

PRELIMINARY EXAMINATION FOR THE Ph.D. DEGREE

*Directions: Answer all questions. Feel free to impose additional structure on the problems below, but please state your assumptions clearly. Point totals for each question are given in parentheses.*

1. (10) Consider a consumer with preferences

$$U = \int_0^{\infty} e^{-\rho t} u(c_t) dt,$$

where  $\rho$  is the subjective discount rate,  $c$  is consumption, and  $u(c) = \ln c$ . She receives an exogenous flow of income  $y$  and can borrow or lend freely at a constant interest rate  $r$ , subject to a no-Ponzi-game condition that rules out infinite debt. Her flow budget constraint is

$$\dot{a} = ra + y - c,$$

where  $a$  represents financial wealth and  $a_0$  is the initial value.

- (a) Derive the first-order conditions for optimal consumption.
- (b) At what rate does consumption grow? Interpret its sign.
- (c) Derive the optimal decision rule for consumption. Provide an interpretation for the case  $\rho = r$ .

ANSWER: The current-value Hamiltonian is

$$H = \log(c) + \lambda(ra + y - c).$$

The FOC are

$$\begin{aligned} \frac{\partial H}{\partial c} &= \frac{1}{c} - \lambda = 0, \\ \frac{\partial H}{\partial a} &= r\lambda = \rho\lambda - \dot{\lambda}, \\ \frac{\partial H}{\partial \lambda} &= ra + y - c = \dot{a}. \end{aligned}$$

After combining the first two conditions, we get

$$\frac{\dot{c}}{c} = r - \rho.$$

Hence consumption grows at a constant rate  $r - \rho$ . The slope of the consumption profile reflects the countervailing forces of interest accumulation and discounting. If the interest rate is greater, it pays to postpone consumption in order to benefit from interest accumulation. Hence consumption is backloaded and growth is positive. When the discount rate is higher, impatience outweighs interest accumulation, so it pays to frontload consumption and have a declining consumption path. To find the decision rule, integrate the consumption growth condition with respect to time to get

$$c_t = c_0 e^{(r-\rho)t}.$$

All that remains is to pin down initial consumption. To do that, integrate the flow budget constraint with respect to time and use the NPG condition to obtain a present-value budget constraint,

$$\int_0^{\infty} e^{-rt} c_t dt = a_0 + h_0,$$

where

$$h_0 \equiv \int_0^{\infty} e^{-rt} y_t dt$$

is the present value of labor income. Substituting  $c_t = c_0 e^{(r-\rho)t}$  into the present-value budget constraint delivers

$$\begin{aligned} a_0 + h_0 &= \int_0^{\infty} e^{-rt} c_0 e^{(r-\rho)t} dt, \\ &= c_0 \int_0^{\infty} e^{-\rho t} dt. \end{aligned}$$

The integral on the right-hand side is easy.

$$\begin{aligned} \int_0^{\infty} e^{-\rho t} dt &= (-\rho^{-1} e^{-\rho t} + \text{constant}) \Big|_{t=0}^{\infty}, \\ &= \rho^{-1}. \end{aligned}$$

Hence, initial consumption is

$$c_0 = \rho(a_0 + h_0),$$

and the consumption decision rule is

$$c_t = \rho(a_0 + h_0) e^{(r-\rho)t}.$$

When  $\rho = r$ , this simplifies to

$$c_t = r(a_0 + h_0)$$

for all  $t$ . I.e., the agent consumes the annuity value of initial total wealth, measured as the sum of initial financial assets plus human capital.

2. (20) Consider the following optimal growth model. Agents' preferences are given by

$$E_0 \left[ \sum_{t=0}^{\infty} \beta^t A_t \ln c_t \right]$$

The term  $A_t$  denotes an *i.i.d.* demand shock. Output is a concave function of beginning of period capital and a technology shock. That is,

$$y_t = z_t k_t$$

It is assumed that  $z_t$  is *i.i.d.* both over time and with respect to  $A_t$ . (That is, the two shocks are independently distributed.) The depreciation rate of capital is 100%. Within this environment, do the following

- (a) Set up the social planner problem as a dynamic programming problem and derive the necessary conditions.
- (b) Conjecture a solution to the policy functions for consumption and savings. Derive the equations that determine these optimal policy functions and characterize their qualitative behavior.
- (c) Suppose one period (real) bonds were introduced into this economy. How do demand shocks affect interest rates in this economy? Explain.

ANSWER: In this model, the state variables are  $(z_t, k_t, A_t)$ . The control variables are:  $(c_t, k_{t+1})$ . The necessary conditions are:

$$\frac{A_t}{c_t} = \beta E \left[ \frac{A_{t+1}}{c_{t+1}} z_{t+1} \right]$$

where the expectations are with respect to the joint distribution for  $z_t$  and  $A_t$ . We showed in class that the consumption and capital policy functions in this environment will be homogeneous of degree one in output,  $y_t = z_t k_t$ . Focusing on the policy function for consumption,

$$c_t = c(z_t, k_t, A_t) = y_t \omega(A_t)$$

Using the conjecture and the resource constraint yields:

$$A_t \frac{(1 - \omega(A_t))}{\omega(A_t)} = \beta E \left[ \frac{A_{t+1}}{\omega(A_{t+1})} \right]$$

This establishes the conjecture; moreover, since the right-hand is a constant given the *i.i.d.* assumption, this requires  $\omega'(A_t) > 0$ . This makes sense: a higher weight on utility implies that current consumption will increase. For one period interest rates, these must satisfy the Euler equation associated with bonds:

$$\frac{A_t}{c_t} = \beta E \left[ \frac{A_{t+1}}{c_{t+1}} \right] R_t$$

where  $R_t$  denotes the gross real interest rate. Using the solution  $c_t = y_t \omega(A_t)$ , this expression becomes:

$$A_t \frac{(1 - \omega(A_t))}{\omega(A_t)} = \beta E \left[ \frac{A_{t+1}}{\omega(A_{t+1})} \frac{1}{z_{t+1}} \right] R_t = \beta E \left[ \frac{A_{t+1}}{\omega(A_{t+1})} \right] E \left[ \frac{1}{z_{t+1}} \right] R_t$$

where the last equality follows from the assumption that the  $A_t$  and  $z_t$  are independently distributed. But the above analysis established that the left hand side is equal to the first term on the right-hand side so that:

$$R_t^{-1} = E \left[ \frac{1}{z_{t+1}} \right]$$

Ignoring Jensen's inequality, this states that the real interest rate is determined by the expected marginal product of capital. But since the technology shock is *i.i.d.* it also implies that the real interest rate is constant in this economy - demand shocks do not influence interest rates. This is in large part due to the assumption of linear technology.

3. (20) Consider the basic real business cycle model in which the representative agent maximizes the expected value of discounted utility given by:

$$E_0 \left[ \sum_{t=0}^{\infty} \beta^t (\ln c_t - \gamma h_t) \right]$$

Technology is given by a standard Cobb-Douglas production function, i.e.  $y_t = z_t k_t^\alpha h_t^{1-\alpha}$  in which  $z_t$  is an *i.i.d.* technology shock with *c.d.f.* given by  $G(z_t)$ . The depreciation rate of capital is 100%. Do the following:

- (a) Solve the model as a social planner problem and derive the optimal policy functions.
- (b) Discuss how well the implied time series characteristics of the model match those seen in the data. (Restrict your discussion to those features discussed in class.) Discuss a minimal set of changes in the model environment that could produce better consistency between the model and business cycle data.

ANSWER: The Euler equations for this model are:

$$\frac{k_{t+1}}{c_t} = \alpha\beta E \left[ \frac{y_{t+1}}{c_{t+1}} \right] \quad (1)$$

$$\gamma = (1 - \alpha) \frac{z_t k_t^\alpha h_t^{-\alpha}}{c_t} \quad (2)$$

We know that eq.(1) implies  $c_t = (1 - \alpha\beta) y_t$ ,  $k_{t+1} = \alpha\beta y_t$ . Multiplying both sides of eq.(2) by  $h_t$  yields

$$h_t = \frac{(1 - \alpha) y_t}{\gamma c_t} = \frac{(1 - \alpha)}{\gamma} \frac{1}{(1 - \alpha\beta)}$$

where the right-hand side is using the eq. policy function for consumption. But this implies that labor is constant in this model. Using this in the optimal policy functions for consumption and capital yields:

$$c_t = (1 - \alpha\beta) \left[ \frac{(1 - \alpha)}{\gamma} \frac{1}{(1 - \alpha\beta)} \right]^{1-\alpha} z_t k_t^\alpha$$

$$k_{t+1} = \alpha\beta \left[ \frac{(1 - \alpha)}{\gamma} \frac{1}{(1 - \alpha\beta)} \right]^{1-\alpha} z_t k_t^\alpha$$

The implications for the time series properties of the model are stark: Labor is invariant over the business cycle (also note that this implies average labor productivity is countercyclical) while consumption, investment ( $k_{t+1}$ ) and output all move together and have equal volatility in percentage terms. Except for the procyclical behavior of consumption and investment, the model's predictions are inconsistent with observation: Labor is procyclical, while investment is more volatile than output which is more volatile than consumption. Since the preferences in this model are the same as used in Hansen's indivisible labor model which WAS consistent with these basic features of the data, assuming a depreciation rate less than 100% (calibrating the model to the data would imply  $\delta = 0.025$  roughly) and introducing a high level of persistence in the technology shock would dramatically increase the consistency between model and data. Of course, other changes such as habit persistence in preferences, endogenous capital utilization, labor adjustment costs would further increase the model's consistency.

4. (20)

5. (20)

(a)

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ANSWER KEY FOR QUESTION 4,5, 6

**Question 4**

a) This part is very similar to the lecture notes. Of course here there are 4 physical commodities. The consumption good, the capital services, the labor services, and the land services. Therefore, we will need 4 prices, say  $p_t, r_t, w_t, q_t$ , and these prices have to be determined in every period. However, things are not any harder than usual, because the supply of labor is fixed, so in every period the equilibrium level of land services will be 1. Hence, all the analysis will go through as usual.

b) The Planner's problem is indeed much easier, because one only needs to describe the allocations, and not the prices of all commodities (which are infinite sequences). What allows us to use this technique here, is the fact that in this environment both Welfare Theorem hold. So we know that the competitive allocation and the Planners allocation will coincide. After characterizing the Planners allocation, we can construct the whole competitive equilibrium, like we did in class.

c) Using any technique you like, you can arrive at the following Euler condition (which is necessary for the dynamic maximization):

$$\frac{1}{Ak_t^{\alpha_1} - k_{t+1}} = \frac{\alpha_1 \beta A k_{t+1}^{\alpha_1 - 1}}{Ak_{t+1}^{\alpha_1} - k_{t+2}}.$$

Impose the guess I gave you as a hint, and after a little some algebra, you will find that

$$k_{t+1} = g k_t^{\alpha_1} = \alpha_1 A \beta k_t^{\alpha_1}.$$

Therefore, in each period agents invest a part equal to  $\beta \alpha_1$  of output and eat the rest.

Moreover, since we know that in equilibrium  $k_t = x_t$ , and we also know  $x_0$ , we can fully characterize the whole capital stock allocation. In particular, for all  $T > 0$ , we have

$$k_T = (\alpha_1 \beta A)^{1 + \alpha_1 + \dots + \alpha_1^{T-1}} (x_0^{\alpha_1})^T.$$

d) This economy has a steady state. No matter where the economy starts, (initial capital stock), it will always converge to this steady state. To find it take the limit as  $T \rightarrow \infty$  in the equation above. We get

$$k^* = \lim_{T \rightarrow \infty} k_T = (\alpha_1 \beta A)^{\frac{1}{1 - \alpha_1}}.$$

The rental rate of capital is given by  $r_t = F_K(k_t, 1, 1)$ . As time goes to infinity, it is easy to verify that

$$r^* = \frac{1}{\beta}.$$

e) If you write down the AD problem of the household and take the first order conditions, you will find an expression very similar to the one obtained in class:

$$\frac{p_{t+1}}{p_t} = \frac{1}{r_{t+1}}.$$

Therefore, we have  $p_1/p_0 = 1/r_1$ , and as we usually do, let us treat consumption at  $t = 0$  as the numeraire, i.e. set  $p_0 = 1$ . Then,  $p_1 = r_1^{-1}$ , and we know that

$$r_1 = F_K(k_1, 1) = (\alpha_1 A)^{\alpha_1} \beta^{\alpha_1 - 1} x_0^{\alpha_1^2 - \alpha_1}.$$

Summing up,

$$p_1 = \left[ (\alpha_1 A)^{\alpha_1} \beta^{\alpha_1 - 1} x_0^{\alpha_1^2 - \alpha_1} \right]^{-1}.$$

f) We know that  $q_t = F_l(k_t, 1, 1)$ . Therefore,

$$q_2 = (1 - \alpha_1 - \alpha_2) A k_2^{\alpha_1}.$$

Since moreover,

$$k_2 = (\alpha_1 \beta A)^{1 + \alpha_1} (x_0^{\alpha_1})^2,$$

we have

$$q_2 = (1 - \alpha_1 - \alpha_2) A \left[ (\alpha_1 \beta A)^{1 + \alpha_1} (x_0^{\alpha_1})^2 \right]^{\alpha_1}.$$

### Question 5

a) The arrival rate of a Type i worker for a typical firm:

$$\frac{m(u, v) \cdot \frac{u_i}{u}}{v} = q(\theta) \frac{u_i}{u}$$

, where  $u = u_1 + u_2$ .

The arrival rate of a firm to a typical Type i worker (the question already tells us that the arrival rate of jobs to a worker does not depend on her type):

$$\frac{m(u, v)}{u} = \theta q(\theta)$$

b) The flows in and out of unemployment are given by

$$u_1 = (\pi - u_1)\lambda_1 - u_1\theta q(\theta)$$

$$u_2 = (1 - \pi - u_2)\lambda_2 - u_2\theta q(\theta)$$

Therefore, the two Beveridge curves are

$$u_1 = \frac{\pi\lambda_1}{\lambda_1 + \theta q(\theta)}$$

$$u_2 = \frac{(1 - \pi)\lambda_2}{\lambda_2 + \theta q(\theta)}$$

c) We have

$$\gamma_1 = \frac{u_1}{\pi} = \frac{\lambda_1}{\lambda_1 + \theta q(\theta)}$$

$$\gamma_2 = \frac{u_2}{1 - \pi} = \frac{\lambda_2}{\lambda_2 + \theta q(\theta)}$$

Since  $\frac{\lambda}{\lambda + \theta q(\theta)} = \frac{1}{1 + \frac{\theta q(\theta)}{\lambda}}$  is increasing in  $\lambda$ , and  $\lambda_1 < \lambda_2$ , we then have  $\gamma_1 < \gamma_2$ .

d)  $i = 1, 2$

$$rV = -pc + q(\theta)\frac{u_1}{u}(J_1 - V) + q(\theta)\frac{u_2}{u}(J_2 - V) \quad (1)$$

$$rJ_i = p - w_i + \lambda_i(V - J_i) \quad (2)$$

$$rU_i = z + \theta q(\theta)(W_i - U_i) \quad (3)$$

$$rW_i = w_i + \lambda_i(U_i - W_i) \quad (4)$$

e) Due to free entry,  $V = 0$ . From eq(2) we get

$$J_i = \frac{p - w_i}{r + \lambda_i}$$

From eq (1) we get

$$\frac{pc}{q(\theta)} = \frac{1}{u}(u_1J_1 + u_2J_2)$$

Combining these two equations:

$$\frac{pc}{q(\theta)} = \frac{1}{u}\left(u_1\frac{p - w_1}{r + \lambda_1} + u_2\frac{p - w_2}{r + \lambda_2}\right)$$

where  $u_i, i = 1, 2$  are in b), and this is the Job Creation curve.

f) Bargaining problem:  $\max_{w_i} [W_i - U_i]^\beta J_i^{1-\beta}$ , and FOC gives us

$$\beta J_i = (1 - \beta)(W_i - U_i) \quad (5)$$



We first substitute in  $J_i$  and  $W_i$ , then substitute in  $rU_i$ , and then use eq(5), and substitute in  $J_i$  again. Steps are as follows.

$$\beta \frac{p - w_i}{r + \lambda_i} = \frac{1 - \beta}{r + \lambda_i} [w_i + \lambda_i U_i - (r + \lambda_i) U_i]$$

$$\begin{aligned} w_i &= \beta p + (1 - \beta) r U_i \\ &= \beta p + (1 - \beta) [z + \theta q(\theta) (W_i - U_i)] \\ &= \beta p + (1 - \beta) z + \theta q(\theta) \beta J_i \\ &= \beta p + (1 - \beta) z + \theta q(\theta) \beta \frac{p - w_i}{r + \lambda_i} \end{aligned}$$

This is the Wage curve, if you want a slightly clearer expression:

$$w_i = \frac{[r + \lambda_i + \theta q(\theta)] \beta p + (1 - \beta) z (r + \lambda_i)}{r + \lambda_i + \beta \theta q(\theta)}$$

g) Calculate  $\frac{dw}{d\lambda}$ : if it is negative, then  $\lambda_1 < \lambda_2$  implies that  $w_2 < w_1$

$$\frac{dw}{d\lambda} = \frac{\beta \theta q(\theta) (1 - \beta) (z - p)}{[r + \lambda_i + \beta \theta q(\theta)]^2}$$

It is negative because every term is positive except for  $z - p$  by assumption.

h) Endogenous variables are  $w_1$ ,  $w_2$ ,  $\theta$ ,  $u_1$ , and  $u_2$ . Using JC, WC1, and WC2, we can solve for  $w_1$ ,  $w_2$ , and  $\theta$ . Then using BC1 and BC2, we get  $u_1$  and  $u_2$ .

## Question 6

a) The state variables for a buyer who enters the CM market are her money holdings,  $m$ , and her (real) asset holdings,  $a$ . Hence,

$$\begin{aligned} W(m, a) &= \max_{X, H, \hat{m}, \hat{a}} \{ U(X) - H + \beta [\pi \Omega^C(\hat{m}, \hat{a}) + (1 - \pi) \Omega^N(\hat{m}, \hat{a})] \} \\ \text{s.t. } & X + \varphi \hat{m} + \psi \hat{a} = H + \varphi m + T + da. \end{aligned}$$

If you substitute for  $H$  in the objective function, and observe that (as always) we have  $X = X^*$  at the optimum, we can re-write

$$W(m, d) = \Lambda + \phi m + da,$$

which establishes the linearity property, and where we have defined

$$\Lambda \equiv U(X^*) - X^* + T + \max_{\hat{m}, \hat{a}} \{ -\phi \hat{m} - \psi \hat{a} + \beta [\pi \Omega^C(\hat{m}, \hat{a}) + (1 - \pi) \Omega^N(\hat{m}, \hat{a})] \}.$$

b) As in the case of  $W$ , the state variables for a buyer who enters the DM are her money

holdings,  $m$  (because they determine how much good she can purchase), and her asset holdings (because this affects her continuation value). We have

$$V(m, a) = u(q(m)) + W(m - p(m), a).$$

For a flawless answer you should explicitly state that the terms  $q, d$  are functions of the buyer's money holdings (as I stressed in class). Notice that the asset holdings  $a$  remain unaltered as the buyer moves from the DM into the CM, because by assumption the buyer cannot use these assets as a medium of exchange in the DM.

c) Recall the terms  $\tau, \chi$  that we defined in the description of the physical environment. If the buyer turns out to be a C-type, we have

$$\Omega^C(m, a) = \frac{f(\pi, 1 - \pi)}{\pi} V(m + \tau, a - \chi) + \left[1 - \frac{f(\pi, 1 - \pi)}{\pi}\right] V(m, a).$$

where the term  $f/\pi$  is the probability with which the typical C-type will match with someone in the SFM. Notice that the matched C-type continues to the DM with her money holdings increased by the term  $\tau$ , but, in return, her asset holdings are reduced by  $\chi$ .<sup>1</sup>

For the typical N-type in the SFM, we have

$$\Omega^N(m, a) = \frac{f(\pi, 1 - \pi)}{1 - \pi} W(m - \tau, a + \chi) + \left[1 - \frac{f(\pi, 1 - \pi)}{1 - \pi}\right] W(m, a).$$

Notice that unlike the C-type, whose continuation value is represented by the function  $V$  (because she is going in the DM to consume), the N-types continuation value is represented by  $W$  because this agent proceeds directly to the CM. Here, the terms  $\tau, \chi$  will typically be functions of the states  $m, a$ . I did not expect you to be too precise about this fact (but if you did, good for you!)

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<sup>1</sup>Of course, this trade is mutually beneficial: while each unit of the asset is worth 1 unit of the numeraire in the hands of either type, the money is worth more in the hands of the C-type (because she has an opportunity to consume). As we sometimes said in class, this trade generates a positive net surplus. A detail that you did not need to worry about, is that this surplus will exist as long as the C-type has some shortage of liquidity, i.e., as long as her initial money holdings are not enough to purchase the first-best DM quantity,  $q^*$ ).