

PRELIMINARY EXAMINATION FOR THE Ph.D. DEGREE

Answer FOUR questions

Question 1.

(a). Let P the set of relevant price-wealth vectors (p, w) , a subset of \mathfrak{R}_{++}^{L+1} , and denote by $\tilde{x}_j : P \rightarrow \mathfrak{R}_{++}$ a consumer's Walrasian demand for good $j, j = 1, \dots, L$, assumed to be strictly positive and differentiable on P .

(a).1. What do we mean when we say that good j is a *necessity* for the consumer at (p, w) ? Same for *luxury* and for *borderline necessity-luxury*.

(a).2. Show that the concepts of *luxury* and *borderline necessity-luxury* can be characterized by a property of the budget share function $b_j(p, w)$ of the good. Can you do the same with the concept of *necessity*? Explain

For the rest of this question we consider the indirect utility function

$$v : P \rightarrow \mathfrak{R} : v(p, w) = \left[\frac{F(p)}{\ln(w/C(p))} + G(p) \right]^{-1}, \quad (1)$$

where $C(p) \gg 0$, and the functions C, F and G are such that $v(p, w)$ has the properties of an indirect utility function on P .

(b). For $j = 1, \dots, L$, obtain the Walrasian demand function $\tilde{x}_j(p, w)$ and the budget share function $b_j(p, w)$ corresponding to (1).

(c). Consider first the case of (1) with $G(p) = 0$, all p . Show that if good j is a luxury at some $(\bar{p}, \bar{w}) \gg 0$, then it is a luxury at (\bar{p}, w) , for all $w > 0$.

(d). Consider now the general case of (1) where $G(p)$ is not always zero.

(d).1. Suppose that good j is a luxury at some $(\bar{p}, \bar{w}) \gg 0$. Does it follow that it is a luxury at (\bar{p}, w) , for all $w > 0$? Explain.

(d).2. Suppose that all consumers in the economy have identical preferences, of the type represented by (1). Under which conditions on the functions C, F and G can the consumers' aggregate demand be a function of (only) prices and aggregate wealth? Argue clearly. Discuss the possibility of a positive representative consumer with these preferences.

Question 2.

This question considers the profit-maximizing assumption for firms with or without market power.

(a). Price-taking firm. Consider a price-taking firm with production set $Y \subset \mathfrak{R}^L$ and facing a strictly positive price vector p .

(a).1. Write the firm's profit-maximizing problem.

(a).2. Normalize all prices with good j as numeraire, and write the profit-maximizing problem under this normalization. Show that the same solution (or set of solutions) obtains no matter which good j is chosen as numeraire.

(b). Firm with market power. Now we specialize to two goods ($L = 2$), where good 1 is an input for the firm, and good 2 is its output. The production set of the firm is $\{(y_1, y_2) \in \mathfrak{R}^2 : y_1 \leq 0, y_2 \leq -y_1 / c\}$, where c is a positive parameter. The firm is a price setter, and faces the following demand function for its output:

$$\tilde{x}_2(p_1, p_2) = \frac{\omega p_1}{p_1 c + \frac{\alpha}{1-\alpha} p_2},$$

where the parameters ω and α satisfy $\omega > 0$ and $\alpha \in (0, 1)$.

(b).1. Write the firm's profits as a function of the prices (p_1, p_2) (disregard the possibility that $y_2 < -y_1 / c$.)

(b).2. Write, analyze and, if possible, solve the firm's profit-maximizing problem when prices are normalized with good 1 as numeraire.

(b).3. Write, analyze and, if possible, solve the firm's profit-maximizing problem when prices are normalized with good 2 as numeraire.

(b).4. Compare (b).2 and (b).3. Compare with (a).2. Comment.

Question 3.

Recently there has been much interest in studying “Social Preferences”, i.e., situations where agents are concerned not only with their own material well-being, but also with the well-being of the other agents in their social environment, because of altruism, envy, preferences for status, inequality aversion... To see how social preferences change the result of the two theorems of welfare economics, consider an economy with L goods and I agents, in which agents’ preferences can be decomposed as follows. Each agent $i \in \{1, \dots, I\}$ has an index m_i of material well-being which is a function of his/her own consumption, i.e., a function from \mathbf{R}_+^L to \mathbf{R} . This function is assumed to be increasing and quasi-concave, like a standard utility function in general equilibrium. However the “true” utility function u_i of agent i is a function of the well-being of all the agents in the economy, i.e., there exists a function $V_i : \mathbf{R}^I \rightarrow \mathbf{R}$ such that

$$u_i(x) = V_i(m_1(x^1), \dots, m_I(x^I)), \quad i = 1, \dots, I$$

where $x = (x^1, \dots, x^I) \in \mathbf{R}_+^{LI}$ is the allocation of the resources to the agents. We assume that for all $i = 1, \dots, I$, V_i is non decreasing, and strictly increasing in m_i (we restrict the study to the case of altruism). All functions are assumed to be continuously differentiable.

Agent i ’s utility depends on the consumption of all the other agents, and there are thus consumption externalities in this economy. However the form of the externality is rather specific. Let $\omega^i \in \mathbf{R}_+^L$ be agent i ’s endowment, $i = 1, \dots, I$, and let $\mathcal{E}(u, \omega)$ denote the economy just described, with $u = (u_1, \dots, u_I)$ and $\omega = (\omega^1, \dots, \omega^I)$. To study the properties of $\mathcal{E}(u, \omega)$ it is convenient to compare it to the fictitious “ego-economy” $\mathcal{E}^{ego}(m, \omega)$, with $m = (m_1, \dots, m_I)$, in which agents are only interested in their own well-being.

- (a) Since $\mathcal{E}^{ego}(m, \omega)$ is a standard convex economy, it has at least one competitive equilibrium. Show that the competitive equilibria of $\mathcal{E}(u, \omega)$ and $\mathcal{E}^{ego}(m, \omega)$ coincide, so that $\mathcal{E}^{ego}(m, \omega)$ has at least one competitive equilibrium.
- (b) A feasible allocation x is said to be materially efficient if it is a Pareto optimal allocation for the ego-economy $\mathcal{E}^{ego}(m, \omega)$. Show that the Pareto optimal allocations of the economy $\mathcal{E}(u, \omega)$ are materially efficient.
- (c) Thus the set of Pareto optimal allocations of the economy $\mathcal{E}(u, \omega)$ is included in the set of materially efficient allocations. To see that the inclusion can be strict consider a two-agent, two-good economy with $m_1(x_1, x_2) = m_2(x_1, x_2) = (x_1 x_2)^{\frac{1}{3}}$, $V_1(m_1, m_2) = m_1 + 0.9m_2$, $V_2(m_1, m_2) = m_2 + 0.9m_1$ and aggregate endowment $\bar{\omega} = (2, 2)$.
 - (c)(i) Show that the materially efficient allocations are $x^1 = (\alpha, \alpha)$, $x^2 = (2 - \alpha, 2 - \alpha)$ for $\alpha \in [0, 2]$. Let $x(\alpha) = ((\alpha, \alpha), (2 - \alpha, 2 - \alpha))$.

- (c)(ii)** By studying the derivatives of $u_1(x(\alpha))$ and $u_2(x(\alpha))$ show that there exist $\underline{\alpha}$ and $\bar{\alpha}$ such that if $\alpha < \underline{\alpha}$ or $\alpha > \bar{\alpha}$, the materially efficient allocation is not Pareto optimal for the economy $\mathcal{E}(u, \omega)$. Explain intuitively this result.
- (d)** Deduce from (a) and (c) that the First Theorem of Welfare Economics does not hold for every economy $\mathcal{E}(u, \omega)$.
- (e)** Deduce from (a) and (b) that the Second Theorem of Welfare Economics holds for all economies $\mathcal{E}(u, \omega)$ (satisfying the assumptions above).

Question 4.

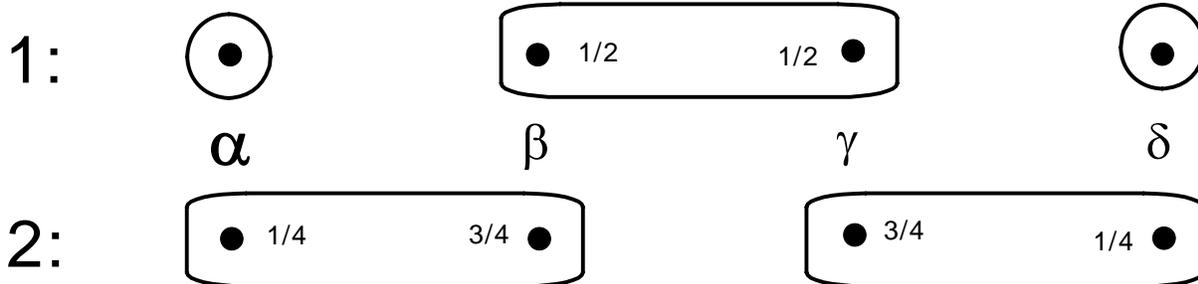
Consider the following two-player situation of incomplete information.

		2	
		L	R
1	T	4,4	0,0
	B	2,0	2,2

		2	
		L	R
1	T	0,4	4,0
	B	2,0	0,2

		2	
		L	R
1	T	0,4	4,0
	B	2,0	0,2

		2	
		L	R
1	T	4,4	0,0
	B	2,0	2,2



- (a) Find the common knowledge partition.
- (b) Associate strategy profiles to states as follows: at α (T,L), at β (B,L), at γ (B,R), at δ (T,R).
 - (b.1) Is it common knowledge at state α that each player knows his own choice of action? Explain your answer.
 - (b.2) Find the events \mathbf{RAT}_1 (player 1 is rational), \mathbf{RAT}_2 (player 2 is rational), \mathbf{RAT} (both players are rational), $\mathbf{K}_1\mathbf{RAT}$ (player 1 knows that both players are rational) and $\mathbf{K}_2\mathbf{RAT}$ (player 2 knows that both players are rational).
- (c) Apply the “Harsanyi transformation” to represent this situation of incomplete information as a game with imperfect information.
- (d) Write down all the pure strategies of player 1 and all the pure strategies of player 2 in the game with imperfect information of part (c).
- (e) Is the following a pure-strategy Bayesian Nash equilibrium: player 1 plays T always and player 2 plays L always? Explain your answer.
- (f) What beliefs, paired with the strategy profile described in part (e), would constitute a consistent assessment? Prove consistency.
- (g) Explain why the above game with imperfect information has at least one subgame-perfect equilibrium.
- (h) In the above game with imperfect information is there a perfect-Bayesian equilibrium where each player’s behavior strategy is always to choose between his two actions with equal probability? Justify your answer.

Question 5.

Rachel wants to borrow \$ X . She cannot offer any property as collateral. The rate of interest is either r_L or r_H , with $0 < r_L < r_H$. Rachel thinks that (1) if she borrows money at the lower rate r_L , the probability that she will completely default on the loan (that is, that she will be unable to repay any money at all to the bank) is p_L and the probability that she will be able to completely repay the loan (principal plus interest) is $(1 - p_L)$; (2) if she borrows at the higher rate r_H , the probability that she will completely default on the loan is p_H and the probability that she will completely repay the loan is $(1 - p_H)$, with $0 < p_L < p_H < 1$. Thus the possible outcomes are:

- z_1 : she borrows at the lower rate r_L and repays the loan (principal + interest),
- z_2 : she borrows at the higher rate r_H and repays the loan (principal + interest),
- z_3 : she does not borrow,
- z_4 : she borrows at the lower rate r_L and defaults on the loan,
- z_5 : she borrows at the higher rate r_H and defaults on the loan.

Her ranking of these outcomes is $z_1 \succ z_2 \succ z_3 \succ z_4 \sim z_5$ (where $x \succ y$ means that she prefers x to y and $x \sim y$ means that she is indifferent between x and y). Rachel satisfies the axioms of expected utility. She is indifferent between outcome z_2 for sure and lottery L_2 , and she is indifferent between outcome z_3 for sure

and lottery L_3 , where $L_2 = \begin{pmatrix} z_1 & z_2 & z_3 & z_4 & z_5 \\ \frac{85}{100} & 0 & 0 & \frac{7}{100} & \frac{8}{100} \end{pmatrix}$ and $L_3 = \begin{pmatrix} z_1 & z_2 & z_3 & z_4 & z_5 \\ 0 & \frac{14}{17} & 0 & \frac{2}{17} & \frac{1}{17} \end{pmatrix}$.

Ross, the lender, is risk neutral and only cares about his expected profits. Assume, for simplicity, that there are no costs in giving a loan. Ross agrees that the possible outcomes are as described above and that the probabilities are as assessed by Rachel. In case of indifference, you can decide what Rachel and Ross would do. Ross decides what interest rate to charge (the choice being between r_L and r_H).

- (a) Find Rachel's normalized von Neumann-Morgenstern utility function.
- (b) For what values of the parameters r_L , r_H , p_L and p_H will Rachel and Ross sign a loan contract at rate r_L ? [Hint: don't forget that it is Ross who chooses the interest rate.]
- (c) If $r_L = 10\%$, $r_H = 20\%$, $p_L = 0.10$ and $p_H = 0.15$, what loan contract will they sign?

Suppose now that there are two types of potential borrowers: type L , who are identical to Rachel, and type H , who are described as follows: (1) they rank the outcomes $z_1 \sim z_2 \succ z_4 \sim z_5 \succ z_3$ (and they also satisfy the axioms of expected utility), (2) their probability of default is q_L (if they borrow at the lower rate r_L) and $q_H = p_H$ (if they borrow at the higher rate r_H), with $p_L < q_L < p_H$ and (3) they too are unable to offer any collateral. There are N_L borrowers of type L and N_H of type H . Ross can only offer $m \leq \min\{N_L, N_H\}$ loans and cannot tell the two types apart (although he knows all of the above). From now on let $X = 10,000$, $r_L = 10\%$, $r_H = 20\%$, $p_L = \frac{2}{25}$, $p_H = \frac{1}{5}$ and $q_L = \frac{3}{25}$.

- (d) If Ross offers n_L loans at the rate r_L and n_H loans at the rate r_H (with $n_L + n_H \leq m$), what are his expected profits?
- (e) How many r_L -loans and how many r_H -loans will Ross offer if $N_L = 5,000$ and $N_H = 1,000$?