University of California, Davis
ARE-Economics
**Microeconomics**

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Time: 5 hours
Reading Time: 20 minutes

**PRELIMINARY EXAMINATION FOR THE Ph.D. DEGREE**

**Please answer four of the following five questions**

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**QESTION 1. WHAT KIND OF COMPENSATION**

An economy is comprised of $I$ persons, numbered 1 to $I$. Person $i$ is endowed with a preference relation $\succsim^i$ defined on an abstract set $A$ of *feasible states*. We assume that $\succsim^i$ can be represented by a utility function $\beta^i : A \to \Re$.

**1(a)**. What do we mean when we say that a feasible state $\bar{\bar{s}}$ is *better than another feasible state* $\bar{s}$ *according to the Pareto criterion*? Graphically illustrate in utility space for the case of two people.

A *feasible redistribution correspondence* is a correspondence Φ: $A \to\to A$ interpreted as follows: Φ($s$) is the *set of states that can be reached from state s by feasible redistributions*.

**1(b)**. Let Φ be given. What do we mean when we say that a feasible state $\bar{\bar{s}}$ is *better than another feasible state* $\bar{s}$ *according to the weak potential compensation criterion for the redistribution correspondence* Φ? Graphically illustrate in utility space for the case of two people.

Now we endow the set $A$ with some structure. Assume that $A \subset \Re^{IN+M}$, i. e., a state $s$ is of the form $s = (s_1^1, ..., s_N^1; s_1^2, ..., s_N^2; ...; s_1^i, ..., s_N^i; ...; s_1^I, ..., s_N^I; s_1, ..., s_M)$, and that, for $i$ = 1,…, $I$, the utility function $\beta^i$ is increasing in $s_j^i$ and constant with respect to $s_j^h$, $h \neq i$, $j$ = 1,…, $N$. The interpretation is that there are $N$ goods, or attributes, that are desirable and do not generate externalities among persons, and that the amount $s_j^i$ of desirable good or attribute $j \in \{1, ..., N\}$ is allocated to Person $i$. In addition, there may be some other $M$ components of the state, e. g., public goods or bads, prices, non-desirable private goods, or private goods that generate externalities in consumption. It follows that Person $i$ may care only about the components $(s_1^i, ..., s_N^i; s_1, ..., s_M)$ of the state. The formulation is general enough to cover the type of states studied in Walrasian demand theory (see 1(d) below) as well as the example in 1(f)-(i) below.

Given $j \in \{1,...,N\}$, we define the correspondence $\Phi_j$: $A \rightarrow\rightarrow A$ as follows:

$$\Phi_j(s) = \{s' \in \Re^{IN+M} : \sum_i s_j^i{}' = \sum_i s_j^i, s_k^i{}' = s_k^i, \textit{ for } k \neq j, i = 1,...,I, s_m{}' = s_m, \textit{ for } m = 1,...,M\}.$$

In words, according to the correspondence $\Phi_j$, state *s*' can be reached from *s* by redistributing good or attribute *j* among the *I* people, leaving unchanged all other components of the vector *s*.

We adopt the notation $s = (s_{-ij}; s_j^i)$, where the vector $s_{-ij}$ comprises all the components of the vector *s* except $s_j^i$.

Given $j \in \{1,...,N\}$, $i \in \{1,...,I\}$ and two feasible states $\bar{s}$ and $\bar{\bar{s}}$, we define the *compensating modification* $CM_j^i(\bar{s},\bar{\bar{s}})$ (in good or attribute *j*, for Person *i*, and for a move from $\bar{s}$ and $\bar{\bar{s}}$) implicitly by the equation

$$\beta^i(\bar{\bar{s}}_{-ij}, \bar{\bar{s}}_j^i - CM_j^i(\bar{s},\bar{\bar{s}})) = \beta^i(\bar{s}). \quad (1)$$

**1(c)**. Interpret $CM_j^i(\bar{s},\bar{\bar{s}})$ in words.

**1(d).** Argue that $CM_j^i(\bar{s},\bar{\bar{s}})$ generalizes the familiar notion of the compensating variation in the usual Walrasian context.

**1(e)**. Prove the following <u>fact</u>. *If* $\sum_h CM_j^h(\bar{s},\bar{\bar{s}}) > 0$, *then state* $\bar{\bar{s}}$ *is better than state* $\bar{s}$ *according to the potential compensation criterion for the redistribution correspondence* $\Phi_j$.

The rest of this question illustrates that choosing different notions of redistribution by choosing different *j*'s has important implications.

To this end, consider an economy with *I* = *N* = 2, and *M* = 0, i. e., two goods, apples (good 1) and bananas (good 2), and two people, John (Person 1) and Mary (Person 2). A feasible state is now a consumption allocation, i. e., a vector in $\Re^4$ of the form $(s_1^1, s_2^1; s_1^2, s_2^2)$, where $s_2^1$ is John's consumption of bananas, $s_1^2$ is Mary's consumption of apples, etc.

John's preferences can be represented by the utility function

$$\beta^1(s) = 3s_1^1 + s_2^1,$$

while Mary's preferences can be represented by the utility function

$$\beta^2(s) = s_1^2 + 3s_2^2 .$$

<u>Remark</u>. The linearity of these functions guarantees that, given Φ, the weak potential compensation criterion is automatically strong, so that if a state is better than another one according to the weak potential compensation criterion for Φ, it cannot be the case that the second one is better than the first one according to the same criterion for the same Φ.

Consider the following two states:

$$\bar{s} = (\bar{s}_1^1, \bar{s}_2^1; \bar{s}_1^2, \bar{s}_2^2) = (100,100,100,100) ,$$

$$\bar{\bar{s}} = (\bar{\bar{s}}_1^1, \bar{\bar{s}}_2^1; \bar{\bar{s}}_1^2, \bar{\bar{s}}_2^2) = (50,150,50,150) .$$

**1(f)**. Is $\bar{s}$ better than $\bar{\bar{s}}$ according to the Pareto criterion? Is $\bar{\bar{s}}$ better than $\bar{s}$ according to the Pareto criterion? Explain, and graphically illustrate in utility space.

**1(g)**. Consider the feasible redistribution correspondence $\Phi_1$, i. e., redistribution is implemented in apples. Compute $CM_1^i(\bar{s}, \bar{\bar{s}})$ , for $i$ = 1, 2, and $CM_1^1(\bar{s}, \bar{\bar{s}}) + CM_1^2(\bar{s}, \bar{\bar{s}})$ . Is $\bar{s}$ better than $\bar{\bar{s}}$ according to the potential compensation criterion for $\Phi_1$? Is $\bar{\bar{s}}$ better than $\bar{s}$ according to the potential compensation criterion for $\Phi_1$? Explain, specifying any relevant compensations, and graphically illustrate in utility space.

**1(h).** Consider now the feasible redistribution correspondence $\Phi_2$, i. e., redistribution is implemented in bananas. Compute $CM_2^i(\bar{s}, \bar{\bar{s}})$ , for $i$ = 1, 2, and $CM_2^1(\bar{s}, \bar{\bar{s}}) + CM_2^2(\bar{s}, \bar{\bar{s}})$ . Is $\bar{s}$ better than $\bar{\bar{s}}$ according to the potential compensation criterion? Is $\bar{\bar{s}}$ better than $\bar{s}$ according to the potential compensation criterion? Explain, specifying any relevant compensation, and graphically illustrate in utility space.

**1(i)**. Compare your answers to 1(g) and 1(h), and comment.

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## QUESTION 2. CONSTRAINED RISK AVERSION

A consumer plans to buy goods 1 and 2 at the market prices $p_1$ and $p_2$, but her wealth $w$ is uncertain. *Ex ante*, she may have to choose among various lotteries that will affect her wealth. Her *ex ante* preferences satisfy the expected utility hypothesis, with the following von Neumann-Morgenstern-Bernoulli (vNMB) utility function defined on consumption vectors $(x_1, x_2)$:

$$u(x_1, x_2) = \frac{[x_1^{\alpha} x_2^{1-\alpha}]^{1-\rho} - 1}{1-\rho}, \qquad (1)$$

where $\alpha \in (0,1)$ and $\rho \in (0,1) \cup (1,\infty)$.

**2(a).** Assume that, once $w$ is known, she can always satisfy her (*ex post*) Walrasian demand for goods 1 and 2 at the (certain) prices $p_1$ and $p_2$. Call her in this case the *unconstrained consumer*.

**2(a).1.** What are her Walrasian demands $\tilde{x}_1(p_1, p_2, w)$ and $\tilde{x}_2(p_1, p_2, w)$? Graphically represent them in Figure 1 for given prices $p_1$ and $p_2$, and wealth $w^0$.

**2(a).2.** Because her only uncertainty concerns her wealth, we can write the vNMB utility of the unconstrained consumer as a function of $w$ (with prices as parameters), to be denoted $\beta(w)$. What is this function?

Compute:

- Its first-order derivative $\beta'(w)$;
- Its second-order derivative $\beta''(w)$;
- Its coefficient of relative risk aversion evaluated at $w$, denoted $CRR[\beta,w]$.

**2(b).** Consider a given wealth level $w^0$, and write $x_1^0 \equiv \tilde{x}_1(p_1, p_2, w^0)$. Assume now that as her wealth changes, she is constrained to consume the exact amount $x_1^0 \equiv \tilde{x}_1(p_1, p_2, w^0)$ of good 1 (perhaps because of previous commitments, or by institutional restrictions), spending what is left of her wealth in good 2. Refer to her in this case as the *constrained consumer*.

**2(b). 1.** Graphically represent her constrained choice in Figure 1.

**2(b). 2**. Now her vNMB utility is a different function of wealth, to be denoted $\beta_C(w)$. What is this function?

**2(b).3.** Compute its first-order derivative $\beta_C'(w)$. How do $\beta_C'(w^0)$ and $\beta'(w^0)$ compare? Explain. Can you graphically illustrate the comparison between $\beta_C'(w^0)$ and $\beta'(w^0)$ in the (wealth, utility) plane?

**2(b).4.** Compute the second-order derivative $\beta_C''(w)$ and evaluate it at $w^0$.

**2(b).5**. Compute the coefficient $CRR[\beta_C, w^0]$ of relative risk aversion of $\beta_C$ evaluated at $w^0$.

**2(c)**. Compare $CRR[\beta, w^0]$ and $CRR[\beta_C, w^0]$. Who is more risk averse (at $w^0$), the constrained consumer, or the unconstrained consumer? Explain.

**QUESTION 3. RISK AVERSION AND EQUILIBRIUM CONSUMPTION**

Consider a (static) exchange economy with uncertainty: there are $S$ "states of nature" indexed by $s = 1, \dots, S$, and the probability that state $s$ occurs is $\rho_s > 0$, with $\sum_{s=1}^S \rho_s = 1$. There are $I$ agents with random endowments, where $\omega^i = (\omega^i_1, \dots, \omega^i_S)$ denotes the endowment of agent $i$ in each state of nature. Agents exchange ex-ante (before knowing which state will occur) their endowment profiles against profiles of consumption $x^i = (x^i_1, \dots, x^i_S)$, $i = 1, \dots, I$, seeking to maximize expected utilities of the form

$$u^i(x^i) = \sum_{s=1}^{S} \rho_s \ln(\alpha^i + x^i_s), \qquad i = 1, \dots, I.$$

That is, the agents' utility functions only differ by the coefficients $\alpha^i$. The exchange (good in state $s$ against good in state $s'$) takes place on the basis of a price system, with $p_s$ denoting the price of the good available in state $s$, $s = 1, \dots, S$. Formally this economy is a standard exchange economy $\mathcal{E}((u^i, \omega^i)_{i=1}^I)$ with $S$ goods. We assume that the possible consumption profiles for agent $i$ are $\{x^i \in \mathbb{R}^S \,|\, \alpha^i + x^i_s > 0, \ \forall\, s = 1, \dots, S\}$, i.e. we do not take into account the non-negativity constraints on consumption.

**(a)** Calculate the coefficient of risk aversion of agent $i$ (absolute or relative risk aversion) and argue that the larger $\alpha^i$, the more risk tolerant the agent.

**(b)** Using the change of variable $X^i_s = \alpha^i + x^i_s$ and using the formula for the demand of an agent with Cobb-Douglas preferences (or doing the calculation long-hand if you prefer) calculate the demand function of agent $i$.

**(c)** Let $\mathbf{1} = (1, \dots, 1) \in \mathbb{R}^S_+$ denote the sure consumption of one unit in each state of nature and let $w = \sum_{i=1}^I \omega^i$ be the aggregate endowment. We assume that the aggregate endowment is risky, i.e. $w$ is not collinear to $\mathbf{1}$. Let $(\bar{x}, \bar{p})$ be a competitive equilibrium of the economy $\mathcal{E}$. Show that the equilibrium consumption of agent $i$ is of the form

$$\bar{x}^i = \bar{b}^i w + \bar{a}^i \mathbf{1}, \quad \text{with } \bar{b}^i = \frac{\bar{p} \cdot (\omega^i + \alpha^i \mathbf{1})}{\bar{p} \cdot (w + \alpha \mathbf{1})}, \quad \alpha = \sum_{i=1}^{I} \alpha^i$$

Calculate $\bar{a}^i$ and show that $\sum_{i=1}^I \bar{a}^i = 0$. Interpret. [Hint: Note that $\alpha^i \sum_s \bar{p}_s = \alpha^i \bar{p} \cdot \mathbf{1}$. You do not need to calculate $\bar{p}$. Use the equilibrium equations to express $\bar{p}_s$ as a function of the aggregate variables and substitute into the demand of agent $i$ to obtain the desired formula.]

**(d)** Noting that the larger $\bar{b}^i$ and the smaller $\bar{a}^i$, the more risky the consumption profile, explain the logic behind the formulas in (c).

**(e)** Suppose $\alpha^1 \to \infty$ while $\alpha^2, \dots, \alpha^I$ stay the same. What happens to the agents' equilibrium consumption? Explain.

**QUESTION 4. PUBLIC GOOD AND GROUP SIZE**

We study how the public good provision depends on the size of the economy, for different assumptions on the way the public good is financed. To make the influence of the size on the amount of public good produced clearly visible, we assume that all agents are identical.

Let us thus consider an economy with $n$ identical agents, one private good and one public good, the preferences of agent $i$ ($i = 1, \ldots, n$) being represented by the utility function

$$u^i(x^i, y) = x^i + h(y), \quad (x^i \text{ amount of private good, } y \text{ amount of public good})$$

where $h$ is concave, differentiable, increasing, and satisfies $\lim_{y\to 0} h'(y) > 1$, $\lim_{y\to\infty} h'(y) = 0$. Each agent has an endowment $\omega$ of private good which is such that the non-negativity constraint on the consumption of private good never binds. The production of public good exhibits constant returns to scale, one unit of private good producing one unit of public good. We study only symmetric allocations.

**(a)** Show that the optimal level of public good is an increasing function of the size $n$ of the economy that tends to infinity when $n$ tends to infinity.

**(b)** Show that the level of public good in a voluntary contribution equilibrium is independent of the size $n$ of the economy. Comment on the result.

**(c)** A method which was proposed in the economic literature to mitigate the free-rider problem is to attach a lottery to contributions to the public good. If agent $i$ contributes $z^i$ then he has a probability

$$\frac{z^i}{\sum_{j=1}^n z^j}$$

to win the lottery prize worth $R$ units of private good (the quasi-linear utility function implies that agents are risk neutral for lotteries in private good). The agents' contributions serve both to finance the public good and the lottery prize, so that only $\sum_{j=1}^n z^j - R$ goes toward the production of the public good. Suppose the lottery prize $R$ is independent of the size $n$ of the economy. Show that, in a symmetric Nash equilibrium in which each agent contributes $z$ (a function of $n$ and $R$)

**(i)** if $n > 1$ the level of public good $y = nz - R$ is always larger than the voluntary contribution level found in (b) for the same economy;

**(ii)** the level of public good $y = nz - R$ increases with $n$;

**(iii)** when $n \to \infty$ the level of public good $y = nz - R$ tends to a finite limit.

**(d)** Given that the optimal level of public good tends to infinity the result of (c)(iii) is disappointing. Suppose that instead of a fixed prize, the size of the prize increases proportionally to the group size: $R = nr$. Show that

**(i)** the level of public good $y = n(z - r)$ increases with $n$,

**(ii)** when $n \to \infty$ the level of public good tends to infinity. Comment.

**QUESTION 5. POTENTIAL ENTRY**

Consider a market where a firm, call it *M*, is currently a monopolist but faces a potential entrant, call it *PE*. They play the following game. *M* chooses a level of investment *k*, where *k* can be any nonnegative number. This decision is observed by *PE*. Next, *PE* decides whether or not to enter the market. If *PE* does not enter, *M* remains a monopolist and earns profits equal to $(2k+1)\pi_M - k$, while *PE* earns zero profits and the game ends. If *PE* enters, *M* observes *PE's* decision and decides whether or not to exit the market. If it exits, it earns $\theta - k$, while *PE* earns profits equal to $\pi_M - F$, where *F* is the cost of entry. If *M* does not exit, then the market is shared by the two firms with corresponding profits of $(2k+1)\pi_d - k$ for *M* and $\pi_d - F$ for *PE*. Assume throughout that $\pi_M > F > \pi_d$ and that, if indifferent between exiting and staying, *M* chooses to stay and this is common knowledge between *M* and *PE*.

**(a)** Assume that the value of $\theta$ is common knowledge. Show the structure of the game by sketching the extensive form.

**(b)** Still assuming that the value of $\theta$ is common knowledge, find the subgame-perfect equilibrium of the game in the following two cases. In each case calculate also the equilibrium profits of both firms.

**(b.1)** $\pi_M = \frac{1}{8}$, $\pi_d = \frac{1}{18}$ and $\theta = \frac{1}{16}$

**(b.2)** $\pi_M = \frac{1}{8}$, $\pi_d = \frac{1}{18}$ and $\theta = \frac{3}{16}$.

For the remaining questions, assume that there are only two possible values of *k*: 0 and $\hat{k}$, that is, $k \in \{0, \hat{k}\}$. Assume also that the value of $\theta$ is private information to *M*. *PE* believes that there are two possibilities: $\theta_H$ and $\theta_L$, with $\theta_H > \theta_L > 0$ and one of these two is in fact the true value of $\theta$. Let $p \in (0,1)$ be the probability that *PE* assigns to $\theta_H$ [and $(1-p)$ the probability that *PE* assigns to $\theta_L$ ]. *PE*'s beliefs are common knowledge between *M* and *PE* as is the fact that *M* knows the true value of $\theta$. Thus we have a situation of incomplete information.

**(c)** Using the Harsanyi transformation sketch the extensive form of the corresponding imperfect-information game. Make sure that information sets are clearly drawn.

**(d)** For the game of part *c* show that under the following parameter restrictions there is no pure-strategy separating weak sequential (or perfect Bayesian) equilibrium (that is, there is no pure strategy equilibrium where the two types of *M* make different investment choices):

$$\pi_M = \frac{1}{8},\ \pi_d = \frac{1}{18},\ \hat{k} = \frac{1}{40},\quad \theta_L = \frac{1}{20},\quad \theta_H = \frac{1}{15}.$$

**(e)** With the parameter values of part *d* and assuming that the players are risk neutral, for what values of $p$ is there a pooling weak sequential (or perfect Bayesian) equilibrium where both types of *M* choose $\hat{k}$ and, observing this, *PE* stays out?