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PRELIMINARY EXAMINATION FOR THE Ph.D. DEGREE

Part I

ANSWER KEY

I(a) There are N goods. Good 1 is leisure, measured in hours, its consumption denoted x_1, and its dollar price (per hour) $p_1 > 0$. Goods 2 to N are consumption goods, with dollar prices $p_2, \ldots, p_N$, all positive. The consumer is endowed with $\omega_1 > 0$ units of leisure and zero units of any other good ($\omega_2 = 0 = \ldots = \omega_N = 0$). The consumer is also endowed with m dollars of price-independent wealth. We define the variable $L \equiv \omega_1 - x_1$ and call it the supply of labor. Her preferences are represented by a differentiable utility function $u : X \rightarrow \Re$, increasing in all its arguments in the interior of X, where $X \subset \Re^N_+$ denotes her consumption set. Denote the Jevonsian supply-of-labor function by $\hat{L}(p_1, p_2, ..., p_N)$ (we take the parameters m and ω_1 as given, so that they do not appear as arguments).

I(a).1. Write the Slutsky equation for the decomposition of the "total effect" $\frac{\partial \hat{L}}{\partial p_1}$ of a change in p_1 on the supply of labor, expressing the substitution and the wealth effects in terms of the partial derivatives of the Hicksian and Walrasian demand-for-leisure functions.

The standard form of the Slutsky equation for the supply of labor is

$$\frac{\partial \hat{L}}{\partial p_1} = \frac{\partial L^H}{\partial p_1} + \frac{\partial \tilde{L}}{\partial w} L \, ,$$

where:

$\frac{\partial L^H}{\partial p_1} = -\frac{\partial h_1}{\partial p_1}$ (h_1 is the Hicksian demand for leisure) is the SUBSTITUTION EFFECT, nonnegative given that $\frac{\partial h_1}{\partial p_1}$ is always nonpositive, by the negative semidefiniteness of the Slutsky matrix.

$\frac{\partial \tilde{L}}{\partial w} = -\frac{\partial \tilde{x}_1}{\partial w}$ ($\tilde{x}_1$ is the Walrasian demand for leisure) is the WEALTH EFFECT.

Hence, we can write the Slutsky equation in terms of the derivatives of the Hicksian and Walrasian demand for leisure as

$$\frac{\partial \hat{L}}{\partial p_1} = -\frac{\partial h_1}{\partial p_1} - \frac{\partial \tilde{x}_1}{\partial w} L \,.$$

I(a).2. Assume that leisure is a normal good. What can you say about the sign of the substitution and wealth effects of I(a).1? What can you say about the sign of the total effect $\frac{\partial \hat{L}}{\partial p_1}$? Interpret in words.

We have that $\frac{\partial \hat{L}}{\partial p_1} = -\frac{\partial h_1}{\partial p_1} - \frac{\partial \tilde{x}_1}{\partial w} L$, where

SUBSTITUTION EFFECT = $-\frac{\partial h_1}{\partial p_1} \geq 0$

WEALTH EFFECT = $-\frac{\partial \tilde{x}_1}{\partial w} L < 0$, because $\frac{\partial \tilde{x}_1}{\partial w} > 0$ by the assumption of normality.

Hence, the substitution and the wealth effect are of (weakly) opposite signs, and the sign of the total effect may in principle be positive, negative or zero.

In words, an increase in the wage p_1 rises the opportunity cost of leisure, (weakly) reducing the demand for it, which is equivalent to (weakly) increasing the supply of labor (SUBSTITUTION EFFECT). On the other hand, an increase in the wage p_1 increases the real wealth of the consumer, increasing her demand for all normal goods, including leisure, which is equivalent to reducing the labor supply (WEALTH EFFECT). Hence, the sign of the total effect can in principle be anything.

I(b). We now consider a new model. We still have prices $(p_1, p_2, \ldots, p_N) >> 0$, an endowment vector $(\omega_1, 0, \ldots 0)$, and the consumer's price-independent wealth *m*. But we introduce two differences.

First, <u>leisure no longer enters the utility function,</u> so that preferences are defined on an open subset $Z \subset \Re_{+}^{N-1}$, with typical element $(x_2, \ldots, x_N)$, and represented by a differentiable, strictly quasiconcave and strictly increasing utility function $\beta : Z \to \Re$.

Second, <u>the consumption of any commodity</u> $j \in \{2,\ldots, N\}$ <u>takes time</u>, so that the available amount of leisure ω_1 must be allocated between the supply of labor in the market, denoted L, which is paid at rate p_1 per hour, and unpaid time devoted to consumption activities. More specifically, the data of the economy include, for $j = 2,\ldots, N$, a nonnegative coefficient t_j expressing the per-unit amount of time required by the consumption of good of j, so that the consumption of x_j units of good j requires spending $t_j\ x_j$ units of time, in addition to spending $p_j\ x_j$ dollars.

I(b).1. Write the consumer optimization problem that yields her demand for goods $\breve{x}_j(p_1, p_2,\ldots, p_N), j = 2,\ldots, N$ and her supply of labor $\breve{L}(p_1, p_2,\ldots, p_N)$ (again, we take the parameters m and ω_1 as given). (Assume that the solution exists.)

$\max_{x_2,\ldots,x_N,L} \beta(x_2,\ldots,x_N)$ subject to

the budget constraint: $p_2x_2 + \ldots + p_Nx_N \le p_1L + m$

and the time constraint $t_2x_2 + \ldots + t_Nx_N + L \le \omega_1$. (1)

I(b).2. Argue that the time endowment ω_1 is totally spent.

It is the usual argument for Walras Law, given the assumed local nonsatiation. If $t_2x_2 + \ldots + t_Nx_N + L < \omega_1$, then the consumer could increase the left hand side of the budget constraint by selling more labor.

I(b).3. Combine the budget constraint and the time constraint into a single equality constraint involving $(x_2,\ldots, x_N)$, and interpret.

Multiplying the time constraint by p_1, we get $p_1t_2x_2 + \ldots + p_1t_Nx_N - p_1\omega_1 = -p_1L$ which added to the budget equality yields

$$(p_2 + p_1t_2)x_2 + \ldots + (p_N + p_1t_N)x_N = p_1\omega_1 + m.$$

<u>Interpretation</u>. It is a budget constraint for goods $2,\ldots, N$, where the "comprehensive price" of good j has two components: the direct dollar price p_j and the indirect cost p_1t_j due to the market value of the time spent consuming good j.

I(b).4. Denote a price vector by $(\pi_2,\ldots,\pi_N)$, and a wealth magnitude by w, and define the Walrasian demand function $(\tilde{x}_2(\pi_2,\ldots,\pi_N, w),\ldots, \tilde{x}_N(\pi_2,\ldots,\pi_N, w))$, as usual, by the solution to the problem $\max \beta(x_2,\ldots,x_N)$ subject to $\sum_{j=2}^{N} \pi_j x_j = w$. Similarly, define the Hicksian demand function $(h_2(\pi_2,...,\pi_N,\bar{u}),...,h_N(\pi_2,...,\pi_N,\bar{u}))$ by the solution to the problem $\min \sum_{j=2}^{N} \pi_j x_j$ subject to $\beta(x_2,...,x_N)=\bar{u}$. Using I.(b)3, express the functions $\breve{x}_j(p_1,p_2,...,p_N), j=2,...,N$ and $\breve{L}(p_1,p_2,...,p_N)$ in terms of the Walrasian demand functions $\tilde{x}_j(\pi_2,...,\pi_N,w), j=2,...,N$.

We can write the argument π_j in the Walrasian demand function as the "comprehensive price" of good j as specified in Ib.(3), $j = 2,\ldots, N$. Similarly, we can write wealth as $w = p_1\omega_1 + m$. Therefore:

$$\breve{x}_j(p_1,p_2,...,p_N)=\tilde{x}_j(p_2+p_1t_2,...,p_N+p_1t_N,p_1\omega_1+m), j=2,...,N.$$

Using the time constraint (1), we can then write

$$\breve{L}(p_1,p_2,...,p_N)=\omega_1-\sum_{j=2}^{N} t_j\tilde{x}_j(p_2+p_1t_2,...,p_N+p_1t_N,p_1\omega_1+m).$$

1(b).5. Compute $\frac{\partial \breve{L}}{\partial p_1}$ in terms of the derivatives of the Walrasian demand functions $(\tilde{x}_2(\pi_2,...,\pi_N,w),...,\tilde{x}_N(\pi_2,...,\pi_N,w))$.

$$\frac{\partial \breve{L}}{\partial p_1}=-\sum_{j=2}^{N} t_j\left[\sum_{k=2}^{N}\frac{\partial \tilde{x}_j}{\partial \pi_k}t_k+\frac{\partial \tilde{x}_j}{\partial w}\omega_1\right]. \qquad (2)$$

1(b).6. By using the Slutsky decomposition of the total effects $\frac{\partial \tilde{x}_j}{\partial \pi_k}$ (j, $k = 2,\ldots, N$), write $\frac{\partial \breve{L}}{\partial p_1}$ as the sum of a "substitution term" involving the derivatives $\frac{\partial h_j}{\partial \pi_k}$ of Hicksian demand, and a "wealth term" involving the derivatives $\frac{\partial \tilde{x}_j}{\partial w}$ of Walrasian demand.

Slutsky decomposition: $\frac{\partial \tilde{x}_j}{\partial \pi_k}=\frac{\partial h_j}{\partial \pi_k}-\frac{\partial \tilde{x}_j}{\partial w}x_k$, which substituted into (2) yields

$$\frac{\partial \breve{L}}{\partial p_1} = -\sum_{j=2}^{N} t_j \left[\sum_{k=2}^{N} \left[\frac{\partial h_j}{\partial \pi_k} - \frac{\partial \tilde{x}_j}{\partial w} x_k \right] t_k + \frac{\partial \tilde{x}_j}{\partial w} \omega_1 \right]$$

$$= -\sum_{j=2}^{N} \sum_{k=2}^{N} t_j \frac{\partial h_j}{\partial \pi_k} t_k + \sum_{j=2}^{N} \sum_{k=2}^{N} t_j \frac{\partial \tilde{x}_j}{\partial w} x_k t_k - \sum_{j=2}^{N} t_j \frac{\partial \tilde{x}_j}{\partial w} \omega_1$$

$$= -\sum_{j=2}^{N} \sum_{k=2}^{N} t_j \frac{\partial h_j}{\partial \pi_k} t_k + \sum_{j=2}^{N} t_j \frac{\partial \tilde{x}_j}{\partial w} \left[-\omega_1 + \sum_{k=2}^{N} t_k x_k \right]$$

$$= -\sum_{j=2}^{N} \sum_{k=2}^{N} t_j \frac{\partial h_j}{\partial \pi_k} t_k - \sum_{j=2}^{N} t_j \frac{\partial \tilde{x}_j}{\partial w} L,$$

where (1) has been used. We may define:

SUBSTITUTION EFFECT: $= -\sum_{j=2}^{N} \sum_{k=2}^{N} t_j \frac{\partial h_j}{\partial \pi_k} t_k$

WEALTH EFFECT: $-L \sum_{j=2}^{N} t_j \frac{\partial \tilde{x}_j}{\partial w}$

and hence $\frac{\partial \breve{L}}{\partial p_1} \equiv$ TOTAL EFFECT $=$ SUBSTITUTION EFFECT $+$ WEALTH EFFECT

1(b).7. Assume that goods 2, …, *N* are normal. What can you say about the sign of the substitution and wealth effects obtained in 1(b).6? What can you say about the sign of the total effect $\frac{\partial \breve{L}}{\partial p_1}$? Compare with 1(a) above.

Because $\frac{\partial h_j}{\partial \pi_k} = \frac{\partial^2 e}{\partial \pi_k \partial \pi_j}$, the (*j*, *k*) entry in the negative semidefinite matrix D^2e (*e* is the expenditure function associated with the Hicksian demand functions), the substitution effect (3) can be written $-(t_2,\dots,t_N)D^2e\,(t_2,\dots,t_N)$, nonnegative .

Because goods 2,…, *N* are normal, $\frac{\partial \tilde{x}_j}{\partial w} > 0$, and therefore the wealth effect $-L \sum_{j=2}^{N} t_j \frac{\partial \tilde{x}_j}{\partial w}$ is positive.

Hence, the substitution and wealth effects have (weakly) opposite signs, and their sum can in principle be positive, negative or zero.

The situation displays a strong parallelism with that in the model of section I(a) above. The modeling of the labor supply is quite different, yet the Slustky equations are quite similar: in both cases, the substitution and wealth effects move in opposite directions. The main difference is that, in the model of I(a), leisure (i.e., time not sold in the market) has direct utility, whereas in that of section I(b) its utility is indirect, in what it enables the consumer to consume goods 2,.., *N*.

Part II **ANSWER KEY**

Two goods, good 1 and good 2, which is the numeraire good. There are *I* consumers. For *i* = 1,…, *I*, consumer *i*'s utility function is $u^i : \Re_+ \times \Re \to \Re : u^i(x_1^i, x_2^i) = b^i(x_1^i) + x_2^i$, where b^i is differentiable, increasing and strictly concave. All consumers are price takers. We assume that the price of the numeraire good is equal to 1, and we denote by *p* the price of good 1. Denote by $\tilde{x}_1^i(p)$ consumer *i*'s Walrasian demand for good 1.

There is a single firm which produces good 1 by using the numeraire as an input, with cost function *C*(*y*), where *y* is the amount of good 1 produced. We assume in what follows that first order equalities characterize the solution to every optimization problem.

Given a price *p*, we define the markup as $\frac{p-C'}{p}$, where the marginal cost *C*' is evaluated at the amount of output equal to aggregate demand.

II(a). Write the first order condition of the consumer optimization problem that yields her Walrasian demand for good 1.

The problem $\max b^i(x_1^i) + x_2^i + w^i - px_1^i$ (where w^i is the wealth of the consumer) yields the first-order condition $\frac{db^i}{dx_1^i} = p$.

II(b). Suppose that prices are regulated in order to maximize the sum of consumer surplus and profits. What can you say about the resulting markup?

Write $X(p) \equiv \sum_{i=1}^{I} \tilde{x}_1^i(p)$. By definition, *i*'s consumer surplus is $b^i(\tilde{x}_1^i(p)) - p\tilde{x}_1^i(p)$, and the firm's profits are $pX(p) - C(X(p))$. Hence, the sum of the consumer surpluses and profits is

$$\sum_{i=1}^{I} b^i(\tilde{x}_1^i(p)) - pX(p) + pX(p) - C(X(p)),$$

with first order equality $\sum_{i=1}^{I} \frac{db^i}{dx_1^i}\frac{d\tilde{x}_1^i}{dp} - C'X' = 0$, which using II(a), can be written

$pX' - C'X' = 0$. Dividing through by $\frac{dX}{dp}$, yields

$$p^E = C'$$

i. e., the maximization of the sum of consumer surpluses and profits implies the equality of prices and marginal costs, and, hence, zero markup. As we know, this is a condition for economic efficiency.

II(c). Let consumer *i* own a share $\theta_i \geq 0$ in the profits of the firm (*i* = 1,…, *I*, $\sum_{i=1}^{I} \theta_t = 1$). As a consumer, she buys the good in the market, where she is a price taker, but she can vote at the shareholders' meeting on the price that the firm will charge. What is the best price for consumer-shareholder *i*?

Consumer-shareholder *i* solves the problem

$\max_p b^i(\tilde{x}_1^i(p)) - p\tilde{x}_1^i(p) + w^i + \theta_i[pX(p) - C(X(p))]$, with first-order equality

$$\frac{db^i}{dp}\frac{d\tilde{x}^i}{dp} - p\frac{d\tilde{x}^i}{dp} - \tilde{x}_1^i + \theta_i[pX' + X - C'X'] = 0,$$

or, using II(a),

$$\tilde{x}_1^i = \theta_i[X + (p - C')X']. \qquad (1)$$

II(d). Suppose that, at the shareholders' meeting, all shareholders unanimously agree on a price. What can you say about the resulting markup? What can you say about the share θ_i of consumer *i*? Interpret. Hint. Add up the FOC.

Adding up (1) for *i* = 1,…,*I*, we obtain

$X = X + (p - C')X'$, i. e., *p* – *C*', and zero markup. Therefore, (1) becomes $\tilde{x}_1^i = \theta_i X$, or $\theta_i = \frac{\tilde{x}_1^i}{X}$, i. e., the profit shares must be equal to the shares in consumption.

Intuitively, the price has two effects on consumer *i*'s welfare. As a consumer, she will prefer a very low price (zero), but as a shareholder she would prefer the monopoly price. Hence, a big consumer who is a small shareholder prefers low prices, whereas a big shareholder who is a small consumer prefers high prices. When the shares in profits equal the shares in consumption, this two effects balance out in a way that all shareholders prefer the surplus maximizing price.

II(e). We now specialize the model to a very simple case, where $b^i(x_1^i) = ax_1^i - \frac{1}{2}(x_1^i)^2$, i = 1,…, I, and $C(y) = cy$, where $a > c$. But we assume that only a fraction σ of the population of I consumers are shareholders in the firm, each owning a share $\theta = \frac{1}{\sigma I}$ in the firm's profits.

II(e).1. Compute the monopoly profit-maximizing price p^M.

Individual demand is given by $p = a - x_1^i$, i. e., $\tilde{x}_1^i(p) = a - p$, and aggregate demand by $X(p) = I(a-p)$. The monopoly profit maximizing problem is

$$\max_p (p-c)I(a-p)$$

with FOC: $(a-p)-(p-c)=0$, i. e., $a-2p+c=0$, or: $p^M = \frac{a+c}{2}$.

II(e).2. Show that all shareholders agree on a price $p(\sigma)$, and compute it. How does $p(\sigma)$ vary with σ? What are the limits of $p(\sigma)$ as $\sigma \to 0$? As $\sigma \to 1$? Comment.

From (1), we can write the FOC for a shareholder as

$$a - p = \frac{1}{\sigma I}[I(a-p)+(p-c)(-I)].$$

Aggregating over the σI shareholders, we obtain

$$\sigma I[a-p] = I(a-p)+(p-c)(-I)],$$

i. e., $\sigma a - \sigma p = a - p - p + c$,

or: $p(2-\sigma) = (1-\sigma)a + c$,

i. e., $p(\sigma) = \frac{(1-\sigma)a+c}{2-\sigma}$.

We compute: $\frac{dp(\sigma)}{d\sigma} = \frac{-a(2-\sigma)+(1-\sigma)a+c}{[2-\sigma]^2} = \frac{-2a+a\sigma+a-\sigma a+c}{[2-\sigma]^2} = \frac{-a+c}{[2-\sigma]^2} < 0$,

$\lim_{\sigma\to 0} p(\sigma) = \frac{a+c}{2} = p^M$,

Answer Key, Prelim Spring 2009.

Question 3

3. (a) convexity of Y: let $(-z_0, y) \in Y$, $(-\tilde{z}_0, \tilde{y}) \in Y$ and $\lambda \in [0,1]$.

Since $\lambda z_0 + (1-\lambda)\tilde{z}_0 \geq \lambda \rho \max\{y_t\} + (1-\lambda)\rho \max\{\tilde{y}_t\} + \gamma \sum(\lambda y_t + (1-\lambda)\tilde{y}_t)$

it suffices to show that

$$\lambda \max_{1\le t\le T}\{y_t\} + (1-\lambda)\max_{1\le t\le T}\{\tilde{y}_t\} \geq \max_{1\le t\le T}\{\lambda y_t + (1-\lambda)\tilde{y}_t\}$$

Let $y_{\bar{t}} = \max\{y_t\}$ $\quad \tilde{y}_{\tilde{t}} = \max\{\tilde{y}_t\}$ and

$$\lambda y_{\hat{t}} + (1-\lambda)\tilde{y}_{\hat{t}} \leq \lambda y_{\bar{t}} + (1-\lambda)\tilde{y}_{\tilde{t}} = \lambda \max_{1\le t\le T}\{y_t\} + (1-\lambda)\max_{1\le t\le T}\{\tilde{y}_t\}$$

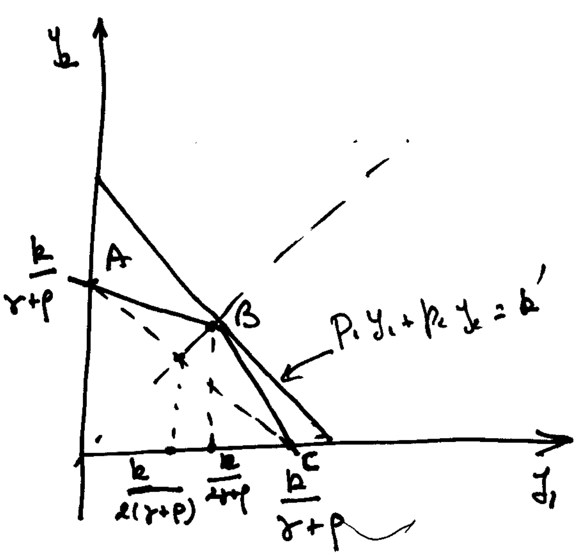
It is obvious that $\lambda y \in Y$, so that Y is convex and exhibits constant returns.

(b) maximization of profit for the firm.

(i) iso cost curve ABC

if $y_1 \le y_2$ $\quad \gamma(y_1+y_2) + \rho y_2 = k$

if $y_1 \geq y_2$ $\quad \gamma(y_1+y_2) + \rho y_1 = k$



the two segments meet at

$$y_1 = y_2 = \frac{k}{2\gamma+\rho}$$

the line joining the two points of the iso curve on the axes crosses the diagonal at $y_1 = y_2 = \frac{k}{2(\gamma+\rho)} < \frac{k}{2\gamma+\rho}$

(ii) If the iso revenue curve is less steep than BC but steeper than AB, the solution to cost minimization on the iso-revenue line ~~is at~~ occurs at $y_1 = y_2$. Analytically: the condition is

$$\frac{\gamma}{\gamma+\rho} < \frac{p_1}{p_2} < \frac{\gamma+\rho}{\gamma}$$

(iii) When cost minimization implies $y_1 = y_2 = y$, the zero profit condition implies

$$\left((p_1 - \gamma) + (p_2 - \gamma) \right) y - \rho y = 0, \text{ and } y \neq 0 \text{ only if}$$

$$\boxed{p_1 + p_2 = 2\gamma + \rho}$$

(c) Looking for an equilibrium with $\bar{y}_1 = \bar{y}_2$. The demand of agent i at prices $(1, p_1, p_2)$ is

$$x_0^i = \alpha_0^i w^i, \qquad x_1^i = \frac{\alpha_1^i w^i}{p_1}, \qquad x_2^i = \frac{\alpha_2^i w^i}{p_2}$$

$$\Sigma x_1^i = \Sigma x_2^i \iff \sum_i \frac{\alpha_1^i w^i}{p_1} = \frac{\sum_i \alpha_2^i w^i}{p_2} \iff p_2 = \frac{\Sigma \alpha_2^i w^i}{\Sigma \alpha_1^i w^i} p_1$$

$$p_1 + p_2 = 2\gamma + \rho \iff p_1 + \frac{\Sigma \alpha_2^i w^i}{\Sigma \alpha_1^i w^i} p_1 = 2\gamma + \rho$$

$$p_1 = \frac{(2\gamma+\rho) \sum_i \alpha_1^i w^i}{\sum_i \alpha_1^i w^i + \sum_i \alpha_2^i w^i} \qquad p_2 = \frac{(2\gamma+\rho) \Sigma \alpha_2^i w^i}{\Sigma \alpha_1^i w^i + \Sigma \alpha_2^i w^i}$$

Since $\frac{\gamma}{\gamma+\rho} < 1 < \frac{p_1}{p_2} = \frac{\Sigma \alpha_1^i w^i}{\Sigma \alpha_2^i w^i} < \frac{\gamma+\rho}{\gamma}$

the profit of the firm is maximized and this is an equilibrium. The price is higher at the time of high demand, (day), lower at night and agents end up consuming the same amount at each period, so that the capacity is always maximally used.

Question 4

4. (a) The Pareto optimal level of public good is obtained by solving:

$$\max \{ x + \gamma \log y \mid Ix + y = Iw \}.$$

which leads to $\boxed{y^* = I\gamma}$

(b) In a voluntary contribution equilibrium, each agent solves

$$\max \{ w - z^i + \gamma \log (z^i + Z^{-i}) \mid z^i \geq 0 \}$$

where Z^{-i} is the total contribution of the other agents

The solution is

$$z^i = \begin{cases} \gamma - Z^{-i} & \text{if } Z^{-i} < \gamma \\ 0 & \text{if } Z^{i} > \gamma \end{cases}$$

In a symmetric equilibrium all agents must contribute. Thus the equilibrium is such that $Iz = \gamma \iff z = \frac{\gamma}{I} \Rightarrow \boxed{\bar{y} = \gamma}$

Agents under estimate the marginal benefit of their contribution since they just consider their private marginal benefit instead of the social marginal benefit, which is much higher since all agents consume the additional public good.

(c) (1) agent i solves the maximum problem

$$\max \left\{ w - z_i + \frac{z_i}{z_i + Z^{-i}} R + \gamma \log (z_i + Z^{-i} - R) \mid z_i \geq 0 \right\} \quad (1)$$

Since we are interested in a symmetric equilibrium, we look for a solution with $z_i > 0$. The first-order condition is

$$-1 + \frac{R}{z_i + Z^{-i}} - \frac{z_i R}{(z_i + Z^{-i})^2} + \frac{\gamma}{z_i + Z^{-i} - R} = 0$$

In a symmetric equilibrium $Z^{-i} = (I-1) z_i$. Let $Z = I z_i$ be the total amount of money collected by the lottery: $Z = z_i + Z^{-i}$, $\forall i$.

Z must satisfy

$$-1 + \frac{R}{Z} - \frac{R}{IZ} + \frac{\gamma}{Z - R} = 0$$

(ii) Eliminating denominators, the expression is equivalent to

$$-IZ(Z-R) + IR(Z-R) - R(Z-R) + I\gamma Z = 0$$

$$\Leftrightarrow \quad (Z-R)(-IZ + IR - R) + I\gamma(Z-R) + I\gamma R = 0$$

$$\Leftrightarrow \quad -I(Z-R)^2 - R(Z-R) + I\gamma(Z-R) + I\gamma R = 0$$

$$\Leftrightarrow \quad I(Z-R)^2 + (R - I\gamma)(Z-R) - I\gamma R = 0.$$

$$\Leftrightarrow \quad I y^2 + (R - I\gamma) y - I\gamma R$$

when $y = Z - R$ is the public good produced when the proceed of the lottery is Z.

$$F(y) = Iy^2 + (R - I\gamma) y - I\gamma R$$

is a second order polynomial with 2 roots of opposite signs (since the constant term is negative). The equilibrium value $y_L = Z_L - R$ is the positive root. If $F(\gamma) < 0$, then $y_L > \gamma$ and if $F(I\gamma) > 0$, then $y_L < I\gamma$.

$$F(\gamma) = I\gamma^2 + (R - I\gamma)\gamma - I\gamma R = -(I-1)\gamma R < 0$$

$$F(I\gamma) = I^3\gamma^2 + (R - I\gamma)I\gamma - I\gamma R = I^2\gamma^2(I-1) > 0$$

(iii) We have thus proved that if the Charity uses a lottery with prize R, then after paying for the prize it can produce more public good than by relying on voluntary contributions, but still less than the Pareto optimal optimal level. One can show that the proceed $Z-R$ is increasing in R, but if R is too large, the assumption of quasi-linearity of the utilities is no longer acceptable.

The inefficiency in (b) comes from the fact that agent i compares its private marginal cost of his contribution with his private marginal benefit, which is much less than the social marginal benefit. The effect of the lottery is to decrease the marginal cost by giving a chance to the agent to win the prize. What would be difficult to guess without the model is that the increase in contributions is larger than the value of the prize.

Answer key for Q5 Micro Prelim June 22 09

(a) Let F be the c.d.f.. Then

$$D_1(p_1,p_2)=\begin{cases}0 & \text{if } p_1>r\\ N & \text{if } p_1\le r \text{ and } p_2>r\\ \frac{N}{2} & \text{if } p_1=p_2\le r\\ F(p_2-p_1)N+\frac{1}{2}[1-F(p_2-p_1)]N & \text{if } p_1<p_2\le r\\ \frac{1}{2}[1-F(p_1-p_2)]N & \text{if } p_2<p_1\le r\end{cases}$$

$$D_2(p_1,p_2)=\begin{cases}0 & \text{if } p_2>r\\ N & \text{if } p_2\le r \text{ and } p_1>r\\ \frac{N}{2} & \text{if } p_1=p_2\le r\\ F(p_1-p_2)N+\frac{1}{2}[1-F(p_1-p_2)]N & \text{if } p_2<p_1\le r\\ \frac{1}{2}[1-F(p_2-p_1)]N & \text{if } p_1<p_2\le r\end{cases}$$

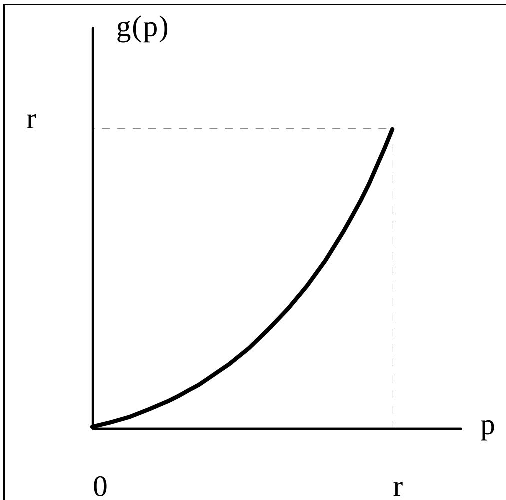
(b) Let p_2 = 120. Then $\pi_1(p_1,120) = p_1 D_1(p_1)$. Thus $\pi_1(120,120) = 120\frac{N}{2}=60N$. Consider a lower price, e.g. 100. Then $\pi_1(100,120) = 100\left[F(20)N+\frac{1}{2}\left(1-F(20)\right)N\right]=100(0.7N)=70N$ > $60N$. Thus (120,120) is **not** a Nash equilibrium.

(c) Fix a firm *i*. If the other firm charges *r*, $p_i = r$ yields a profit of $\frac{rN}{2}$. For this to be a Nash equilibrium it is necessary and sufficient that firm *i* cannot increase its profits by choosing a price $p_i < r$. If the firm charges $p_i < r$ then its profits will be: $p_i\, F(r-p_i)\, N + \frac{1}{2}\,[1-F(r-p_i)]Np_i$
Thus we need

$$\frac{rN}{2} \ \ge\ p_i\, F(r-p_i)\, N + \frac{1}{2}\,[1-F(r-p_i)]Np_i \qquad \textit{for all } p_i \le r \quad \text{that is,}$$

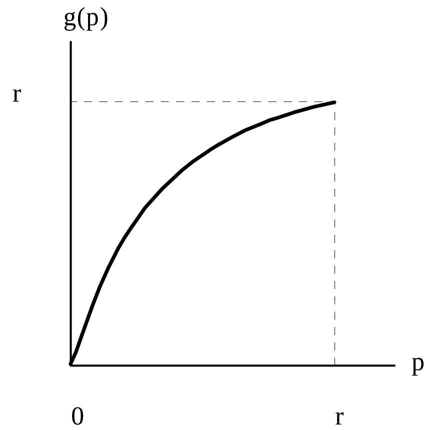
$$r \ \ge\ p_i\,[1 + F(r-p_i)] \qquad \textit{for all } p_i \le r. \tag{1.1}$$

Let us drop the subscript *i* and define the RHS of (1.1) as $g(p)$. Thus $g(p)=p+pF(r-p)$. Then $g(0) = 0$ and $g(r) = r$. Furthermore, $g'(p)=1+F(r-p)-pf(r-p)$. Thus $g'(0)=1+F(r)>0$. If $g(p)$ is convex then $g'(p)$ is non-decreasing and $g(p)$ looks like



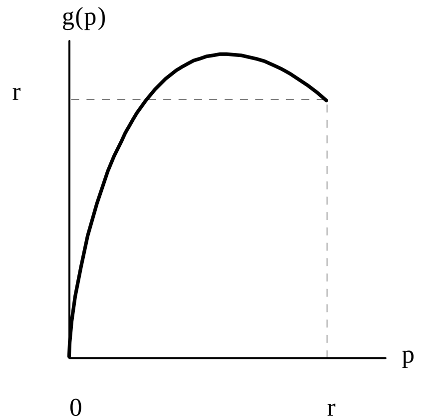
and therefore (1.1) is satisfied and (r,r) is a Nash equilibrium.

(d) If the function $g(p)$ is concave then there are two possibilities. CASE 1: $g'(r) \geq 0$. Then $g(p)$ looks like



and therefore (1.1) is satisfied, hence (r,r) is a Nash equilibrium.

CASE 2: $g'(r) < 0$ then $g(p)$ looks like



In this case there is a $p \in (0,r)$ such that $g(p) > r$ and therefore (1.1) is violated and (r,r) is not a Nash equilibrium.

Thus a necessary and sufficient condition for (r,r) to be a Nash equilibrium is $g'(r) \geq 0$. Since $g'(r) = 1 - rf(0)$ the condition can also be written as $f(0) \leq \frac{1}{r}$

(e) If f is constant, then it must be $f(x) = \frac{1}{r}$ for all x. Then $F(x) = \frac{x}{r}$ so that the function $g(p)$ of part (ii) becomes $g(p) = p\left(1 + \frac{r-p}{r}\right)$. Thus $g''(p) = -\frac{2}{r}$, i.e. $g(p)$ is concave. Hence, by the results of part (ii), (r,r) is a Nash equilibrium if and only if $f(0) \leq \frac{1}{r}$ which is of course true. So (r,r) is a Nash equilibrium in this case.

(f) Assume that $F(x) < 1$ for sufficiently small x. Then if firm 2 charges 0, firm 1 gets zero profits if it also charges 0, but positive profits if it charges a little bit more than zero (its demand is positive since $F(x) < 1$ for x small). Thus $p_1 = p_2 = 0$ is not a Nash equilibrium.

(g) Intuitively, the Bertrand paradox corresponds to the case where all the mass is concentrated at 0. One might be able to show the Bertrand paradox as a limit result: consider a family f_t of density functions such that, as $t \to \infty$, the smallest x at which $F_t(x) = 1$ tends to zero. Then the Nash equilibrium might tend to zero.