $p_2$, facing the budget constraint

$$p_1x_1 + p_2x_2 \leq w. \qquad (1)$$

The twist is that her consumption of good 2 creates a positive externality from which she, and everybody else in society, benefits. Conversely, she benefits from the positive externality generated by every other consumer. Formally, her utility function, assumed to be differentiable, concave and with strictly positive gradient in the interior of its domain, is written $u(x_1, x_2, E)$, with the three arguments $x_1$ (the amount of good 1 that she consumes, $x_2$ (the amount of good 2 that she consumes) and $E$, the total amount of the externality generated, to be also called the *level of the public good*. The dependence of $E$ on both her consumption $x_2$ of good 2 and the amount $E_{-i}$ of positive externality aggregately generated by everybody else is defined by

$$E \leq E_{-i} + g\, x_2, \qquad (2)$$

where $g > 0$ is a parameter.

Our consumer treats $E_{-i}$ (as well as $p_1$, $p_2$ and $w$) parametrically. The maximization of her utility function subject to the constraints (1) and (2) yields her market demand functions $\hat{x}_j(p_1, p_2, w, E_{-i}),\ j = 1,2$ for goods 1 and 2, as well as her desired amount of the public good $\hat{E}_j(p_1, p_2, w, E_{-i})$.

**1(a).** Write the maximization problem the solution of which is $(\hat{x}_1(p_1, p_2, w, E_{-i}), \hat{x}_2(p_1, p_2, w, E_{-i}), \hat{E}(p_1, p_2, w, E_{-i}))$, and call it Problem *PG*. Write its Kuhn-Tucker conditions. Assuming that the solution is interior and unique, and that constraints (1) and (2) are satisfied with equality, write a system of equations, not involving the multipliers, the solution of which defines $(\hat{x}_1(p_1, p_2, w, E_{-i}), \hat{x}_2(p_1, p_2, w, E_{-i}), \hat{E}(p_1, p_2, w, E_{-i}))$.
(Just write the system of equations: there is no need to work towards its solution.)

**1(b).** For the rest of this part we consider the utility function

$$u : \Re \times \Re^2_+ : u(x_1, x_2, E) = x_1 + [a_2, a_E]\begin{bmatrix} x_2 \\ E \end{bmatrix} - \frac{1}{2}[x_2, E]B\begin{bmatrix} x_2 \\ E \end{bmatrix},$$

where $(a_2, a_E) >> 0$ and $B \equiv \begin{bmatrix} b_{22} & b_{2E} \\ b_{E2} & b_{EE} \end{bmatrix}$ is a symmetric, positive definite matrix. We are interested in the comparative statics expressed by the partial derivatives $\frac{\partial \hat{x}_2}{\partial E_{-i}}$ and $\frac{\partial \hat{E}}{\partial E_{-i}}$. Again, assume that the solutions are interior and the constraints satisfied with equality.

**1(b) (i).** Specialize to this utility function the system of equations obtained in 1(a).

**1(b) (ii).** Verbally interpret the partial derivatives $\frac{\partial \hat{x}_2}{\partial E_{-i}}$ and $\frac{\partial \hat{E}}{\partial E_{-i}}$, as well as their signs. Compute $\frac{\partial \hat{x}_2}{\partial E_{-i}}$ and $\frac{\partial \hat{E}}{\partial E_{-i}}$.

**1(b) (iii).** Let $b_{2E} \geq 0$. Find the signs of $\frac{\partial \hat{x}_2}{\partial E_{-i}}$ and of $\frac{\partial \hat{E}}{\partial E_{-i}}$.

**1(b) (iv).** Let $b_{2E} < 0$. Argue that one cannot determine, without additional assumptions on the parameters, the signs of $\frac{\partial \hat{x}_2}{\partial E_{-i}}$ and of $\frac{\partial \hat{E}}{\partial E_{-i}}$. Find additional assumptions on the parameters guaranteeing determinate signs for $\frac{\partial \hat{x}_2}{\partial E_{-i}}$ and $\frac{\partial \hat{E}}{\partial E_{-i}}$.

**1(c).** We continue with the utility function of 1(b) above. Our objective is now to interpret the conditions analyzed in 1(b) (iii)-(iv) in terms of the net complementarity-substitutability relation between good 2 and the public good $E$. Because these concepts are based on the entries of the Slutsky equation, we must first obtain the Walrasian demand functions $(\tilde{x}_1(\overline{p}_1, \overline{p}_2, \overline{p}_E, \overline{w}), \tilde{x}_2(\overline{p}_1, \overline{p}_2, \overline{p}_E, \overline{w}), \tilde{E}(\overline{p}_1, \overline{p}_2, \overline{p}_E, \overline{w}))$ defined by the maximization of $u$ subject to the budget constraint $\overline{p}_1 x_1 + \overline{p}_2 x_2 + \overline{p}_E E \leq \overline{w}$ (as if the consumer, endowed with wealth $\overline{w}$, bought the amount $E$ at the "market price" $\overline{p}_E$ together with the amounts $x_1$ and $x_2$ at the market prices $\overline{p}_1$ and $\overline{p}_2$, respectively). Call this program Problem $W$.

**1(c) (i).** Compute $(\tilde{x}_2(\overline{p}_1, \overline{p}_2, \overline{p}_E, \overline{w}), \tilde{E}(\overline{p}_1, \overline{p}_2, \overline{p}_E, \overline{w}))$.

**1(c) (ii).** Denote by $\begin{bmatrix} s_{11} & s_{12} & s_{1E} \\ s_{21} & s_{22} & s_{2E} \\ s_{E1} & s_{E2} & s_{EE} \end{bmatrix}$ the Slutsky matrix corresponding to this Walrasian demand. Compute the submatrix $\begin{bmatrix} s_{22} & s_{2E} \\ s_{E2} & s_{EE} \end{bmatrix}$.

**1(c) (iii).** Interpret the sign of $b_{2E}$ in terms of whether good 2 and good $E$ are net complements or net substitutes.

**1(c) (iv)**. Use 1(c) (iii) to verbally discuss the conditions analyzed in 1(b) (iii)-(iv).

**Question 2.**

Consider an exchange economy $\mathcal{E}\left((u^i, \omega^i)_{i=1}^I\right)$ with $I$ agents and $L$ goods, where the agents' utility functions $u^i$ are continuous and increasing in all components, and the endowments $\omega^i$ are such that $\sum_{i=1}^I \omega^i \gg 0$. We want to provide a characterization of the Pareto optimal allocations as allocations where the benefits from trade have been exhausted. To do this, we choose a bundle of goods $g \in \mathbb{R}_+^L$, $g \neq 0$, and we define the *benefit function* of each agent (in units of the bundle $g$) as follows: if $x^i \in \mathbb{R}_+^L$ is a consumption vector for agent $i$ and if $v^i \geq u^i(0)$ is a utility level, the benefit $b^i(x^i, v^i)$ of $x^i$ over $v^i$ is defined by

$$b^i(x^i, v^i) = \max\{\beta \in \mathbb{R} \mid u^i(x^i - \beta g) \geq v^i\}$$

**(a)** Interpret $b^i(x^i, v^i)$ and give a geometric representation of it in a two-good economy. Show that $u^i(x^i) \geq v^i$ implies $b^i(x^i, v^i) \geq 0$, with strict inequality if $u^i(x^i) > v^i$.

**(b)** Consider a feasible allocation $x = (x^1, \ldots, x^I) \in \mathbb{R}_+^{LI}$ (i.e. such that $\sum_i x^i \leq \sum_i \omega^i$), and let $v^i = u^i(x^i)$. Suppose that there is a feasible allocation $\tilde{x}$ such that

$$\sum_{i=1}^I b^i(\tilde{x}^i, v^i) > 0$$

Show that the allocation $x$ is not Pareto optimal.

**(c)** Illustrate the result of (b) in the Edgeworth box, taking a case where $b^1(\tilde{x}^1, v^1) > 0,\ b^2(\tilde{x}^2, v^2) < 0.$

**(d)** Using (b), prove the following result, often referred to as the "zero-maximum principle": If $x^*$ is a Pareto optimal allocation, and $v^{i*} = u^i(x^{i*})$, then $x^*$ is the solution of

$$\max\left\{ \sum_{i=1}^I b^i(x^i, v^{i*}) \;\middle|\; x \in \mathbb{R}_+^L,\ \sum_{i=1}^I x^i \leq \sum_{i=1}^I \omega^i \right\} \qquad (1)$$

Show that the value of the maximum in (1) is then equal to zero, and interpret the result.

**(e)** Prove the converse property: if $x^*$ is a solution to (1) then $x^*$ is a Pareto optimal allocation.

**(f)** Show that if, in addition to satisfying the assumptions made above, the utility function $u^i$ is quasi-concave, then the benefit function $b^i(x^i, v^i)$ is concave in $x^i$, so that the solutions of the maximum problem (1) are characterized by the first-order conditions (this question does not require calculating the first-order conditions).

**Question 3.**

Consider an economy in which agents are averse to inequality. More precisely the utility function of agent $i$ ( $1 \le i \le I$ ) is of the form $u_i(x_i, m)$, where $x_i$ is agent $i$ 's consumption and $m$ is the minimum consumption in the economy, i.e. $m = \min\{x_1, x_2, \ldots, x_I\}$. To simplify the analysis we assume that there are only two types of agents: all rich agents have endowment $\omega_R$, and there are $n_R$ of them; all poor agents have endowment $\omega_P < \omega_R$ and there are $n_P$ of them, with $n_P > n_R > 1$. All the agents have the same utility function $u(x, m) = \ln(x) + \gamma \ln(m)$ with $\gamma < 1$. In order to improve on the unequal distribution of income, the rich agents decide to create a charity, which receives the gifts of the agents who are willing to contribute, and redistributes the proceed so as to maximize $m$. The consumption of each agent is thus of the form $x_i = \omega_i - z_i + t_i$, where $z_i \ge 0$ is the gift to the charity and $t_i \ge 0$ is the transfer received from the charity.

**(a)** Let us first study the optimal redistribution of the charity. Suppose the charity has $Z$ to distribute and let $(\bar{t}_i)_{i \in I}$ be an optimal redistribution of $Z$. Let $(\bar{x}_i)_{i \in I}$ be the resulting consumption allocation.

(i) Show that if $\bar{t}_i > 0$ then $\bar{x}_i = m$.

(ii) Consider the case where each rich agent contributes $z_R$, each poor agent contributes $z_P$, $\omega_R - z_R > \omega_P - z_P$, and the charity has $Z$ to distribute. Find $t_R(Z)$, $t_P(Z)$, $m(Z)$, the optimal transfer to a rich agent, to a poor agent, and the minimum consumption in the economy. Show that these functions are differentiable, except when $Z = n_P\left[(\omega_R - z_R) - (\omega_P - z_P)\right]$, in which case the right and left derivatives of the functions are different.

**(b)** An equilibrium $(\bar{x}_i, \bar{z}_i, \bar{t}_i)$ with voluntary contributions to the charity is such that each agent chooses his/her optimal gift, given that the charity redistributes any amount $Z$ of income that it receives so as to maximize $m$.

(i) Show that in an equilibrium with voluntary contributions an agent whose consumption is equal to the minimum consumption does not contribute to the charity. [Small hint: look at the marginal cost and the marginal benefit of a change in the contribution of such an agent at equilibrium. ]

(ii) Show that an equilibrium with voluntary contributions in which identical agents are treated equally is of one of two types: either all the contributions are zero and agents consume their endowment (equilibrium with no redistribution), or rich agents contribute and poor agents receive a transfer (positive redistribution). Find the condition on the parameter values $(\omega_R, n_R, \omega_P, n_P, \gamma)$ which implies that the equilibrium has no redistribution.

(iii) In the case where the equilibrium has no redistribution (which corresponds to the most intuitive values of the parameters) show that, if $\gamma n_R \omega_R > n_P \omega_P$, then the equilibrium is not Pareto optimal. (You are not required to find the Pareto optimal allocations. Just show that it is feasible to marginally improve all the agents.)

**(c)** The rich people realize that the charity system with voluntary contributions does not function properly, so they decide to change the system. Instead of making voluntary gifts, they accept to pay a percentage of their income to the charity and put some representatives (of the rich agents) in charge of choosing this percentage. Find the percentage that will be chosen, and show that, as long as $\gamma n_R \omega_R > n_P \omega_P$, there will be a positive redistribution of the rich to the poor. Explain why this system works better than the gift system.

# Question 4.

Consider the market for second-hand cars. There are four possible qualities with the corresponding proportions, values to seller and buyer indicated in the table below, with $0 < p_i < 1$ and $b_i > s_i \geq 0$, for all $i \in Q = \{A, B, C, D\}$, and $\sum_{i \in Q} p_i = 1$. Furthermore, $s_A > s_B > s_C > s_D$.

| Quality of car | A | B | C | D |
|---|---|---|---|---|
| proportion | $p_A$ | $p_B$ | $p_C$ | $p_D$ |
| value to seller | $s_A$ | $s_B$ | $s_C$ | $s_D$ |
| value to buyer | $b_A$ | $b_B$ | $b_C$ | $b_D$ |

The utility-of-money function of a seller is $\begin{cases} m + s_i & \text{if he owns a car of quality } i \\ m & \text{if he does not own a car} \end{cases}$ and the utility of a buyer is $\begin{cases} m + b_i & \text{if he owns a car of quality } i \\ m & \text{if he does not own a car} \end{cases}$. In the initial situation each seller owns exactly one car and all sellers and buyers have the same initial endowment of money, which you can assume to be "sufficiently large".

All sellers and buyers are risk neutral. While each seller knows the quality of his car, each buyer is unable to determine the quality of any car offered for sale. Thus there can be only one price for used cars, call it *p*. If indifferent between selling and not selling (respectively, buying and not buying), a seller (respectively, a buyer) would choose to sell (respectively, buy).

**(a)** **(a.1)** Give necessary and sufficient conditions for there to exist a price *p* such that all cars are traded.
**(a.2)** In the special case where $p_i = \frac{1}{4}$ and $b_i = \alpha s_i$, for all $i \in Q$, express the necessary and sufficient condition in terms of $\alpha$.

**(b)** **(b.1)** Give necessary and sufficient conditions for there to exist a price *p* such that all and only cars of qualities B, C and D are traded.
**(b.2)** In the special case where $p_i = \frac{1}{4}$ and $b_i = \alpha s_i$, for all $i \in Q$, express the necessary and sufficient condition in terms of $\alpha$.

**(c)** **(c.1)** Give necessary and sufficient conditions for there to exist a price *p* such that all and only cars of quality D are traded.
**(c.2)** In the special case where $p_i = \frac{1}{4}$ and $b_i = \alpha s_i$, for all $i \in Q$, express the necessary and sufficient condition in terms of $\alpha$.

**(d)** Are any of the equilibria of parts (a)-(d) Pareto inefficient?

Now suppose that a car of quality $i \in Q = \{A, B, C, D\}$ has a probability $q_i$ of requiring a major repair within the next 5 years, with $0 < q_A < q_B < q_C < q_D \leq 1$. A major repair costs $R. The buyer's valuation reflects this, in the sense that $b_i = V - q_i R$, for all $i \in Q = \{A, B, C, D\}$, with $V > 0$. Now each seller can offer his car for sale either with or without warranty. If a car is sold with warranty, the seller credibly undertakes to pay for the repair himself, should a repair become necessary within the next 5 years. In this new situation, there can be two prices in the market: a price $p_N$ for cars sold without warranty and a price $p_W$ for cars sold with warranty.

**(e)** Find sufficient conditions for a two-price equilibrium to exist, where only qualities *A* and *D* are traded.

**(f)** For the case where $V = 24$, $R = 150$, $s_A = 15$, $s_B = 10$, $s_C = 8$, $s_D = 6$ and $q_A = \frac{1}{100}$, $q_B = \frac{5}{100}$, $q_C = \frac{8}{100}$, $q_D = \frac{10}{100}$ verify that the sufficient condition you gave is satisfied and find a two price equilibrium $(p_W, p_N)$ where only qualities *A* and *D* are traded.

**(g)** Consider the parameter values given under (f) with the addition of $p_i = \frac{1}{4}$, for all $i \in \{A, B, C, D\}$. Suppose that $p_W = 17$ and $p_N = 11$. What cars are sold with warranty and what cars are sold without warranty?

## Question 5.

Consider a market with two firms, both of which have **zero production costs**. We will consider two scenarios, one involving price competition and the other involving competition in output levels.

SCENARIO 1 The two firms play the following two-stage game. In stage 1 they simultaneously and independently choose between H and L, where H means "produce a high-quality product" and L means "produce a low-quality product". At the end of the first stage the two decisions become common knowledge and we proceed to the second stage. In the second stage the firms simultaneously and independently choose the price of their product. Payoffs are given by profits (= revenue). If both firms choose H in stage 1 then they are producing a homogeneous product for which industry demand is $Q = 80 - 8P$, while if they both choose L in stage 1 then they are producing a homogeneous product for which industry demand is $Q = 80 - 10P$. If one chooses H and the other L, then (letting $p_H$ be the price charged by the firm that chose H and $p_L$ the price charged by the firm that chose L) demand is as follows: for the H-firm $q_H = 80 - 40p_H + 40p_L$ and for the L firm $q_L = 40p_H - 50p_L$.

**(a)** Sketch the extensive-form game for the case where only two prices are possible: p and p′ (don't worry about payoffs, only about the structure of the game).

**(b)** Write down one strategy of firm 1 in the game of part (a) above. How many strategies does firm 1 have in the game of part (a) above?

*From now on allow for all non-negative prices*, that is $p_1 \in [0,\infty)$ and $p_2 \in [0,\infty)$.

**(c)** Write the demand function of firm 1, as a function of the two prices $p_1$ and $p_2$, for the case where they both choose H in stage 1.

**(d)** Find the pure-strategy subgame-perfect equilibria of the two-stage game.

**(e)** Find a Nash equilibrium of the two-stage game which is not subgame perfect.

SCENARIO 2 The two firms play the following two-stage game. The first stage is as in Scenario 1. In the second stage the firms simultaneously and independently choose their output levels. Demand for the case where both firms choose H and the case where both choose L is as given in Scenario 1. For the case where one chooses H and the other chooses L, inverting the demand system given under Scenario 1 yields the following:

$p_H = 10 - \frac{q_H}{8} - \frac{q_L}{10}$ and $p_L = 8 - \frac{q_H}{10} - \frac{q_L}{10}$.

**(f)** Find the pure-strategy subgame-perfect equilibria of the two-stage game.

**(g)** What is the main difference in the firms' behavior when we switch the second-stage game from price competition to output competition?