MACROECONOMICS PRELIM, AUGUST 2018

ANSWER KEY FOR QUESTIONS 1 AND 2

Question 1

a) We have the following value functions:

$$\begin{aligned} rV &= -pc + q(\theta)(M - V), && (1)\\ rM &= a(J - M), && (2)\\ rJ &= p - w - \lambda J. && (3) \end{aligned}$$

The intuition is straightforward. For instance, V is as in the lectures, except when the worker arrives, the firm goes to state M not J. At state M the firm is not paying any wage but also not producing. It is basically waiting for the training to be over, in which case it moves to the state J. J is as in the lectures.

b) For the worker we have

$$\begin{aligned} rU &= z + \theta q(\theta)(T - U), && (4)\\ rT &= a(W - T), && (5)\\ rW &= w + \lambda(U - W), && (6) \end{aligned}$$

where the interpretation of these functions is the same as in part (a).

c) The free-entry condition $V = 0$ implies that

$$M = \frac{pc}{q(\theta)}. \qquad (7)$$

Combining this with (2) implies that

$$J = \frac{a + r}{a} \frac{pc}{q(\theta)}. \qquad (8)$$

Now combine (8) and (3) to get the job creation curve:

$$w = p - \frac{(r + \lambda)pc}{q(\theta)} \frac{a + r}{a}.$$

As I hinted, this gives you the standard JC curve of the textbook as $a \to \infty$. Regardless of the term a showing up, this is still a negatively-sloped curve in the (θ, w) space.

d) The firm and worker are solving[1]

$$\max_w (T - U)^\beta M^{1-\beta}. \tag{9}$$

Like in the lecture notes, this problem leads to

$$(1 - \beta)(T - U) = \beta M. \tag{10}$$

After a little algebra (the steps of which are identical to the ones seen many times in class and homeworks), we get:

$$w = \beta p + (1 - \beta) rU \left(1 + \frac{r + \lambda}{a}\right). \tag{11}$$

As in the lecture notes, the last task is to get rid of rU. Use (10) to write (4) as

$$rU = z + \theta q(\theta) \frac{\beta}{1 - \beta} M,$$

and use (7) to write the last expression as

$$rU = z + \frac{\beta \theta p c}{1 - \beta}.$$

Plugging the last expression for rU back into (11) yields the wage curve:

$$w = \beta p + \left(1 + \frac{r + \lambda}{a}\right) \left[\beta p c \theta + (1 - \beta) z\right],$$

which also coincides with the textbook formula as $a \to \infty$. Regardless of the term a showing up, this is still a positively-sloped curve in the (θ, w) space.

e) We have seen that the WC is positively sloped and the JC negatively sloped. So for existence, we basically need to make sure that the intercept of the JC curve when $\theta = 0$ lies above the intercept of the WC curve when $\theta = 0$. If this is true the equilibrium will exist and it will be unique (I asked you to not get into too much detail, so basically there is no need to see what happens as $\theta \to \infty$). When $\theta = 0$ the WC gives:

$$w_{WC}(0) = \beta p + \left(1 + \frac{r + \lambda}{a}\right) (1 - \beta) z.$$

[1] This is exactly what I was describing in the footnote. Although the two parties are negotiating over something that will be paid in the future, the problem is to split the surplus generated at the moment when the two are bargaining. That surplus is $T - U$ for the worker and M for the firm. Notice that this makes prefect sense, since the term T includes the term W (the value of a productive worker) and it is only adjusted to reflect the waiting time till production starts (a similar comment applies for the terms M and J).

When $\theta = 0$ the JC gives:

$$w_{JC}(0) = p.$$

So for existence we need

$$w_{WC}(0) > w_{JC}(0) \;\Leftrightarrow\; \beta p + \left(1 + \frac{r+\lambda}{a}\right)(1-\beta)z > p,$$

which after a little algebra yields the much simpler and intuitive condition

$$p > \left(1 + \frac{r+\lambda}{a}\right) z.$$

With $a < \infty$ having $p > z$ is not good enough for an equilibrium to exist. The p has to be even bigger in order to compensate for the fact that jobs do not start immediately, but have to go through the training period. The lower the a, the higher the p must be (for given z). Of course, as $a \to \infty$, we get the usual sufficient condition $p > z$.

f) If a goes down it is easy (and intuitive) to see that both the WC and the JC will shift to the left. Hence, without any doubt we will have a reduction in θ, which is not surprising because a decrease in a is essentially a decrease in productivity (so fewer firms wish to enter in this market).

Since both curves shift to the left, the effect on equilibrium wage is unclear, which is an interesting result. What is going on here is that there are two opposing forces. On the one hand, the lower a, which effectively means a lower productivity, lowers θ and tends to lower the wages too (through the JC effect). On the other hand, the lower a means that workers must wait longer before they start getting paid, and so they will demand higher compensations in order to agree to work for the firm. Which force "wins" is unclear. Identifying these two forces is more than enough to get full credit here.

g) Let u, t, e be the measure of workers in the states of unemployment, training, and employment, respectively ("employment" here means workers who are producing and getting paid). We have $\theta q(\theta)u$ workers moving out of U and into the T state (no worker goes directly from U to W). We also have at workers moving from T into the state of employment. Finally, we have λe workers moving from employment to unemployment. To equate the inflow and outflow of workers at states U and T, respectively, we need

$$\theta q(\theta)u = \lambda e. \tag{12}$$

$$\theta q(\theta)u = at. \tag{13}$$

These two equations together with $u+t+e = 1$ form a very simple system of 3 equations and three unknowns. Solving it with respect to u yields the Beveridge curve:

$$u = \frac{a\lambda}{a\lambda + \theta q(\theta)(a+\lambda)}.$$

As always, taking the limit as a goes to infinity will give you the well-known textbook Beveridge curve. Also, this is a standard downward sloping curve in the (v, u) space.

Regaridng the effect of a on unemployment, it is easy to see that a lower a will shift the BC to the right. Moreover, we know from part (f) that the equilibrium θ went down. Hence, with no doubt equilibrium unemployment will go up.

Question 2

a) The buyer's value function is given by

$$W(m,d) = \max_{X,H,\hat{m}} \Big\{U(X) - H + \beta V(\hat{m})\Big\},$$
$$s.t.\;\; X + \phi\hat{m} = \phi m + H - d + T,$$

where all variables are explained in the question, and T is the lump-sum monetary transfer to the buyer. Using the standard procedure described in class, we will find that

$$W(m,d) = \Lambda + \phi m - d,$$

where Λ just summarizes a number of terms that are not related to the state variables.

b) As we know, in these types of models the seller never wants to carry money out of the CM. The reason is very simple: money is costly to carry due to its liquidity, but the sellers, by default, can never take advantage of that liquidity. (So why pay the unnecessary cost?) Thus, as in the case of the buyer, the seller's VF will be linear in its argument. The only question is what is that argument? In type-0 meetings, the argument of the seller's value function will be the amount of CM good that the buyer has promised to reply the seller, say,

$$W_S^0(d) = \Lambda_S^0 + d,$$

where Λ_S^0 is a constant of no direct interest to us. On the contrary, in type-1 meetings, the argument of the seller's value function will be the amount of money that the buyer paid to the seller in exchange for goods:

$$W_S^1(m) = \Lambda_S^1 + \phi m,$$

where Λ_S^1 is a constant. This is all we need in order to solve the bargaining problem that follows.

c) As I hinted in the question, here I wanted you to use your intuition: In a type-0 meeting, the buyer is not constrained. So, of course, she will want to trade the first-best, q^*. Now, the question is what will she give in return? The answer is, a promise for an amount of numeraire good that will make the seller just happy to produce the first-best (recall that the buyer makes a TIOLI offer). Hence, the payment must be such that the seller's surplus, $d - q$, is zero. Summing up, we will have $q_0 = q^*$ and $d = q^*$.[2]

In a type-1 meeting, again the buyer would love to go to the first-best, but now she may be constrained by her cash holdings. Thus, the bargaining solution is as follows:

[2] This is a special (and easier) case of what we saw in class, except now the compensation function $z(q)$ is just q because the buyer has all the bargaining power. (Thus, she is suppressing the seller to work for a payment equal to her cost.)

$q_1 = \min\{\phi m, q^*\}$ and $x = \min\{m, q^*/\phi\}$. In words, given the price of money, ϕ, q^*/ϕ is the amount of money that allows the buyer to buy the first best q^*. Then, either $m \geq q^*/\phi$ and the buyer gets $q_1 = q^*$, or $m < q^*/\phi$, and the buyer gives up all her money just to get $q_1 = \phi m < q^*$.

d) As I explained, there is no need to spend time on deriving the objective because it is very intuitive (assuming one has understood the environment well). The objective function here is:[3]

$$J(\hat{m}) = (-\phi + \beta\hat{\phi})\hat{m} + \beta(1-\sigma)[u(q_1(\hat{m})) - \hat{\phi}x(\hat{m})],$$

where $q_1(\hat{m}), x(\hat{m})$ are the terms of trade in a type-1 meeting, explained in the bargaining solution above. But as we know, in these types of models, if the cost of carrying money is positive (an assumption maintained here), the buyer will never bring more than the amount that would buy her q^*. Another way of saying this is that "we will be in the binding branch of the bargaining solution" (this is the language we often used in class), i.e., where $q_1 = \phi m$ and $x(m) = m$. So we can re-write this objective as

$$J(\hat{m}) = (-\phi + \beta\hat{\phi})\hat{m} + \beta(1-\sigma)[u(\hat{\phi}\hat{m}) - \hat{\phi}\hat{m}].$$

e) The next step is to obtain the FOC and evaluate it in equilibrium. The FOC yields:

$$\phi = \beta\hat{\phi} + \beta(1-\sigma)[\hat{\phi}u'(\hat{\phi}\hat{m}) - \hat{\phi}].$$

In the steady state equilibrium, we have $z = \phi M = \hat{\phi}\hat{M}$, and, moreover, here due to the TIOLI offer assumption, we will also have $z = q_1$ (because $z(q) = q$).

Summing up, in the steady state equilibrium, we have $q_0 = q^*$, and q_1 is given implicitly by[4]

$$i = (1-\sigma)[u'(q_1) - 1].$$

f) The equilibrium condition for $z = q_1$ is given by

$$i = (1-\sigma)[u'(q_1) - 1],$$

or after a little algebra,

$$u'(q_1) = 1 + \frac{i}{1-\sigma}.$$

[3] Clearly, there is no term that starts with σ, because with that probability the buyer meets a seller who accepts credit and her money is not useful in that meeting.

[4] Originally, you should find an expression that has exactly the same RHS, and on the LHS you should have $\frac{1+\mu-\beta}{\beta}$. But through the Fisher equation that last expression is simply equal to i.

So whether the equilibrium will be monetary really depends on the functional form of the u function. This is the short answer, and if you wrote this, you got the vast majority of the points.

For those more interested in the details, notice that if u satisfies the so-called Inada condition, i.e., if $\lim_{q \to 0} = \infty$, then a monetary equilibrium always exists. The intuition is simple. Even as $\sigma \to 1$, which means that money is only used in a tiny fraction of meetings, carrying no money at all is still very costly (because of the huge marginal benefit that the very first units of money give you, due to the Inada condition). Hence, even as $\sigma \to 1$, money will still be held, and the equilibrium will be monetary, i.e., $z = q_1 > 0$.

But consider some value function that does not satisfy the Inada condition, e.g., $u(q) = -\frac{q^2}{2} + (1+\gamma)q$ (for more details, see the July 2017 Prelim). In this case, the benefit of bringing money at the limit is not so large anymore (the Inada condition is gone), and it may very well be the case that σ is so large that it may discourage agents from carrying money altogether. In this case, $q_1 = \gamma - i/(1-\sigma)$. Thus, the upper bound for which a monetary equilibrium will be sustained is given by

$$\bar{\sigma} \equiv 1 - \frac{i}{\gamma}.$$

g) As I explained in the question, the welfare function here is

$$\mathcal{W} = \sigma[u(q_0) - q_0] + (1-\sigma)[u(q_1) - q_1],$$

where q_0, q_1 were determined in the previous parts. Notice that σ does not change the level of q_0 (because it is equal to q^* always) but it effects how often agents get it. On the other hand, σ affects both the likelihood of getting q_1 *and* its value (through the money demand channel). More precisely, the first derivative of $\mathcal{W}$ with respect to σ is

$$\begin{aligned} \frac{\partial \mathcal{W}}{\partial \sigma} &= [u(q^*) - q^*] - [u(q_1) - q_1] \\ &+ (1-\sigma)[u'(q_1) - 1]\, \frac{\partial q_1}{\partial \sigma}. \end{aligned} \tag{14}$$

Since $q_1 < q^*$ for all $i > 0$ (a maintained assumption in this question), the first line of this expression will be positive. (After all $u(q) - q$ is simply the net surplus generated in a KW meeting, and that meeting is maximized at $q = q^*$). In the second line, the expression $u'(q_1) - 1$ is positive (for all $q_1 < q^*$), but the partial derivative $\frac{\partial q_1}{\partial \sigma}$ is negative. You can see this by applying total differentiation in the equilibrium condition for q_1, or, simply, by using your intuition and pointing out that a larger σ (more frequent credit meetings) lowers the demand for money and depresses q_1.

Summing up, the sign of $\frac{\partial \mathcal{W}}{\partial \sigma}$ is impossible to pin down without knowing more about u. All you needed to say here is that there are two opposing forces, and one cannot be certain about the sign.

For the interested reader, in the July 2017 Prelim, and under a quadratic utility function, we showed that $\frac{\partial \mathcal{W}}{\partial \sigma}$ will actually be negative for all parameter values (a result which is quite unintuitive)! But here, without a specific functional form for u, we cannot say much more.

University of California, Davis
Department of Economics
Macroeconomics

PRELIMINARY RESIT EXAMINATION FOR THE Ph.D. DEGREE,
August, 2018

Directions: Answer all questions. Feel free to impose additional structure on the problems below, but please state your assumptions clearly. Point totals for each question are given in parentheses. You have 5 hours to complete the exam and an additional 20 minutes of reading time.

Question 1 (*20 points*)

Consider a *decentralized* real business cycle model. The representative household chooses consumption (c) and leisure ($L = 1 - N$, where N is hours worked) to maximize (detrended) lifetime utility:

$$E_0 \sum_{t=0}^{\infty} \beta^t \left(\log c_t + \frac{\theta}{1-\eta}(L_t^{1-\eta} - 1) \right) \tag{1}$$

subject to the detrended household budget constraint:

$$c_t + \gamma k_{t+1} = w_t N_t + (1 - \delta(u_t))k_t + r_t^k u_t k_t + \pi_t \tag{2}$$

where u_t is the degree of capital utilization and is also chosen by the household. w is the real wage, N is hours worked, k is capital, r^k is the rental price of capital and π are profits from firms. Detrended capital evolves as follows:

$$\gamma k_{t+1} = (1 - \delta(u_t))k_t + i_t$$

where $X_t/X_{t-1} = \gamma$ is the deterministic growth rate of labor augmenting technological change. Assume $\gamma = 1$.

Competitive firms produce output using capital services $u_t k_t$ and labor N_t. The detrended production function is

$$y_t = A_t(u_t k_t)^{\alpha}(N_t)^{1-\alpha}$$

where TFP, denoted by A_t, is stochastic and follows an AR(1) process in logs. In steady state $A = 1$.

Households own the capital stock, invest and decide on the degree of utilization. Utilization is costly and causes capital to depreciate more quickly, as indicated by the depreciation cost function $\delta(u_t)$, where $\delta(1) = \delta$ in steady state. Firms choose capital to rent from households, understanding that they will be able to utilize it to a certain degree. You can therefore think of the firm as choosing their demand for labor N_t and their demand for capital services $u_t k_t$.

a) Write down the household's problem in recursive form and write down the firm's maximization problem. Derive the household's first order conditions and the firm's optimal hiring rules.

Sketch answer:

The household's problem is:

$$\max_{N_{S,t}, k_{S,t+1}, c_t, u_t} V(A_t, K_t, k_{S,t}) = \log(c_t) + \frac{\theta}{1-\eta}(1 - N_{S,t})^{1-\eta} + \beta E_t[V(A_{t+1}, K_{t+1}, k_{S,t+1})] \tag{3}$$

subject to:

$$c_t + \gamma k_{S,t+1} = w_t(N_{S,t}) + (r_t^k u_t + 1 - \delta(u_t))k_{S,t} + \pi_t \tag{4}$$

The FOCs are:

$$\frac{1}{c_t} = \lambda_t \tag{5}$$

$$\gamma\frac{1}{c_t} = \beta E_t[(r_{t+1}^k u_{t+1} + 1 - \delta(u_{t+1}))\frac{1}{c_{t+1}}] \tag{6}$$

$$\frac{1}{c_t}w_t = \theta L_{S,t}^{-\eta} \tag{7}$$

The main new FOC is with respect to u_t:

$$r_t^k k_{S,t} = \delta'(u_t)k_{S,t} \tag{8}$$

The firm's profit maximization is:[1]

$$\max_{N_{D,t}, u_t k_{D,t}} Z_t(u_t k_{D,t})^\alpha N_{D,t}^{1-\alpha} - w_t N_{D,t} - r_t^k u_t k_{D,t} \tag{9}$$

The FOC with respect to effective capital is:

$$r_t^k = \alpha Z_t(u_t k_{D,t})^{\alpha-1} N_{D,t}^{1-\alpha} \tag{10}$$

b) Carefully define a recursive competitive equilibrium. Take care to distinguish between the aggregate and individual state variables and explain any market clearing conditions.

Sketch answer:

A recursive competitive equilibrium is a value function $V(A_t, k_{S,t}, K_t)$, decision rules $k_{S,t+1} = g_k(k_{S,t}, K_t, A_t)$, $c_t = g_c(k_{S,t}, K_t, A_t)$, $u_t = g_u(k_{S,t}, K_t, A_t)$, $N_t = g_n(k_{S,t}, K_t, A_t)$, a law of motion for the aggregate capital stock $K_{t+1} = G(K_t, A_t)$ and prices $\{w(K_t, A_t), r^k(K_t, A_t)\}$ Such that:

1. Given the pricing functions and the law of motion for K, the value function and decision rules solve the household's problem (the allocation satisfies all the first order conditions)

2. The firm's optimality conditions are satisfied.

[1]Note: Firms choose effective capital $u_t k_t$

3. All markets clear:

$$k_{D,t} = k_{S,t} = K$$

$$G(K_t, A_t) = g_k(k_t, K_t, A_t)$$

$$N_{S,t} = N_{D,t} = N_t$$

$$c_t = C, y_t = Y_t, i_t = I_t$$

and

$$Y_t = C_t + I_t$$

Firms make zero profits. To see this substitute the equilibrium conditions for the wage and the rental price of capital back into the profit function, this produces:

$$\pi_t = 0$$

c) By linearizing the equilibrium condition governing the optimal degree of utilization, show that

$$\hat{y}_t = \hat{a}_t + \alpha\hat{k}_t + (1-\alpha)\hat{N}_t + \frac{\alpha}{(1-\alpha+\xi)}(\hat{a}_t - (1-\alpha)\hat{k}_t + (1-\alpha)\hat{N}_t) \tag{11}$$

where $\xi = \frac{\delta''}{\delta'}$ and δ' and δ'' refer to the first and second derivatives of the δ function with respect to u_t in steady state. As usual, variables with a hat denote percentage deviations from steady state. (**Hint:** once you have linearized the equilibrium condition governing the optimal degree of utilization, you can combine this with the linearized production function $\hat{y}_t = \hat{a}_t + \alpha\hat{u}_t + \alpha\hat{k}_t + (1-\alpha)\hat{N}_t$).

Sketch answer:

We then combine 10 and 8 and impose market clearing conditions to obtain:

$$\alpha A_t u_t^{\alpha-1} k_t^{\alpha} N_t^{1-\alpha} = \delta'(u_t)k_t \tag{12}$$

Linearizing the equation yields:

$$[\frac{\delta''(u)u}{\delta'(u)} + 1 - \alpha]\hat{u}_t = \hat{a}_t + (\alpha-1)\hat{k}_t + (1-\alpha)\hat{N}_t \tag{13}$$

The linearized production function is simply:

$$\hat{y}_t = \hat{a}_t + \alpha\hat{k}_t + \alpha\hat{u}_t + (1-\alpha)\hat{N}_t \tag{14}$$

We now combine these equations, which provides the following equation:

$$\hat{y}_t = \hat{a}_t + \alpha\hat{k}_t + (1-\alpha)\hat{N}_t + \frac{\alpha}{1-\alpha+\epsilon}[\hat{a}_t - (1-\alpha)\hat{k}_t + (1-\alpha)\hat{N}_t] \tag{15}$$

d) Discuss how the addition of variable capital utilization helps the RBC model explain the business cycle facts in the data? If you can, explain why in the limit $\xi = \infty$ and $\xi = 0$ correspond to the cases of no variable capital utilization and full variable capital utilization.

Sketch answer:

$\epsilon = \frac{\delta''(u)u}{\delta'(u)}$. $\epsilon = \infty$ yields the standard linearized production function which corresponds to zero capital utilization while $\epsilon = 0$ implies full variable capital utilization. For more detail on how the standard model changes, see slides 25-31 in Lecture Notes 8. In general, the simple RBC model usually requires large TFP shocks and a high elasticity of labor supply to match the business cycle dynamics in the data. Adding VCU makes output more responsive to TFP shocks, as can be seen in the case where $\epsilon \to 0$. Also see Lecture Notes 8 for a discussion of how utilization can imply a more plausible Solow Residual.

e) *Briefly* explain how you would solve this model using a linearization-based method and how you would produce impulse response functions for the effects of a temporary one percent shock to $\hat{a}_t$.

Sketch answer:

The answer could outline the method of Blanchard and Kahn. The full details were covered in the lectures but to summarize: (1) linearize all equilibrium conditions (2) calibrate steady state and structural parameters (3) rewrite the linearized system in matrix format such that

$$\underbrace{\mathbf{B^{-1}A}}_{C} E_t \begin{bmatrix} \hat{x}_{t+1} \\ \hat{w}_{t+1} \end{bmatrix} = \begin{bmatrix} \hat{x}_t \\ \hat{w}_t \end{bmatrix} \tag{16}$$

(4) Check the Blanchard-Kahn conditions are satisfied. The number of unstable eigenvalues needs to equal the number of controls/jumps for there to be a unique stable solution. (5) Applying the Blanchard Kahn algorithm yields matrices **F** and **P** where **P** represents the transition matrix for state variables and **F** contains the policy functions both as percentage deviations from steady state.

$$\hat{x}_{t+1} = P\hat{x}_t \tag{17}$$

$$\hat{w}_t = F\hat{x}_t \tag{18}$$

These are the policy functions and we can use them to simulate an impulse response function. We can initialize the $\hat{a}_t$ variable at 1, this will produce a 1 percent shock to TFP. All other values in the $\hat{x}_t$ vector are zero initially. We then apply the matrices P and F repeatedly to simulate the impulse response function.

Question 2 (*20 points*)

This question considers a TFP shock in the New Keynesian model.

The representative household's utility function is:

$$\frac{C_t^{1-\sigma}}{1-\sigma} - \frac{N_t^{1+\psi}}{1+\psi} \tag{19}$$

In linearized form, the equilibrium conditions for this model are as follows. The household's Euler equation and labor supply conditions are:

$$E_t\hat{c}_{t+1} - \hat{c}_t = \frac{1}{\sigma}(\hat{i}_t - E_t\hat{\pi}_{t+1}) \tag{20}$$

$$\hat{w}_t = \sigma\hat{c}_t + \psi\hat{n}_t \tag{21}$$

The linearized equilibrium conditions for firms are:

$$\hat{y}_t = \hat{a}_t + \hat{n}_t \tag{22}$$

$$\hat{mc}_t = \hat{w}_t - \hat{a}_t \tag{23}$$

$$\hat{\pi}_t = \beta E_t(\hat{\pi}_{t+1}) + \lambda\hat{mc}_t \tag{24}$$

The resource constraint is:

$$\hat{y}_t = \hat{c}_t \tag{25}$$

Monetary policy follows a simple Taylor Rule:

$$\hat{i}_t = \phi_\pi\hat{\pi}_t \tag{26}$$

(Linearized) TFP follows an AR(1) process

$$\hat{a}_t = \rho\hat{a}_{t-1} + e_t \tag{27}$$

e_t is i.i.d.

In percentage deviations from steady state: $\hat{mc}_t$ is real marginal cost, $\hat{c}_t$ is consumption, $\hat{w}_t$ is the real wage, $\hat{n}_t$ is hours worked, $\hat{y}_t$ is output and $\hat{a}_t$ is Total Factor Productivity. In deviations from steady state: $\hat{i}_t$ is the nominal interest rate, $\hat{\pi}_t$ is inflation. λ is a function of model parameters, including the degree of price stickiness.[2] Assume that $\phi_\pi > 1$, $0 < \rho < 1$ and $0 < \beta < 1$

a) Using the equilibrium conditions above, show that this model can be represented by the standard 3 equations

$$E_t\tilde{y}_{t+1} - \tilde{y}_t = \frac{1}{\sigma}(\hat{i}_t - E_t\hat{\pi}_{t+1} - \hat{r}_t^n) \tag{28}$$

[2] $\lambda = \frac{(1-\theta)(1-\beta\theta)}{\theta}$ where θ is the probability that a firm cannot adjust its price.

$$\hat{\pi}_t = \beta E_t(\hat{\pi}_{t+1}) + \kappa \tilde{y}_t \tag{29}$$

$$\hat{i}_t = \phi_\pi \hat{\pi}_t \tag{30}$$

plus expressions for the natural rate of output:

$$\hat{y}_t^n = \frac{1+\psi}{\sigma+\psi}\hat{a}_t \tag{31}$$

the natural real rate of interest:

$$\hat{r}_t^n = -\sigma(1-\rho)\frac{1+\psi}{\sigma+\psi}\hat{a}_t \tag{32}$$

and the output gap

$$\tilde{y}_t = \hat{y}_t - \hat{y}_t^n \tag{33}$$

where $\hat{a}_t$ follows the process in equation 27. (**Hint:** You may want to start by finding the natural rate of output (equation 31) and then writing the household Euler equation in terms of (linearized) output, the natural rate of output, the nominal interest rate, TFP and expected inflation.)

Sketch answer:

Let's first find the natural rate of output. This means the model with flexible prices, so $\hat{mc}_t = 0$ and $\hat{a}_t = \hat{w}_t$ (from the linearized labor demand condition for the firm). Combining this with the production function, the resource constraint and the labor supply curve yields the expression for the natural rate in the question.

Next, we can substitute $\hat{y}_t = \hat{c}_t$ into the Euler equation and then add and subtract the natural rate of output at t and $t+1$ from both sides. Rearranging yields:

$$E_t\tilde{y}_{t+1} - \tilde{y}_t = \frac{1}{\sigma}(\hat{i}_t - E_t\hat{\pi}_{t+1}) - (\rho-1)\frac{1+\psi}{\sigma+\psi}\hat{a}_t \tag{34}$$

and we have also made use of the following property of the stochastic process:

$$E_t\hat{a}_{t+1} - \hat{a}_t = (\rho-1)\hat{a}_t$$

Finally, let's find the natural rate of interest. With flexible prices the output gap is zero. The LHS of equation 34 therefore 0. Note that the real rate is defined as

$$\hat{r}_t = \hat{i}_t - E_t\pi_{t+1}$$

and after imposing flexible prices the real rate we're solving for is the natural rate of interest.

$$0 = \frac{1}{\sigma}(\hat{r}_t^n + \sigma(1-\rho)\frac{1+\psi}{\sigma+\psi}\hat{a}_t)$$

which can be rearranged to yield the natural rate of interest given in the question.

By combining the labor supply equation, labor demand equation, production function and resource constraint with the Phillips Curve, this will produce equation 29 above.

b) Using the method of undetermined coefficients, find the response of the output gap and inflation to an exogenous increase in $\hat{a}_t$ when prices are sticky and monetary policy follows the Taylor Rule above. To do this, guess that the solution for each variable is a linear function of the shock $\hat{a}_t$:

$$\tilde{y}_t = \Lambda_y \hat{a}_t$$
$$\hat{\pi}_t = \Lambda_\pi \hat{a}_t$$

Sketch answer:

Substitute the guesses into the Phillips Curve and solving for Λ_y in terms of Λ_π yields:

$$\Lambda_y = \frac{\Lambda_\pi(1-\beta\rho)}{\kappa}$$

Next, substitute the guesses into the dynamic IS curve, and make use of the definition of the natural rate of interest. Also make use of the expression for Λ_y that we just derived. Solve for Λ_π:

$$\Lambda_\pi = -\frac{1+\psi}{\sigma+\psi}\kappa\sigma(1-\rho)\left(\sigma(1-\beta\rho)(1-\rho)+\kappa(\phi_\pi-\rho)\right)^{-1} < 0$$

Combining this with the solution for Λ_y we found above:

$$\Lambda_y = -\frac{1+\psi}{\sigma+\psi}\sigma(1-\rho)(1-\beta\rho)\left(\sigma(1-\beta\rho)(1-\rho)+\kappa(\phi_\pi-\rho)\right)^{-1} < 0$$

c) Interpret your results in (b). In particular, carefully explain how, and why, TFP shocks affect the output gap and inflation in this model.

Sketch answer:

A positive TFP shock lowers inflation and the output gap. The economy is more productive and firms would like to raise output and cut prices. With sticky prices some firms cannot cut prices and therefore can't raise output as much as they'd like. When firms do re-optimize they cut prices, which is why inflation is negative. Because output doesn't rise as much as in the flexible price model, the output gap is negative.

d) Instead of following the Taylor Rule above, policy is now set optimally. One result is that, under optimal policy, the real interest rate tracks the natural real interest rate. From your knowledge of this model and optimal policy, what is the optimal path for the output gap and inflation in response to a TFP shock and why (you do not need to derive anything)?

Sketch answer:

The basic idea can be seen from:

$$E_t\tilde{y}_{t+1} - \tilde{y}_t = \frac{1}{\sigma}(\hat{i}_t - E_t\hat{\pi}_{t+1} - \hat{r}_t^n) \tag{35}$$

If the real rate tracks the natural real rate, the right hand side is zero. As a result the left hand side is zero. With a closed output gap, inflation is also zero. To show this properly, you could iterate forward on this expression and the Phillips Curve to obtain expressions for the output gap and inflation today. If the real rate gap is always zero from today until infinity, the output gap and inflation will be zero.

This replicates optimal policy. Sticky prices lead to welfare losses. If policy can completely stabilize the output gap and inflation, the welfare loss is zero. Optimal policy therefore sets the path for the output gap and inflation to zero. This is possible for TFP shocks, it would not be possible for cost-push shocks. Because of this "divine coincidence", it does not matter whether the policy is set under commitment or discretion.

e) Suppose the monetary policymaker wants to implement optimal policy using an interest rate rule for $\hat{i}_t$. Explain why the policymaker cannot simply use a rule which attempts to set the nominal interest rate ($\hat{i}_t$) equal to the natural real interest rate ($\hat{r}_t^n$). What other component(s) should the rule contain and why (you do not need to derive anything)?

Sketch answer:

A policy rule of the following form:

$$\hat{i}_t = \hat{r}_t^n$$

is like an interest rate peg (because the natural rate is exogenous). Without any feedback from inflation, this rule does not deliver a unique stable solution for inflation (or any endogenous variables in this model). Mathematically, this rule does not produce the right number of stable eigenvalues. To rule out multiple equilibria, the policy rule must specify how it handles deviations from the desired equilibrium. A rule including inflation will work:

$$\hat{i}_t = \hat{r}_t^n + \phi\hat{\pi}_t$$

as long as $\phi > 1$ and therefore satisfies the Taylor Principle. Ex-post, in this model, all variables end up being zero because this rule replicates optimal policy.

Solutions to Macro Prelim Exam

Nicolas Caramp

August 2018

1 The Choice of Project

1. If effort is observable then entrepreneurs will never shirk[1]

   Then

   - Expected return of project 1: $p_H R$
   - Expected return of project 2: $q_H R$
   - Denote R^i_j the return of project j for agent i in case of success. Then for any N

   $$p_H R^F_1 = I - N$$
   $$q_H R^F_2 = I - N$$

   Since $p_H > q_H$, then

   $$p_H R^E_1 = p_H R - p_H R^F_1 = p_H R - (I - N) > q_H R - (I - N) = q_H R - q_H R^F_2 = q_H R^E_2$$

   Which establishes that project 2 is strictly dominated by project 1 when there is no information problem.

2. For project 1 the constraints are:

   - The resource constraint:

   $$R = R^F_1 + R^E_1$$

   - Financier participation constraint:

   $$p_H R^F_1 \geqslant I - N$$

[1] We assume that if they shirk, this is verifiable by a court and the entrepreneur can be punished so that it won't be optimal for her to exert low effort/shirk

- Entrepreneur incentive compatibility:

$$p_H R_1^E \geqslant p_L R_1^E + B \;\Rightarrow\; R_1^E \geqslant \frac{B}{\Delta p}$$

For project 2 the constraints are:

- The resource constraint:

$$R = R_2^F + R_2^E$$

- Financier participation constraint:

$$q_H R_2^F \geqslant I - N$$

- Entrepreneur incentive compatibility:

$$q_H R_2^E \geqslant q_L R_2^E + b \;\Rightarrow\; R_2^E \geqslant \frac{b}{\Delta q}$$

Conditional on undertaking project 1, the contracting problem the entrepreneur solves is:

$$\begin{aligned} \max_{\{R_1^E, R_1^F\}} \quad & p_H R_1^E \\ \text{subject to} \\ & R = R_1^F + R_1^E \\ & p_H R_1^F \geqslant I - N \\ & R_1^E \geqslant \frac{B}{\Delta p} \end{aligned} \tag{1}$$

The objective function can be rewritten as $p_H(R - R_1^F)$, then the incentive compatibility constraint of financiers will always bind in the optimal contract. Therefore the objective function can be rewritten as

$$p_H R_1^E = p_H(R - R_1^F) = p_H R + (N - I)$$

which is strictly positive since the project has positive NPV. Thus, E can undertake project 1 if and only if the IC constraint is satisfied

$$I - N \leqslant p_H \left(\underbrace{R - \frac{B}{\Delta p}}_{\equiv \rho_1} \right) \iff N \geqslant I - p_H \rho_1 \equiv \overline{N}_p$$

Similar calculations show that the project is feasible for E if and only if $N \geqslant I - q_H \rho_2 \equiv \overline{N}_q$.

3. Note that $\overline{N}_q < \overline{N}_p$ since $p_H \rho_1 < q_H \rho_2$ by assumption.

   Thus, we can partition $[0, I)$ into three regions:

- $[0, \overline{N}_q)$ where N is not enough to invest in any project. Then the optimal policy in $[0, \bar{N}_q)$ is to not invest
- $[\overline{N}_q, \overline{N}_p)$ where N is enough only to invest in project 2. Since project 2 has positive NPV, E chooses to undertake project 2 in this region
- $[\bar{N}_p, I)$ where N is enough to invest in project 1 and 2. The optimal choice is to invest in project 1 since by assumption $p_H > q_H$, which implies :

$$p_H R_1^E = p_H R - (I - N) > q_H R - (I - N) = q_H R_2^E$$

Project 2 is worse in terms of payoffs but better in terms of incentives. Intuitively the contract is doing 3 things: paying the financier, paying the entrepreneur and making sure incentives are aligned (prevent misbehavior). Project 2 pledgeable output is higher, which implies that is less costly for the contract to prevent misbehavior, and less "skin in the game" is needed. Hence under asymmetric information there is a role for project 2 (remember that in point 1, there was no role for project 2 since there was no information problem, hence the only dimension in which project 2 was batter than project 1 was not relevant).

4. Assume that now, financiers cannot see the project E chooses.

If $N \geqslant \bar{N}_p$, the contract designed for project 1 imply that if project 1 is chosen the entrepreneur will behave (not shirk). The new source of potential misbehavior is choosing to pursue project 2 when funding for project 1 was provided by the financiers. But this is not privately optimal for the entrepreneur since $p_H R + (N - I) > q_H R + (N - I)$. This means, when $N \geqslant \bar{N}_p$, financiers know that entrepreneurs will undertake project 1 without the need of effectively observing them

If $N \in [\bar{N}_q, \bar{N}_p)$, by what we discussed previously financiers will only want to give funds to entrepreneurs if it is privately optimal for entrepreneurs to pursue project 2 conditional on getting the loan from financiers and they will ask for $R_2^F = \frac{I-N}{q_H}$.

If entrepreneurs have incentives to deviate and choose project 1 and shirk, then financiers will not lend to them. If E undertakes project 2 and doesn't shirk, she gets

$$q_H \left(R - \frac{I - N}{q_H} \right)$$

If after signing the contract and getting the funding she decides to undertake project 1, she gets

$$p_H \left(R - \frac{I - N}{q_H} \right) > q_H \left(R - \frac{I - N}{q_H} \right)$$

That is, she wants to deviate and undertake project 1. However, when $N < \overline{N}_p$, we know that

$$p_H \left(R - \frac{I - N}{p_H} \right) < p_L \left(R - \frac{I - N}{p_H} \right) + B$$

Since $p_H\left(R-\frac{I-N}{q_H}\right) < p_H\left(R-\frac{I-N}{p_H}\right)$, we have

$$q_H\left(R-\frac{I-N}{q_H}\right) < p_H\left(R-\frac{I-N}{p_H}\right) < p_L\left(R-\frac{I-N}{p_H}\right)+B$$

which implies that when $N \in [\overline{N}_q, \overline{N}_p)$, if E gets funding, she will undertake project 1 and shirk. But we know that in that case the project has negative NPV, so the financiers will not lend to Es with this level of wealth.

5. Now we assume:

$$p_H R - c > q_H R > I$$

Hence it is more profitable to pay c and implement the project 1 rather than not paying c and implementing project 2.

Note that $R - \frac{b}{\Delta p}$ is the pledgable income of the monitored project, $p_H\left(R-\frac{b}{\Delta p}\right)$ is the maximum compensation that a financier can get.

There is going to be a level of N for which financing the monitored project is possible, call it $\tilde{N}$:

$$p_H\left(\underbrace{R-\frac{b}{\Delta p}}_{\equiv\tilde{\rho}_1}\right) = I - \tilde{N} + c$$

$$\tilde{N} = I + c - p_H\tilde{\rho}_1$$

Monitoring is useful only if $\tilde{N} < \overline{N}_p$ (that is, it allows some "low" wealth Es to get funding for project 1). Formally:

$$\tilde{N} < \overline{N}_p \iff p_H\left(R-\frac{b}{\Delta p}\right) - c > p_H\left(R-\frac{B}{\Delta p}\right)$$

$$p_H\frac{B-b}{\Delta p} > c$$

Note that project 2 is going to be implemented if the required skin in the game is smaller than under project 1 monitored. Hence, we need to compare $\bar{N}_q = I + \frac{b}{\Delta q} - R$ with $\tilde{N} = I + p_H\frac{b}{\Delta p} - (p_H R - c)$

$$\bar{N}_q < \tilde{N} \iff I - q_H\left(R-\frac{b}{\Delta q}\right) < I - p_H\left(R-\frac{b}{\Delta q}\right) + c$$

Using $\Delta p = \Delta q$

$$c > (p_H - q_H)\left(R-\frac{b}{\Delta p}\right)$$

Project 2 will be undertaken by entrepreneurs with net worth $N \in [\bar{N}_q, \tilde{N})$

2 Bubbles

1. Given prices, the problem of the agent is:

$$\begin{aligned} \max_{\{c_{t+1}^t, c_{t+1}^{t+1}, m_t\}} \quad & \frac{(c_t^t)^{1-\sigma}}{1-\sigma} + \frac{(c_{t+1}^t)^{1-\sigma}}{1-\sigma} \\ \text{subject to} \quad & \\ & c_t^t + p_t m_t \leqslant w_1 \qquad (\lambda_t^t) \\ & c_{t+1}^t \leqslant w_2 + p_{t+1} m_t \qquad (\lambda_{t+1}^t) \end{aligned} \tag{2}$$

Since the utility function satisfies inada conditions, consumption will be strictly positive, hence the FOC characterizing the solution of the agents problem are:

$$\begin{aligned} (c_t^t)^{-\sigma} &= \lambda_t^t \\ (c_{t+1}^t)^{-\sigma} &= \lambda_{t+1}^t \\ \lambda_t^t p_t &= \lambda_{t+1}^t p_{t+1} \end{aligned} \tag{3}$$

2. An equilibrium for this economy is sequence of prices $\{p_t\}_{t=0}^{\infty}$ and allocations of consumption $(c_0, \{(c_t^t, c_{t+1}^t\}_{t=1}^{\infty})$ and money $\{m_t^t\}_{t=0}^{\infty}$ such that:
   - $\forall\ t \geqslant 1$, given prices c_t^t, c_{t+1}^t, m_t solves the problem of the agent born at t
   - c_0 solves the problem of the initial generation
   - Markets clear:

$$M = m_t$$

$$c_t^t + c_t^{t-1} = w_1 + w_2$$

Equilibrium is characterized by the following system of equations:

$$\begin{aligned} (c_t^t)^{-\sigma} &= \lambda_t^t \\ (c_{t+1}^t)^{-\sigma} &= \lambda_{t+1}^t \\ \lambda_t^t p_t &= \lambda_{t+1}^t p_{t+1} \\ c_t^t + p_t m_t &= w_1 \\ c_{t+1}^t &= w_2 + p_{t+1} m_t \\ M &= m_t \\ c_t^t + c_t^{t-1} &= w_1 + w_2 \end{aligned} \tag{4}$$

3. Before guessing, let's work the equilibrium characterization:

Using the optimality conditions:

$$(c_t^t)^{-\sigma} p_t = (c_{t+1}^t)^{-\sigma} p_{t+1}$$

And the budget constraint:

$$(w_1 - p_t m_t)^{-\sigma} p_t = (w_2 + p_{t+1} m_t)^{-\sigma} p_{t+1}$$

Combined with market clearing:

$$(w_1 - p_t M)^{-\sigma} p_t = (w_2 + p_{t+1} M)^{-\sigma} p_{t+1}$$

Now, guess $p_t = p_{t+1} = p > 0$, then:

$$(w_1 - pM)^{-\sigma} p = (w_2 + pM)^{-\sigma} p$$

$$w_1 - w_2 = 2pM$$

$$p = \frac{w_1 - w_2}{2M}$$

This verify our guess, p does not depend on t and by assumption $w_1 > w_2$, hence $p > 0$.

4.

Claim 1. *If* $p_t = 0$ *then* $p_{t+k} = 0 \ \forall k \geqslant 1$

Proof. Let $p_t = 0$ and assume to the contrary that $p_{t+1} > 0$. Then at today's prices of $p_t = 0$ we would have excess demand since we can get money for free today and then trade it for goods tomorrow. Then for $p_t = 0$ to be an equilibrium, agent's must believe also that $p_{t+1} = 0$.

What is more, assume $p_{t+k} > 0$ from t point's of view, then it must be that $p_{t+k-1} > 0$ by our previous argument... and iterating we would get that $p_t > 0$. Hence $p_t = 0$, imposes $p_{t+k} = 0 \ \forall k \geqslant 1$. □

The previous claim says, once the bubble burst it burst forever. Hence there are two possible cases $p_t = 0$ that implies that autarky is the only equilibrium allocation or $p_t > 0$. We proceed to characterize the second case.

Let $p_t > 0$, denote $c_{t+1,a}^t, c_{t+1,b}^t$ the consumption next period when the bubble burst and continues respectively. Therefore, given the agents beliefs $1 - \pi$ that the bubble will bursts the agent solves:

$$\begin{aligned}
\max_{\{c_{t+1}^t, c_{t+1}^{t+1}, m_t\}} \quad & \frac{(c_t^t)^{1-\sigma}}{1-\sigma} + \pi \frac{(c_{t+1,b}^t)^{1-\sigma}}{1-\sigma} + (1-\pi)\frac{(c_{t+1,a}^t)^{1-\sigma}}{1-\sigma} \\
\text{subject to} \quad & \\
& c_t^t + p_t m_t \leqslant w_1 \qquad\qquad (\lambda_t^t) \\
& c_{t+1,b}^t \leqslant w_2 + p_{t+1} m_t \qquad (\lambda_{t+1,b}^t) \\
& c_{t+1,a}^t \leqslant w_2 \qquad\qquad (\lambda_{t+1,a}^t)
\end{aligned} \tag{5}$$

Inada and monotonicty imply:

$$\begin{aligned}
c_t^t &= w_1 - p_t m_t \\
c_{t+1,b}^t &= w_2 + p_{t+1} m_t \\
c_{t+1,a}^t &= w_2
\end{aligned}$$

Therefore we can rewrite the agent's program as:

$$\max_{\{m_t\}} \quad \frac{(w_1 - p_t m_t)^{1-\sigma}}{1-\sigma} + \pi \frac{(w_2 + p_{t+1} m_t)^{1-\sigma}}{1-\sigma} + (1-\pi)\frac{(w_2)^{1-\sigma}}{1-\sigma} \tag{6}$$

$$p_t (w_1 - p_t m_t)^{-\sigma} = \pi \, p_{t+1} (w_2 + p_{t+1} m_t)^{-\sigma} \tag{7}$$

Guess $p_t = p_{t+1} = p$, then using market clearing $m_t = M$ we get:

$$\begin{aligned}
(w_1 - pM)^{-\sigma} &= \pi \, (w_2 + pM)^{-\sigma} \\
(w_2 + pM) &= \pi^{\frac{1}{\sigma}} (w_1 - pM)
\end{aligned}$$

$$p = \frac{\pi^{\frac{1}{\sigma}} w_1 - w_2}{(\pi^{\frac{1}{\sigma}} + 1)M}$$

$$\frac{\partial p}{\partial \pi} = \frac{\pi^{\frac{1-\sigma}{\sigma}} (w_1 - w_2)}{\sigma (\pi^{\frac{1}{\sigma}} + 1)^2 M} > 0$$

Where we used $w_1 > w_2$

5.

$$\begin{aligned}
\max_{\{c_{t+1}^t, c_{t+1}^{t+1}, m_t\}} \quad & \ln c_t^t + \pi_t \ln c_{t+1,H}^t + (1-\pi_t) \ln c_{t+1,L}^t \\
\text{subject to} \quad & \\
& c_t^t + p_t m_t \leqslant w_1 \qquad\qquad (\lambda_t^t) \\
& c_{t+1,H}^t \leqslant p_H m_t \qquad\qquad (\lambda_{t+1,H}^t) \\
& c_{t+1,L}^t \leqslant p_L m_t \qquad\qquad (\lambda_{t+1,L}^t)
\end{aligned} \tag{8}$$

$$\max_{\{m_t\}} \quad \ln(w_1 - p_t m_t) + \pi_t \ln(p_{t+1} m_t) + (1-\pi_t)\ln(p_{t+1} m_t) \tag{9}$$

If we continue with the case analyzed in point 4.

$$\max_{\{m_t\}} \quad \ln(w_1 - p_t m_t) + \pi_t \ln(p_{t+1} m_t) \tag{10}$$

$$p_t(w_1 - p_t m_t)^{-1} = \pi_t \; p_{t+1}(p_{t+1} m_t)^{-1} \tag{11}$$

$$p_t(w_1 - p_t M)^{-1} = \pi_t \; p_{t+1}(p_{t+1} M)^{-1}$$

$$p_t = \pi_t \frac{w_1}{M} - \pi_t \frac{p_t M}{M}$$

$$p_t = \frac{\pi_t}{1+\pi_t}\frac{w_1}{M}$$

Note $\frac{\partial p_t}{\partial \pi_t} = \frac{1}{(1+\pi_t)^2}\frac{w_1}{M} > 0$, hence when π_t is high/low prices today are high/low. The only driver of the price fluctuations are beliefs.

Let's consider two levels of beliefs optimists: π^H and pessimists π^L. $\pi^H = \pi^L + \Delta\pi$ where $\Delta\pi > 0$ and $\pi^H \leqslant 1$

Consider any arbirary sequence of beliefs $(\pi_1, \ldots, \pi_t, \ldots)$. Whenever $\pi_t = \pi^H$ we have:

- Equilibrium prices $p_H = \frac{\pi_H}{1+\pi_H}\frac{w_1}{M}$
- Consumption allocations of generation t given by:

$$c^t_{t,H} = w_1 - p_H M = \frac{1}{1+\pi_H} w_1$$

  Now it's key the beliefs of generation $t+1$ that are going to buy the money:

  - If $\pi_{t+1} = \pi^H$

$$c^t_{t+1,H} = \frac{\pi_H}{1+\pi_H} w_1$$

  - If $\pi_{t+1} = \pi^L$

$$c^t_{t+1,H} = p_L M = \frac{\pi_L}{1+\pi_L} w_1$$

  Now markets at $t+1$ will clear since

  - If generation $t+1$ have beliefs π^H

$$c^t_{t+1,H} + c^{t+1}_{t+1,H} = \frac{\pi_H}{1+\pi_H} w_1 + \frac{1}{1+\pi_H} w_1 = w_1$$

– If generation $t+1$ have beliefs π^L

$$c^t_{t+1,H} + c^{t+1}_{t+1,L} = \frac{\pi_L}{1+\pi_L} w_1 + \frac{1}{1+\pi_L} w_1 = w_1$$

Analogous argument show us that if $\pi_t = \pi^L$ we get individual rationality and aggregate consistency.