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PRELIMINARY EXAMINATION FOR THE Ph.D. DEGREE

ANSWER KEYS
Question 1

1(a). Equations defining $(\hat{x}_1(p_1,p_2,w,E_{-i}),\hat{x}_2(p_1,p_2,w,E_{-i}),\hat{E}(p_1,p_2,w,E_{-i}))$

Problem *PG.* $\max_{(x_1,x_2,E)} u(x_1,x_2,E)$ subject to $p_1x_1+p_2x_2\le w$ and $E - E_{-i} - g x_2 \le 0$. (Plus any nonnegativity constraints, ignored in what follows).

Lagrangian. $u(x_1,x_2,E)-\lambda_1(p_1x_1+p_2x_2-w)-\lambda_2(E-E_{-i}-gx_2)$

KT Conditions

$$\frac{\partial u(x_1,x_2,E)}{\partial x_1}-\lambda_1 p_1=0,\qquad \frac{\partial u(x_1,x_2,E)}{\partial x_2}-\lambda_1 p_2+\lambda_2 g=0,$$

$$\frac{\partial u(x_1,x_2,E)}{\partial E}-\lambda_2=0,\qquad \lambda_1(p_1x_1+p_2x_2-w)=0,\qquad \lambda_2(E-E_{-i}-gx_2)=0.$$

Substituting the values for the multipliers found in the first and third condition into the second condition, and appealing to the assumption that constraints (1) and 2 are satisfied with equality, we obtain the following system of three equations in the three unknowns (x_1,x_2,E)

$$\begin{cases}\dfrac{\partial u(x_1,x_2,E)}{\partial x_2}-\dfrac{\partial u(x_1,x_2,E)}{\partial x_1}\dfrac{p_2}{p_1}+\dfrac{\partial u(x_1,x_2,E)}{\partial E}g=0,\\ p_1x_1+p_2x_2-w=0,\\ E-E_{-i}-gx_2=0.\end{cases}\qquad (3)$$

Because PG is a concave program (concave objective function and linear constraints), the KT conditions are sufficient. Hence, a solution to the above system of equations solves Problem *PG*.

1(b). For the rest of this part we consider the utility function

$$u:\Re\times\Re^2_+ : u(x_1,x_2,E)=x_1+[a_2,a_E]\begin{bmatrix}x_2\\E\end{bmatrix}-\frac{1}{2}[x_2,E]B\begin{bmatrix}x_2\\E\end{bmatrix},$$

where $(a_2,a_G)>>0$ and $B\equiv\begin{bmatrix}b_{22} & b_{2E}\\ b_{E2} & b_{EE}\end{bmatrix}$ is a symmetric, positive definite matrix.

1(b)(i). Specialize to this utility function the system of equations obtained in 1(a).

We compute $\dfrac{\partial u}{\partial x_1}=1$ and $\begin{bmatrix}\dfrac{\partial u}{\partial x_2}\\ \dfrac{\partial u}{\partial E}\end{bmatrix}=\begin{bmatrix}a_2\\a_E\end{bmatrix}-\begin{bmatrix}b_{22} & b_{2E}\\ b_{E2} & b_{EE}\end{bmatrix}\begin{bmatrix}x_2\\E\end{bmatrix}$. Therefore the system of equations (3) specializes to

$$\begin{cases}(a_2-b_{22}x_2-b_{2E}E)-\dfrac{p_2}{p_1}+(a_E-b_{E2}x_2-b_{EE}E)g=0,\\ p_1x_1+p_2x_2-w=0,\\ E-E_{-i}-gx_2=0.\end{cases}$$

1(b)(ii). Verbal interpretation of $\frac{\partial \hat{x}_2}{\partial E_{-i}}$. It is the rate at which our consumer's demand for good 2 increases as the amount of the externality created by everybody else increases. Because the contribution of our consumer to the total amount of the public good is $g\hat{x}_2$, the rate at which her contribution to the public good responds to an increase in the amount contributed by everybody else has the same sign as $\frac{\partial \hat{x}_2}{\partial E_{-i}}$. If $\frac{\partial \hat{x}_2}{\partial E_{-i}} > 0$, then our consumer increases her demand for good 2, together with her contribution to the public good, as the amount contributed by everybody else increases.

Verbal interpretation of $\frac{\partial \hat{E}}{\partial E_{-i}}$. It is the rate at which our consumer's desired amount of the public good increases as the amount of the externality created by everybody else increases. Because $\hat{E} = E_{-i} + g\hat{x}_2$, $\frac{\partial \hat{E}}{\partial E_{-i}} = 1 + \frac{\partial \hat{x}_2}{\partial E_{-i}}$, i. e., $\frac{\partial \hat{x}_2}{\partial E_{-i}} > 0 \Rightarrow \frac{\partial \hat{E}}{\partial E_{-i}} > 0$. But when $\frac{\partial \hat{x}_2}{\partial E_{-i}} < 0$, then $\frac{\partial \hat{E}}{\partial E_{-i}}$ can in principle be positive or negative. If, say, $\frac{\partial \hat{x}_2}{\partial E_{-i}} > 0$ and $\frac{\partial \hat{E}}{\partial E_{-i}} < 0$, then our consumer decreases her demand for good 2 but ends up increasing enjoying a higher amount of the public good as the amount contributed by everybody else increases.

Computation of $\frac{\partial \hat{x}_2}{\partial E_{-i}}$ and $\frac{\partial \hat{E}}{\partial E_{-i}}$.

The partial derivatives can found either by computing the explicit solutions or by implicit differentiation. This Answer Key follows the first approach.
The first and third equations of 1(b) (i) do not involve x_1 (this is a consequence of quasilinearity). Hence, we can solve these two equations for x_2 and E, i. e.,

$$\begin{cases} (a_2 - b_{22}x_2 - b_{2E}E) - \dfrac{p_2}{p_1} + (a_E - b_{E2}x_2 - b_{EE}E)g = 0, \\ E - E_{-i} - gx_2 = 0, \end{cases}$$

or in matrix notation

$$\begin{bmatrix} -b_{22} - b_{E2}g & -b_{2E} - b_{EE}g \\ -g & 1 \end{bmatrix} \begin{bmatrix} x_2 \\ E \end{bmatrix} = \begin{bmatrix} -a_2 + (p_2/p_1) - a_E g \\ E_{-i} \end{bmatrix}, \quad \text{i. e.,}$$

$$\begin{bmatrix} b_{22} + b_{E2}g & b_{2E} + b_{EE}g \\ g & -1 \end{bmatrix} \begin{bmatrix} x_2 \\ E \end{bmatrix} = \begin{bmatrix} a_2 - (p_2/p_1) + a_E g \\ -E_{-i} \end{bmatrix},$$

which can be solved as

$$\begin{bmatrix} x_2 \\ E \end{bmatrix} = \begin{bmatrix} b_{22} + b_{E2}g & b_{2E} + b_{EE}g \\ +g & -1 \end{bmatrix}^{-1} \begin{bmatrix} a_2 - (p_2/p_1) + a_E g \\ -E_{-i} \end{bmatrix},$$

$$= \frac{1}{-\Delta} \begin{bmatrix} -1 & -b_{2E} - b_{EE}g \\ -g & b_{22} + b_{E2}g \end{bmatrix} \begin{bmatrix} a_2 - (p_2/p_1) + a_E g \\ -E_{-i} \end{bmatrix},$$

where $-\Delta = -(b_{22} + b_{E2}g) - (b_{2E} + b_{EE}g)g = -b_{22} - b_{E2}g - b_{2E}g - b_{EE}g^2$,
yielding

$$\begin{bmatrix} \frac{\partial \hat{x}_2}{\partial E_{-i}} \\ \frac{\partial \hat{E}}{\partial E_{-i}} \end{bmatrix} = \begin{bmatrix} \frac{-b_{2E} - b_{EE}g}{\Delta} \\ \frac{b_{22} + b_{E2}g}{\Delta} \end{bmatrix}. \qquad (4)$$

1(b)(iii). The positive definiteness of the matrix B, implies

$$b_{22} > 0,\ b_{EE} > 0, \qquad (5)$$

as well as

$$\Delta = b_{22} + b_{E2}g + b_{2E}g + b_{EE}g^2 = [1, g]\begin{bmatrix} b_{22} & b_{2E} \\ b_{E2} & b_{EE} \end{bmatrix}\begin{bmatrix} 1 \\ g \end{bmatrix} > 0. \qquad (6)$$

Let $b_{2E} \geq 0$. Using (5) and (6), the signs of (4) can be ascertained as:

$$\begin{bmatrix} \frac{\partial \hat{x}_2}{\partial E_{-i}} \\ \frac{\partial \hat{E}}{\partial E_{-i}} \end{bmatrix} = \begin{bmatrix} \frac{-b_{2E} - b_{EE}g}{\Delta} \\ \frac{b_{22} + b_{E2}g}{\Delta} \end{bmatrix} \text{of sign pattern} \begin{bmatrix} \frac{-\{+ \text{ or } 0\} - \{+\}}{\{+\}} \\ \frac{\{+\} + \{+ \text{ or } 0\}}{\{+\}} \end{bmatrix} = \begin{bmatrix} - \\ + \end{bmatrix},$$

i. e., $\frac{\partial \hat{x}_2}{\partial E_{-i}} < 0$ and i. e., $\frac{\partial \hat{E}}{\partial E_{-i}} > 0$. In words, an increase of the contributions to the public good by everybody else induces our consumer to decreases her contribution (and her consumption of good 2), at a rate that still allows her to enjoy a higher level of the public good.

1(b)(iv). Let $b_{2E} < 0$. Now the two terms in the numerators of the expressions $\begin{bmatrix} \frac{\partial \hat{x}_2}{\partial E_{-i}} \\ \frac{\partial \hat{E}}{\partial E_{-i}} \end{bmatrix} = \begin{bmatrix} \frac{-b_{2E} - b_{EE}g}{\Delta} \\ \frac{b_{22} + b_{E2}g}{\Delta} \end{bmatrix}$ are of opposite sign, and hence the partial derivatives cannot be signed without additional assumptions on the entries of the matrix B and g.

Maintain $b_{2E} < 0$. $\frac{\partial \hat{x}_2}{\partial E_{-i}}$ is then negative when $-b_{2E} - b_{EE}g < 0$, while $\frac{\partial \hat{x}_2}{\partial E_{-i}}$ is positive if $b_{22} + b_{E2}g > 0$. In this case, b_{2E} is negative but relative small in absolute value, and the signs are those of 1(b)(iii), where $b_{2E} \geq 0$.

If $-b_{2E} - b_{EE}g < 0$ but $b_{22} + b_{E2}g < 0$ (i. e., $-b_{EE}g < b_{E2} < -b_{22}/g < 0$, meaning that b_{E2} takes an intermediate value) then both derivatives are negative.

If $-b_{2E} - b_{EE}g > 0$, then $\frac{\partial \hat{x}_2}{\partial E_{-i}} > 0$, which implies that $\frac{\partial \hat{x}_2}{\partial E_{-i}}$ is also positive, because $\hat{E} = E_{-i} + g\hat{x}_2$.

1(c) (i) We first define the Walrasian demand functions $(\tilde{x}_1(\bar{p}_1, \bar{p}_2, \bar{p}_E, \bar{w}), \tilde{x}_2(\bar{p}_1, \bar{p}_2, \bar{p}_E, \bar{w}), \tilde{E}(\bar{p}_1, \bar{p}_2, \bar{p}_E, \bar{w}))$ by the maximization of u subject to the budget constraint $\bar{p}_1 x_1 + \bar{p}_2 x_2 + \bar{p}_E E \leq \bar{w}$. Compute $(\tilde{x}_2(\bar{p}_1, \bar{p}_2, \bar{p}_E, \bar{w}), \tilde{E}(\bar{p}_1, \bar{p}_2, \bar{p}_E, \bar{w}))$.

It is the usual quasilinear Walrasian demand, with Lagrangian

$$x_1 + [a_2, a_E]\begin{bmatrix} x_2 \\ E \end{bmatrix} - \frac{1}{2}[x_2, E]B\begin{bmatrix} x_2 \\ E \end{bmatrix} - \rho(\bar{p}_1 x_1 + \bar{p}_2 x_2 + \bar{p}_E E - \bar{w})$$

and first-order conditions

$$1 - \rho\bar{p}_1 = 0,$$

$$\begin{bmatrix} a_2 \\ a_E \end{bmatrix} - B\begin{bmatrix} x_2 \\ E \end{bmatrix} - \rho\begin{bmatrix} \bar{p}_2 \\ \bar{p}_E \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad \rho(\bar{p}_1 x_1 + \bar{p}_2 x_2 + \bar{p}_E E - \bar{w}) = 0.$$

From the first equation, we obtain $\rho = 1/\bar{p}_1$. Hence, $(\tilde{x}_2, \tilde{E})$ can be found by solving

$$\begin{bmatrix} a_2 \\ a_E \end{bmatrix} - B\begin{bmatrix} x_2 \\ E \end{bmatrix} = \begin{bmatrix} \bar{p}_2/\bar{p}_1 \\ \bar{p}_E/\bar{p}_1 \end{bmatrix}, \text{ i. e., } -B\begin{bmatrix} x_2 \\ E \end{bmatrix} = \begin{bmatrix} -a_2 + (\bar{p}_2/\bar{p}_1) \\ -a_E + \bar{p}_E/\bar{p}_1 \end{bmatrix} \text{ or } B\begin{bmatrix} x_2 \\ E \end{bmatrix} = \begin{bmatrix} a_2 - (\bar{p}_2/\bar{p}_1) \\ a_E - \bar{p}_E/\bar{p}_1 \end{bmatrix}.$$

Multiplying both sides by the inverse B^{-1} we obtain

$$\begin{bmatrix} x_2 \\ E \end{bmatrix} = \frac{1}{\bar{\Delta}}\begin{bmatrix} b_{EE} & -b_{2E} \\ -b_{E2} & b_{22} \end{bmatrix}\begin{bmatrix} a_2 - (\bar{p}_2/\bar{p}_1) \\ a_E - (\bar{p}_E/\bar{p}_1) \end{bmatrix}.$$

Hence, neither the Walrasian for good 2 nor that for E depends on wealth, and therefore

$$\begin{bmatrix} s_{22} & s_{2E} \\ s_{E2} & s_{EE} \end{bmatrix} = \begin{bmatrix} \frac{\partial \tilde{x}_2}{\partial \bar{p}_2} & \frac{\partial \tilde{x}_2}{\partial \bar{p}_E} \\ \frac{\partial \tilde{E}}{\partial \bar{p}_2} & \frac{\partial \tilde{E}}{\partial \bar{p}_E} \end{bmatrix} = \frac{1}{\bar{\Delta}}\begin{bmatrix} -(b_{EE}/\bar{p}_1) & (b_{2E}/\bar{p}_1) \\ (b_{E2}/\bar{p}_1) & -(b_{22}/\bar{p}_1) \end{bmatrix} = \frac{1}{\bar{p}_1\bar{\Delta}}\begin{bmatrix} -b_{EE} & b_{2E} \\ b_{E2} & -b_{22} \end{bmatrix},$$

where $\bar{\Delta} \equiv b_{22}b_{EE} - b_{2E}b_{E2} > 0$ by the positive-definiteness of B.

1(c) (iii). It follows that goods 2 and E are net (and gross) complements if $s_{2E} < 0$, i.e., $b_{2E} < 0$, and net (and gross) substitutes if $s_{2E} > 0$, i. e., $b_{2E} > 0$.

1(c) (iv). Use 1(c) (iii) to verbally discuss the conditions analyzed in 1(b) (iii)-(v).

As seen in 1(b)(iii), if $b_{2E} \geq 0$, then $\frac{\partial \hat{x}_2}{\partial E_{-i}} < 0$ and i. e., $\frac{\partial \hat{E}}{\partial E_{-i}} > 0$. In view of 1(c)(iii), this is the case where goods 2 and E are net substitutes (or borderline substitute-complements). An increase in the amount E_{-i} of the public good exogenously made available to the consumer leads her to decrease her demand for the substitute good 2.

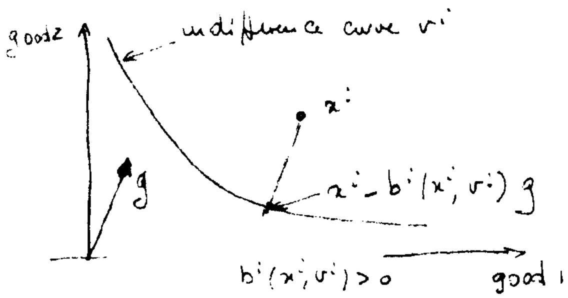
If $b_{2E} < 0$, then goods 2 and E are net complements, and the signs of the partial derivatives $\frac{\partial \hat{x}_2}{\partial E_{-i}}$ and $\frac{\partial \hat{E}}{\partial E_{-i}}$ are ambiguous, depending on the magnitudes of the various Slutsky terms. If the complementarity is strong ($b_{2E} < 0$ and large in absolute values), then it may be that the consumer ends up demanding a higher amount of good 2 when E_{-i} increases.

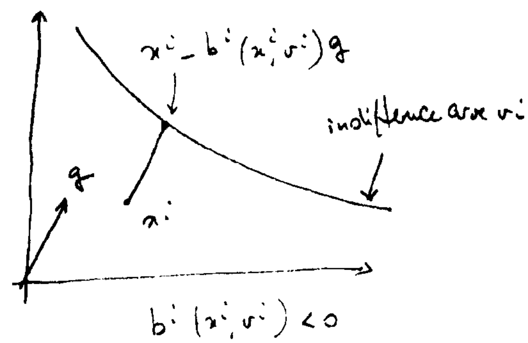
The table summarizes the comparative statics results.

	Goods 2 and E substitutes ($s_{2E} \geq 0$, $b_{2E} \geq 0$)	Goods 2 and E complements ($s_{2E} < 0$, $b_{2E} < 0$)		
		$-b_{EE}g < b_{2E} < 0$		$b_{2E} < -b_{EE}g < 0$
		$-b_{22} < b_{2E}g < 0$	$b_{2E} < -b_{22}g < 0$	
$\frac{\partial \hat{x}_2}{\partial E_{-i}}$	–	–	–	+
$\frac{\partial \hat{E}}{\partial E_{-i}}$	+	+	–	+

Answer key June, Questions 2,3

2(a) If $u^i(x^i) > v^i$, $b^i(x^i, v^i)$ is the maximum number of units of the bundle g which can be subtracted from x^i still keeping agent i at the utility level v^i. If $u^i(x^i) < v^i$, and $b^i(x^i, v^i)$ is negative and $|b^i(x^i, v^i)|$ is the minimum units of bundle g which must be added to the consumption x^i in order that agent i reaches the utility level v^i. Formally, if $u^i(x^i) \geq v^i$ and $b^i(x^i, v^i) < 0$, by strict monotonicity $u^i(x^i - b^i(x^i, v^i)g > u^i(x^i) \geq v^i$ and by continuity $u^i(x^i - (b^i(x^i, v^i) + \epsilon)g > v^i$ for ϵ sufficiently small, which contradicts the maximum property of $b^i(x^i, v^i)$. Thus $b^i(x^i, v^i) \geq 0$ The same reasoning shows that when $u^i(x^i) > v^i$ then $b^i(x^i, v^i) > 0$.





(b) Let $\tilde{\beta}^i = b^i(\tilde{x}^i, v^i)$. By definition,

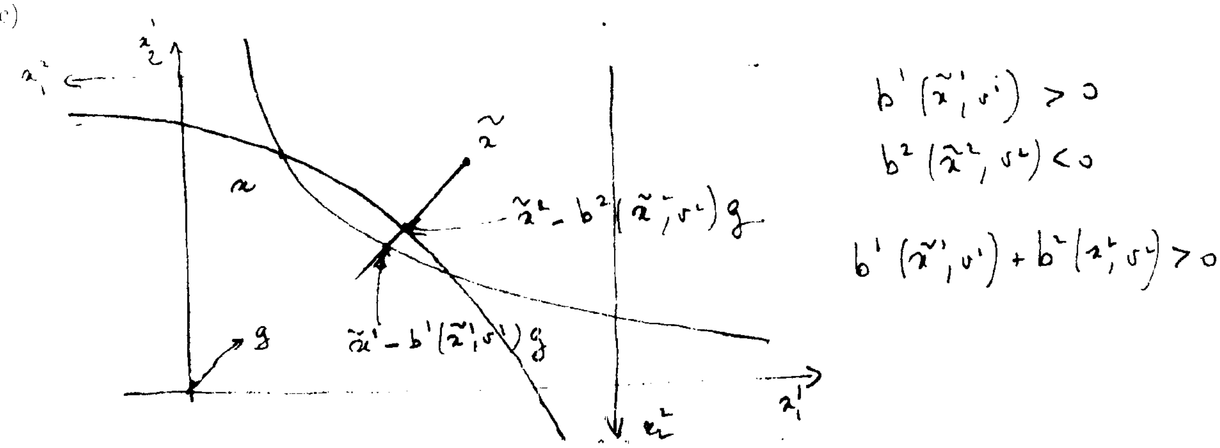
$$u^i(\tilde{x}^i - \tilde{\beta}^i g) \geq u^i(x^i)$$

On the other hand the allocation $(\tilde{x}^i - \tilde{\beta}^i g)_{i=1}^I$ does not use all the available resources since

$$\sum_i (\tilde{x}_l^i - \tilde{\beta}^i g_\ell) \leq \sum_i \tilde{x}_\ell^i \leq \sum_i \omega_\ell^i$$

where the second inequality comes from the fact that $\tilde{x}$ is feasible, and where the first inequality is strict if $g_\ell > 0$. Let ϵ be such that $I\epsilon \leq \sum_i \tilde{\beta}^i$. Since the preference are strongly monotone, the allocation $(\tilde{x}^I - \tilde{\beta}^i g + \epsilon g)_{i=1}^I$ is feasible and strictly preferred by each agent to x^i, so that x^i is not Pareto optimal.

(c)



(d) Since $b(x^{i*}, v^{i*}) = 0$, and x^* satisfies the constraint of the program (1), the maximum in (1) is larger or equal to 0. If the maximum is strictly positive, then there exist an allocation $\tilde{x}$ which satisfies the constraint, i.e. which is feasible, and which is such that $\sum_i b^i(\tilde{x}^i, v^{i*}) > 0$. By (b) this contradicts that x^* is Pareto optimal.

(e) Suppose x^* is solution of (1) (with a maximum value of zero for the objective) and that x^* is not Pareto optimal. There exists a feasible allocation $\tilde{x}$ such that $u^i(\tilde{x}^i) \geq u^i(x^{*i})$ with at least a strict inequality. By (a), $b^i(\tilde{x}^i, v^{*i} \geq 0$ with at least one strict inequality. The allocation $\tilde{x}$ satisfies the constraints of (1), since it is feasible, and gives a positive value to the objective, which contradicts that x^* is a solution of (1).

(f) Let x^i and $\tilde{x}^i$ be two consumption bundles, v^i a utility level, $\beta^i = b(x^i, v^i)$, $\tilde{\beta}^i = b^i(\tilde{x}^i, v^i)$ and $t \in [0,\ 1]$. By quasi-concavity of u^i

$$u^i(tx^i + (1-t)\tilde{x}^i - (t\beta^i + (1-t)\tilde{\beta}^i)g) \geq \min\{u^i(x^i - \beta^i g), u^i(\tilde{x}^i - \tilde{\beta}^i g)\} \geq v^i$$

Thus $b^i(tx^i + (1-t)\tilde{x}^i, v^i) \geq t\beta^i + (1-t)\tilde{\beta}^i$, so that b^i is concave in x^i.

3(a)(i). If $\bar{t}_i > 0$ and $\bar{x}_i > m$, then the charity could increase m by decreasing $\bar{t}_i$ by ϵ and distributing ϵ to the agents who have the minimum consumption. So only agents at the minimum consumption level receive transfers from the charity.

(a)(ii) There are two cases to consider:

(α) If $Z \le n_P[(\omega_R - z_R) - (\omega_P - z_P)]$, the charity does not have enough resources (or at the plimit has just enough resources) to equalize the consumption of all agents. By (i) only the poor receive a transfer, thus

$$t_R(Z) = 0, \quad t_P(Z) = \frac{Z}{n_P}, \quad m(Z) = \omega_P - z_P + \frac{Z}{n_P}$$

(β) If $Z > n_P[(\omega_R - z_R) - (\omega_P - z_P)]$ and the charity give transfers only to the poor agents, they will have more consumption than the rich, which contradicts (i). Thus both types of agents must receive transfers and by (i) they must all have the minimum (same) consumption. Thus the transfers t must satisfy

$$\omega_R - z_R + t_R = \omega_P - z_P + t_P, \qquad n_R t_R + n_P t_P = Z$$

Solving the two linear equations give

$$t_R(Z) = \frac{Z - n_P[(\omega_R - z_R) - (\omega_P - z_P)]}{n_R + n_P}, \qquad t_P(Z) = \frac{Z + n_R[(\omega_R - z_R) - (\omega_P - z_P)]}{n_R + n_P}$$

$$m(Z) = \frac{n_R(\omega_R - z_R) + n_P(\omega_P - z_P) + Z}{n_R + n_P}$$

The functions are linear in Z but there is a change of slope when $Z = n_P[(\omega_R - z_R) - (\omega_P - z_P)]$.

(b)(i) In an equilibrium with voluntary contribution, $\bar{z}_i \ge 0$ must maximize $\ln(\omega_i + \bar{t}_i - z_i + \gamma \ln(m(\bar{Z}^{-i} + z_i))$, with obvious notation. The marginal cost of increasing z^i is MC$= \frac{1}{\bar{x}^i}$ and the marginal benefit is $\gamma \frac{m'(\bar{Z})}{m(\bar{Z})}$. Since $\gamma < 1$ and $m'(\bar{Z}) \le 1$ (the minimum consumption cannot increase by more than the additional contribution), if $\bar{x}^i = m(\bar{Z})$ the marginal cost exceeds the marginal benefit. If $\bar{z}_i$ was positive agent i could increase his utility by decreasing his contribution. Thus $\bar{z}_i$ must be equal to 0.

(b)(ii) Consider an equilibrium with voluntary contributions $(\bar{x}_R, \bar{z}_R, \bar{t}_R)$, $(\bar{x}_P, \bar{z}_P, \bar{t}_P)$ in which agents of the same type are treated equally. The minimum consumption is either that of the poor, or that of the rich, or that of both types of agents who then have identical consumption. Let us show that the last two cases are not possible. Suppose all agents are at minimum consumption. Then by (i) nobody contributes, and it is thus impossible to equalize the consumption of the rich and the poor. Thus this is not possible in equilibrium. Suppose that the rich have the minimum consumption. Then by (i) they do not contribute and only the poor contribute. But then $\bar{x}_R \geq \omega_R > \omega_P - \bar{z}_P = \bar{x}_P$, which contradicts the assumption that the rich have the lower consumption. Thus it must be that the poor have the minimum consumption and we are in the case of question (a)(ii)(α) with $Z = n_R \bar{z}_R \geq 0$.

The FOCs for the maximum problem of a rich agent

$$\max \ln(\omega_R - z) + \gamma \ln(m(Z^{-i} + z)), \quad z \geq 0$$

is

$$-\frac{1}{\omega_R - z} + \gamma \frac{m'(Z^{-i} + z)}{m(Z^{-i} + z)} \leq 0, \quad = \text{ if } z > 0$$

By (a)(ii)

$$m(Z^{-i} + z) = \omega_P + \frac{Z^{-i} + z}{n_P}, \qquad m'(Z^{-i} + z) = \frac{1}{n_P}$$

Thus if $z > 0$ and $Z^{-i} + z = n_R z$ (equal contributions)

$$-\frac{1}{\omega_R - z} + \gamma \frac{1/n_P}{\omega_P + (n_R z)/n_P} \iff (n_R + \gamma) z = \gamma \omega_R - n_P \omega_P$$

Thus if $\gamma \omega_R - n_P \omega_P > 0$, the rich contribute, and if $\gamma \omega_R - n_P \omega_P \leq 0$, the equilibrium involves no redistribution. Note that the case where there are positive contributions to the charity is very restrictive: it must be that the total income of the poor is less than the income of one rich agent.

(b)(iii) Suppose that the equilibrium has no redistribution. Suppose that a planner takes ϵ from each rich agent and distributes $n_R \epsilon / n_P$ to each poor agent. The utility of the poor agents increase since both their own consumption and m increase. If ϵ is small the change is utility of a rich agent is approximately $-1/\omega_R \epsilon + (1/\omega_P)(n_R \epsilon / n_P)$, which is positive if $\gamma n_R \omega_R > n_P \omega_P$. Thus it is possible to increase the utility of all agents by redistributing from the rich to the poor.

(c) A representative rich agent will choose the tax rate $t \geq 0$ which maximizes $\ln(\omega_R(1 - t) + \gamma \ln(\omega_P + \frac{n_R \omega_R t}{n_P})$. Assuming $t > 0$, writing the FOC, and solving for t leads to

$$t = \frac{\gamma n_R \omega_R - n_P \omega_P}{(1 + \gamma) n_R \omega_R}$$

which is positive if $\gamma n_R \omega_R - n_P \omega_P > 0$. Thus this system where the rich tax themselves involves more redistribution than the voluntary contribution equilibrium. One can show that actually the equilibrium with tax is Pareto optimal. The reason why it works better is that when computing the marginal benefit of increasing the tax to choose the optimal tax rate, a representative of the rich takes into account that each rich person will contribute $t\omega_R$, which in good case like this one amounts to summing the marginal benefits for all rich agents of one agent's contribution. This is different from a voluntary contribution equilibrium where each rich agent only considers his own marginal benefit, forgetting the marginal benefit of his contribution for all other agents. The marginal benefit considered in choosing the tax being closer to the true social marginal benefit, the level of the contributions is closer to being optimal.

Microeconomics Prelim June 2008
Answer Keys for Questions 4 and 5

4.

(a.1) We need to be able to find a p such that $p \geq s_A$ and $p \leq \sum_{i \in Q} p_i b_i$. Thus a necessary and sufficient condition is $s_A \leq \sum_{i \in Q} p_i b_i$.

(a.2) $\alpha \geq \dfrac{4s_A}{s_A + s_B + s_C + s_D}$.

(b.1) We need to be able to find a p such that $s_B \leq p < s_A$ (this is always possible since, by hypothesis, $s_B < s_A$) and $p \leq \sum_{i \in \{B,C,D\}} \left(\dfrac{p_i}{p_B + p_C + p_D} \right) b_i$. Thus a necessary and sufficient condition is $s_B \leq \left(\dfrac{1}{p_B + p_C + p_D} \right) \sum_{i \in \{B,C,D\}} p_i b_i$ (if this condition is satisfied an appropriate price exists).

(b.2) $\alpha \geq \dfrac{3s_B}{s_B + s_C + s_D}$.

(c.1) We need $s_D \leq p < s_C$ and $p \leq b_D$. Thus a necessary and sufficient condition is $s_D \leq b_D$, which is one of our hypotheses.

(c.2) $\alpha \geq 1$.

(d) Any situation where there are unsold cars is Pareto inefficient, since buyers value cars more than the sellers. Thus, while the equilibria of part (a) are Pareto efficient, those of part (b), and (c) are Pareto inefficient.

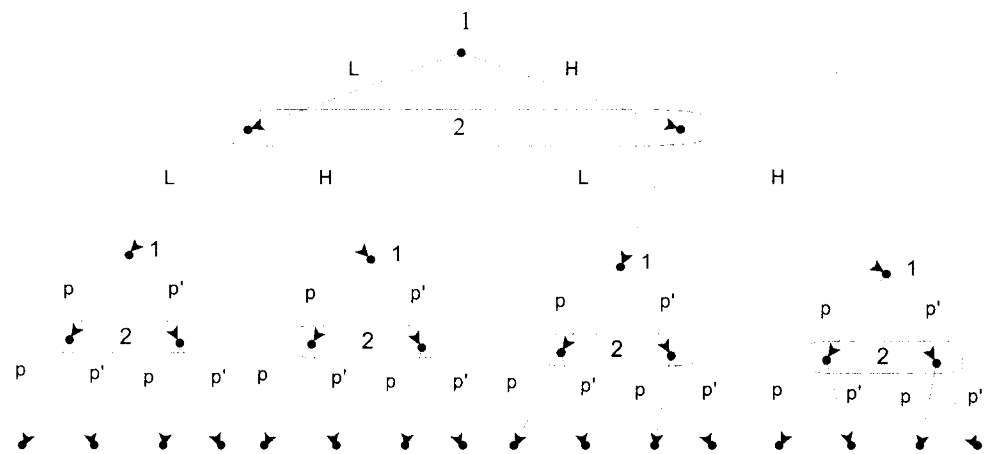
(e) If $p_W \geq s_A + q_A R$ then the owner of an A car is willing to sell with warranty at price p_W; if, furthermore, $p_W < s_i + q_i R$ for all $i \in \{B,C,D\}$ then the owners of cars of qualities B, C and D are not willing to sell with warranty at price p_W. Only cars of quality D are offered for sale without warranty at price p_N if $s_D \leq p_N$ and $p_N < s_i$ for all $i \in \{A,B,C\}$ and such a price exists since, by hypothesis, $s_D < s_i$ for all $i \in \{A,B,C\}$. On the buyer side we need $p_W \leq V$ for the buyer to be willing to buy with warranty and $p_N \leq b_D$ for the buyer to be willing to buy a D car without warranty; the latter condition is always possible since, by hypothesis, $s_D < b_D$. Thus a sufficient condition is $s_A + q_A R < \min\left\{V, \min_{i \in \{B,C,D\}} \{s_i + q_i R\}\right\}$.

(f) In this case, $s_A + q_A R = 16.5$, $s_B + q_B R = 17.5$, $s_C + q_C R = 20$, $s_D + q_D R = 21$ so that the sufficient condition of part (f) is satisfied. Any pair (p_W, p_N) with $16.5 \leq p_W < 17.5$ and $6 \leq p_N \leq 8 = \min\{b_C = V - q_C R = 9,\ s_C = 8\}$ will yield the desired equilibrium.

(g) First we compute $b_i = V - q_i R$: $b_A = 22.5$, $b_B = 16.5$, $b_C = 12$, $b_D = 9$. Since $p_W = 17 > s_A + q_A R = 16.5$, while $p_N = 11 < s_A = 15$, cars of quality A will be sold with warranty. Since $p_W = 17 < s_i + q_i R$ for all $i \in \{B,C,D\}$ (respectively, 17.5, 20 and 21: see part (f)), no other qualities will be sold with warranty. Since $p_N = 11 > s_B = 10$, all other qualities will be offered for sale. Since $\left(\frac{1}{p_B+p_C+p_D}\right)\sum_{i\in\{B,C,D\}} p_i b_i = \frac{1}{3}(16.5+12+9)=12.5$ buyers are willing to buy a car without warranty at price $p_N = 11$, realizing that it will be of either quality *B* or *C* or *D*.

5.

(a) The extensive form is as follows.



(b) (L, p, p´, p´, p) (going from left to right). Firm 1 has $2^5 = 32$ strategies.

(c) $D_1(p_1,p_2)=\begin{cases} 80-8p_1 & if\ p_1<p_2 \\ \frac{1}{2}(80-8p_1) & if\ p_1=p_2 \\ 0 & if\ p_1>p_2 \end{cases}$

(d) In the subgame where they both choose H as well as in the subgame where they both choose L, by Bertrand's theorem the unique Nash equilibrium is $p_1 = p_2 = 0$ with corresponding profits of zero for both firm. Now consider a subgame where one firm chooses H and the other chooses L. The profit functions are $\pi_H = p_H(80-40p_H+40p_L)$ and $\pi_L = p_L(40p_H-50p_L)$. To find the Nash equilibrium solve $\frac{\partial \pi_H}{\partial p_H}=0$ and $\frac{\partial \pi_L}{\partial p_L}=0$. The solution is $\left(p_H=\frac{5}{4}=1.25, p_L=\frac{1}{2}=0.5\right)$ with corresponding profits $\pi_H=\frac{125}{2}=62.5$ and $\pi_L=\frac{25}{2}=12.5$. Thus the game can be reduced to the following one-stage game:

		Firm 2 H	Firm 2 L
Firm 1	H	0 , 0	62.5 , 12.5
	L	12.5 , 62.5	0 , 0

Thus there are two subgame-perfect equilibria:

$$\left(\underbrace{(H, \text{if HH}\, p_1=0,\ \text{if HL}\, p_1=1.25,\ \text{if LH}\, p_1=0.5,\ \text{if LL}\, p_1=0)}_{\text{firm 1's strategy}}, \underbrace{(L, \text{if HH}\, p_2=0,\ \text{if HL}\, p_2=0.5,\ \text{if LH}\, p_2=1.25,\ \text{if LL}\, p_2=0)}_{\text{firm 2's strategy}}\right)$$

where firm 1 chooses H and sets a price of 1.25 and firm 2 chooses L and sets a price of 0.5, and

$$\left(\underbrace{(L, \text{if HH } p_1 = 0, \text{ if HL } p_1 = 1.25, \text{ if LH } p_1 = 0.5, \text{ if LL } p_1 = 0)}_{\text{firm 1's strategy}}, \underbrace{(H, \text{if HH } p_2 = 0, \text{ if HL } p_2 = 0.5, \text{ if LH } p_2 = 1.25, \text{ if LL } p_2 = 0)}_{\text{firm 2's strategy}}\right)$$

where firm 1 chooses L and sets a price of 0.5 and firm 2 chooses H and sets a price of 1.25.

(e) There are many. For example, pick one of the above equilibria and change both prices in the unreached second-stage games to zero.

(f) In the subgame where they both choose H, inverse demand is $P = 10 - \frac{Q}{8}$ so that the profit functions are $\pi_1 = q_1\left(10 - \frac{q_1 + q_2}{8}\right)$ and $\pi_2 = q_2\left(10 - \frac{q_1 + q_2}{8}\right)$. To find the Nash equilibrium solve $\frac{\partial \pi_1}{\partial q_1} = 0$ and $\frac{\partial \pi_2}{\partial q_2} = 0$. The solution is $q_1 = q_2 = \frac{80}{3} = 26.67$ with corresponding profits $\pi_1 = \pi_2 = \frac{800}{9} = 88.89$.

In the subgame where they both choose L, inverse demand is $P = 8 - \frac{Q}{10}$ so that the profit functions are $\pi_1 = q_1\left(8 - \frac{q_1 + q_2}{10}\right)$ and $\pi_2 = q_2\left(8 - \frac{q_1 + q_2}{10}\right)$. To find the Nash equilibrium solve $\frac{\partial \pi_1}{\partial q_1} = 0$ and $\frac{\partial \pi_2}{\partial q_2} = 0$. The solution is $q_1 = q_2 = \frac{80}{3} = 26.67$ with corresponding profits $\pi_1 = \pi_2 = \frac{640}{9} = 71.11$

Now consider a subgame where one firm chooses H and the other chooses L. The profit functions are $\pi_H = q_H\left(10 - \frac{q_H}{8} - \frac{q_L}{10}\right)$ and $\pi_L = q_L\left(8 - \frac{q_H}{10} - \frac{q_L}{10}\right)$. To find the Nash equilibrium solve $\frac{\partial \pi_H}{\partial q_H} = 0$ and $\frac{\partial \pi_L}{\partial q_L} = 0$. The solution is $(q_H = 30, q_L = 25)$ with corresponding profits $\pi_H = \frac{225}{2} = 112.5$ and $\pi_L = \frac{125}{2} = 62.5$. Thus the game can be reduced to the following one-stage game:

		Firm 2 H	Firm 2 L
Firm 1	H	88.89 , 88.89	112.5 , 62.5
	L	62.5 , 112.5	71.11 , 71.11

Thus there is a unique subgame-perfect equilibrium where they both choose H and produce 26.67 units each:

strategy of firm 1: $(H, \text{if HH } q_1 = 26.67, \text{ if HL } q_1 = 30, \text{ if LH } q_1 = 25, \text{ if LL } q_1 = 26.67)$

strategy of firm 2: $(H, \text{if HH } q_2 = 26.67, \text{ if HL } q_2 = 25, \text{ if LH } q_2 = 30, \text{ if LL } q_2 = 26.67)$

(g) The main difference is that in scenario 1 the firms choose to differentiate their products, while in scenario 2 they choose to produce a homogeneous product.