MACROECONOMICS PRELIM, AUGUST 2019

ANSWER KEY FOR QUESTIONS 1 AND 2

Question 1

a) An important observation is that matching here is unbiased. This is a term I have used numerous times in past exams, and it simply means that the two types of workers find jobs at the same rate. Given this, for the state E we have

$$\begin{aligned} outflows &: \lambda(1-u), \\ inflows &: \theta q(\theta)(u-u_n)+\theta q(\theta)u_n. \end{aligned}$$

For the state U_n, we have

$$\begin{aligned} outflows &: \theta q(\theta)u_n, \\ inflows &: \delta(u-u_n). \end{aligned}$$

Finally, for the state U, we have

$$\begin{aligned} outflows &: \theta q(\theta)(u-u_n)+\delta(u-u_n), \\ inflows &: \lambda(1-u). \end{aligned}$$

b) Set the inflows and outflows for state E equal to each other, to obtain:

$$u=\frac{\lambda}{\lambda+\theta q(\theta)},$$

which is the usual Beveridge curve. To obtain a formula that links u_n with the other variables, set the inflows and outflows for state U_n equal to each other. This yields

$$u_n=\frac{\delta}{\delta+\theta q(\theta)}u=\frac{\delta}{\delta+\theta q(\theta)}\,\frac{\lambda}{\lambda+\theta q(\theta)}.$$

For future reference, let us call these two equations BC and BC_n, respectively.

c) A typical worker can be in one of four possible states. Unemployed with benefits; unemployed without benefits; employed at a wage that was negotiated while she was still eligible for benefits; and employed at a wage that was negotiated when the worker was not eligible for benefits. The value functions for these four states are, respectively,

$$\begin{aligned} rU &= z + \theta q(\theta)(W - U) + \delta(U_n - U), && (1) \\ rU_n &= \theta q(\theta)(W_n - U_n), && (2) \\ rW &= w + \lambda(U - W), && (3) \\ rW_n &= w_n + \lambda(U - W_n). && (4) \end{aligned}$$

d) A typical firm can be in one of three possible states. Vacant and waiting to match with a worker; matched with an unemployed worker who had benefits when the match took place; and matched with an unemployed worker who benefits had expired at the time of the match. The value functions for these three states are, respectively,

$$\begin{aligned} rV &= -pc + q(\theta)\left[\frac{\theta q(\theta)}{\delta + \theta q(\theta)} J + \frac{\delta}{\delta + \theta q(\theta)} J_n\right], && (5) \\ rJ &= p - w - \lambda J, && (6) \\ rJ_n &= p - w_n - \lambda J_n. && (7) \end{aligned}$$

Notice that the multiplier of the term J (J_n) in (5) is the probability that, conditional on having met a worker, that worker is eligible (ineligible) for benefits. These expressions follow directly from our work in part (b).

e) The free-entry condition simply means that $V = 0$ in any equilibrium. Setting $V = 0$ in (5), and replacing for J and J_n from (6) and (7) yields the JC condition of out economy:

$$\frac{\theta q(\theta)}{\delta + \theta q(\theta)} \frac{p - w}{r + \lambda} + \frac{\delta}{\delta + \theta q(\theta)} \frac{p - w_n}{r + \lambda} = \frac{pc}{q(\theta)}.$$

f) First, consider a meeting with a worker who is eligible for benefits. In this type of meeting Nash bargaining dictates that

$$(1 - \beta)(W - U) = \beta J.$$

The first step is to replace W and J from the expressions provided earlier. If you do that you will find:

$$w = \beta p + (1 - \beta) rU.$$

This expression of course is not very useful because we want to get rid of all value functions. To that end, replace U from (1) to obtain:[1]

$$w = \beta p + (1 - \beta) z + \theta q(\theta) \beta J + (1 - \beta)\delta(U_n - U).$$

[1] Notice that we have also replaced $(1 - \beta)(W - U)$ with βJ, exploiting the bargaining solution.

Now we can easily get rid of J using (6). How about getting rid of $U_n - U$? We can take the difference between equations (1) and (2), and after some algebra we find

$$U_n - U = \frac{1}{r+\delta+\theta q(\theta)}\left[\frac{\theta q(\theta)}{r+\lambda}(w_n - w) - z\right].$$

Putting all these facts together, we can eventually find the first of the two wage curves, which is given by

$$w = \beta p + (1-\beta)z + \theta q(\theta)\beta\frac{p-w}{r+\lambda} + \frac{(1-\beta)\delta}{r+\delta+\theta q(\theta)}\left[\frac{\theta q(\theta)}{r+\lambda}(w_n - w) - z\right].$$

Now, consider the second type of bargaining, i.e., the one with a worker who is not eligible for benefits. Nash bargaining dictates that

$$(1-\beta)(W_n - U_n) = \beta J_n.$$

Following similar steps as above, we can once again get rid of the various value functions and write the second wage curve as:

$$w_n = \beta p + \theta q(\theta)\beta\frac{p-w_n}{r+\lambda} + \frac{(1-\beta)\lambda}{r+\delta+\theta q(\theta)}\left[\frac{\theta q(\theta)}{r+\lambda}(w_n - w) - z\right].$$

g) Now, let's combine the two WC that we derived in part (f), and in particular, let's focus on the expression $w - w_n$. After a little algebra, we can write:

$$(w - w_n)\left\{1 + \frac{\theta q(\theta)}{r+\lambda}\left[\beta + \frac{(1-\beta)(\delta-\lambda)}{r+\delta+\theta q(\theta)}\right]\right\} = \frac{r+\lambda+\theta q(\theta)}{r+\delta+\theta q(\theta)}(1-\beta)z.$$

Clearly, the expression on the right-hand side is positive and the same is true about the multiplier of $w - w_n$ on the left-hand side.[2] We therefore conclude that $w - w_n > 0$ or $w > w_n$. Of course, this is precisely what *intuition suggests*: the wage paid to the worker whose benefits have expired should be lower because that worker is more desperate to start working, having a lower outside option.

h) As I explained, there is no need for details here. I needed a "strategy" along the following lines:

Step 1: Take the two wage curve equations together with the JC, and this is a system of three equations that should give us the three equilibrium variables w, w_n, θ.

Step 2: Having derived θ, we can now use the conditions BC and BC_n from part (b) in order to describe the last two equilibrium variables, namely, u and u_n.

[2] But notice that for this result to become obvious, we needed the maintained assumption $\delta \geq \lambda$. Without it, proving that this multiplier is positive would be very hard.

Question 2

a) Write down the Social Planner's problem, letting $a \in (0,1)$ denote the Pareto weight that the planner assigns on the typical agent of country 1. The planner wishes to maximize the expression

$$\sum_{t=0}^{\infty} \left[a\beta_1^t ln(c_t^1) + (1-a)\beta_2^t ln(c_t^2) \right],$$

subject to $c_t^1 + c_t^2 = e$ for all t, by choosing sequences of consumptions for the typical agent of each country. The Langrangian function for this problem is

$$L = \sum_{t=0}^{\infty} \left[a\beta_1^t ln(c_t^1) + (1-a)\beta_2^t ln(c_t^2) \right] + \sum_{t=0}^{\infty} \mu_t (e - c_t^1 - c_t^2),$$

where μ_t is the Langrangian multiplier for period t (there will be infinitely many such multipliers). The first order conditions withe respect to c_t^1 and c_t^2, respectively, are given by:

$$a\beta_1^t = \mu_t c_t^1,$$
$$(1-a)\beta_2^t = \mu_t c_t^2,$$

and combining these we obtain the following relationship between consumption in the two countries:

$$c_t^1 = \frac{a}{1-a} \left(\frac{\beta_1}{\beta_2} \right)^t c_t^2. \tag{8}$$

But we also know that the consumptions must obey the feasibility constraint: $c_t^1 + c_t^2 = e$, for all t. Combining these facts, we can obtain the following result: For all t, the consumption of the typical agent in the two countries is given by:

$$\begin{aligned} c_t^1(a) &= (1-\gamma_t)\, e, \\ c_t^2(a) &= \gamma_t\, e, \end{aligned}$$

where

$$\gamma_t \equiv \frac{1}{1 + \frac{a}{1-a} \left(\frac{\beta_1}{\beta_2} \right)^t}.$$

It is easy to see that $\gamma_t \in (0,1)$, for all t. But it is also easy to check that γ_t is increasing in t and that, as t grows infinitely large, $\gamma_t \to 1$. That is, the agents who are more patient will eventually become so wealthy that they will consume the whole endowment available in this economy. Notice that we can make all these statements without even knowing the equilibrium yet (we have not solved for the "correct" a).

To fully characterize equilibrium, we need to find which value of a each type of agent can "afford". To that end, define the usual transfer functions for agent i, using as prices the Langragian multipliers μ_t (this is the so-called Negishi method). We have

$$t^i(a) = \sum_{t=0}^{\infty} \mu_t \left[c_t^i(a) - e_i\right],$$

where

$$\mu_t = \frac{(1-a)\beta_2^t + a\beta_1^t}{e}.$$

As is usual in these cases, we have an extra degree of freedom, which means we can work with either type of agent. Hence, let's focus on agents in country 2. After combining the last two equations and a little bit of algebra, we can show that

$$t^2(a) = \frac{1-a}{1-\beta_2} - \frac{a\beta_1}{1-\beta_1^2} - \frac{(1-a)\beta_2}{1-\beta_2^2}.$$

Of course, the value of a associated with equilibrium is the one for which $t^2(a) = 0$, so that the type 2 agents do not need any transfer in order to afford that specific allocation. Solving $t^2 = 0$ with respect to a yields the unique solution:[3]

$$\hat{a} = \frac{1-\beta_1^2}{1-\beta_1^2 + \beta_1(1-\beta_2^2)}.$$

After a little more algebra, we find that

$$\hat{c}_t^2 = \left[1 + \frac{1-\beta_1^2}{\beta_1(1-\beta_2^2)}\left(\frac{\beta_1}{\beta_2}\right)^t\right]^{-1} e.$$

Of course, $\hat{c}_t^1 = e_t - \hat{c}_t^2$.

b) Based on Negishi's Theorem, we know that the equilibrium prices will be given by the Langrangian multiplier, again, evaluated at the "correct" $a = \hat{a}$. It is, therefore, easy to see that

$$\hat{p}_t = \mu_t(\hat{a}) = \frac{(1-\hat{a})\beta_2^t + \hat{a}\beta_1^t}{e}.$$

c) It is very easy to see that in early periods the typical agent 1 consumes more than the typical agent 2. For instance, set $t = 0$. In that period, the consumption for the typical agent in country 2 is

$$\hat{c}_0^2 = \frac{1}{1 + \frac{1-\beta_1^2}{\beta_1(1-\beta_2^2)}} e.$$

[3] If you have reached this point and you are wondering whether this result is correct, there is a nice test that can be give you a strong indication of the fact that the result is correct. Setting $\beta_1 = \beta_2 = \beta$ in this expression, will give you $\hat{a} = 1/(1+beta)$, which is exactly what we saw in class for the symmetric case.

and this is lower than what agents in country 1 consume, because the multiplier of e in the last expression is smaller than 1/2. Hence, for early periods, $\hat{c}_t^1 > \hat{c}_t^2$. But we have also seen that $\hat{c}_t^2$ is increasing in t, and, actually, as $t \to \infty$, $\hat{c}_t^2 \to e$. Hence, there must be some t in between, where $\hat{c}_t^1 = \hat{c}_t^2$. This t solves

$$\frac{1-\beta_1^2}{\beta_1(1-\beta_2^2)}\left(\frac{\beta_1}{\beta_2}\right)^t = 1,$$

which implies that the t we are looking for is

$$t^* = \frac{ln\left(\frac{\beta_1(1-\beta_2^2)}{1-\beta_1^2}\right)}{ln\beta_1 - ln\beta_2}.$$

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PRELIMINARY EXAMINATION FOR THE Ph.D. DEGREE,

Answer Key: 200E Questions

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Question 3 (*20 points*)

Consider the following decentralized real business cycle model. The representative household has preferences over consumption and leisure. Expected lifetime utility is:

$$E_0 \sum_{t=0}^{\infty} \beta^t \left(\frac{(C_t H_t^{-h})^{1-\sigma}}{1-\sigma} - \frac{N_t^{1+\eta}}{1+\eta} \right) \tag{1}$$

where C is consumption and N is hours worked. These preferences include a form of habit formation and the "habit stock", H_t, is simply equal to consumption in $t-1$: $H_t = C_{t-1}$. h governs the importance of habits and $0 \leq h < 1$.

The household maximizes lifetime utility subject to their budget constraint:

$$C_t + I_t = w_t N_t + (1 + r_t^k - \delta) K_t + \Pi_t \tag{2}$$

where w is the real wage, N is hours worked, K is capital, I is investment, r^k is the rental price of capital and Π are profits from firms. Capital evolves as follows:

$$K_{t+1} = (1-\delta) K_t + I_t$$

but there is full depreciation each period and $\delta = 1$.

Competitive firms produce output using capital and labor. The production function is

$$Y_t = Z_t K_t^{\alpha} (X_t N_t)^{1-\alpha}$$

where TFP is stochastic and follows the following Markov process

$$\log Z_t = \rho \log Z_{t-1} + \varepsilon_t$$

$X_t / X_{t-1} = \gamma$ is the deterministic growth rate of labor augmenting technological change but assume $\gamma = 1$ and $X = 1$. You do not need to de-trend the model.

a) Write down the household's problem in recursive form and write down the firm's maximization problem. Derive the household's first order conditions and the firm's optimal hiring rules.

Answer:

The state variables are the habit stock, capital and TFP: C_{t-1}, K_t and Z_t. $\mathbf{S}_t$ is the associated vector of aggregate endogenous state variables. The recursive formulation of the household problem is as follows.

$$V(C_{t-1}, K_t, Z_t, \mathbf{S}_t,) = \max_{C_t, K_{t+1}, N_t} \left\{ \left(\frac{(C_t C_{t-1}^{-h})^{1-\sigma}}{1-\sigma} - \frac{N_t^{1+\eta}}{1+\eta} \right) + \beta E_t V(C_t, K_{t+1}, Z_{t+1}, \mathbf{S}_{t+1}) \right\}$$

$$s.t. \quad C_t + K_{t+1} = w_t N_t + r_t^k K_t + \Pi_t$$

In principle there is a typo in the budget constraint, which was meant to be $C_t + K_{t+1} = w_t N_t + (1 + r_t^k - \delta)K_t + \Pi_t$ *as usual. Note, however, this makes no difference in this question because* $\delta = 1$ *so* $K_{t+1} = I_t$ *and the RHS of the budget constraint is* $w_t N_t + r_t^k K_t + \Pi_t$.

Households FOCs (λ_t is the Lagrange multiplier on the budget constraint).

$$[C_t] \quad C_{t-1}^{-h}(C_t C_{t-1}^{-h})^{-\sigma} + \beta E_t \frac{\partial V(C_t, K_{t+1}, Z_{t+1}, \mathbf{S}_{t+1})}{\partial C_t} = \lambda_t$$

$$\frac{\partial(C_{t-1}, K_t, Z_t, \mathbf{S}_t,)}{\partial C_{t-1}} = -hC_t C_{t-1}^{-h-1}(C_t C_{t-1}^{-h})^{-\sigma} \quad \text{(by Envelope theorem)}$$

Note that, when $\sigma = 1$, combining these we get $\lambda_t = \dfrac{1 - \beta h}{C_t}$ (3)

$$[N_t] \quad N_t^{\eta} = \lambda_t w_t$$

$$[K_{t+1}] \quad \beta E_t \frac{\partial V(C_t, K_{t+1}, Z_{t+1}, \mathbf{S}_{t+1})}{\partial K_{t+1}} = \lambda_t$$

$$\frac{\partial V(C_{t-1}, K_t, Z_t, \mathbf{S}_t)}{\partial K_t} = \lambda_t r_t^k \quad \text{(by Envelope theorem)}$$

Which implies,

$$\beta E_t \left[\lambda_{t+1} r_{t+1}^k \right] = \lambda_t \tag{4}$$

The firm's problem is standard:

$$\max \Pi = \max(Z_t K_t^{\alpha} N_t^{1-\alpha} - w_t N_t - r_t^k K_t)$$

And the FOCs are

$$w_t = (1 - \alpha) Z_t K_t^{\alpha} N_t^{-\alpha}$$
$$r_t^k = \alpha Z_t K_t^{\alpha - 1} N_t^{1-\alpha}$$

b) Carefully define a recursive competitive equilibrium.

Answer:

A recursive competitive equilibrium is a value function $V(C_{t-1}, K_t, Z_t, \mathbf{S}_t)$ decision rules $K_{t+1} = k(C_{t-1}, K_t, Z_t, \mathbf{S}_t)$, $c_t = c(C_{t-1}, K_t, Z_t, \mathbf{S}_t)$, a law of motion for the aggregate capital stock, $\mathbf{K}_{t+1} = g(Z_t, \mathbf{S}_t)$, a law of motion for the stock of habits ($H_{t+1} = C_t$), and prices $\{w(Z_t, \mathbf{S}_t), r^k(Z_t, \mathbf{S}_t)\}$

Such that

- Given the pricing functions and the law of motion for capital, the value function and decision rules solve the households problem (the allocation satisfies all the first order conditions)
- The firms optimality conditions are satisfied
- All markets clear:

$$K_t^d = K_t^s = K_t$$
$$N_t^d = N_t^s = N_t$$
$$Y_t = C_t + I_t$$

and the law of motion for the individual endogenous state variables is consistent with the law of motion for the aggregate endogenous state variables (rational expectations):

$$S(Z_t, \mathbf{S}_t) = s(C_{t-1}, K_t, Z_t, \mathbf{S}_t)$$

c) Assume $\sigma = 1$. Using guess and verify, find the policy functions for investment, consumption, hours worked and output and show that these are independent of C_{t-1} (**Hints:** Guess that consumption and investment are a constant share of output. You will also find it easier to combine various equilibrium conditions from part (a) before applying the guess and verify method). How, and why, do TFP shocks affect output, consumption, investment and hours worked in this model?

Answer

Note that, by combining the FOCs above and assuming $\sigma = 1$, the key equilibrium conditions are:

$$N_t^\eta = \frac{1-\beta h}{C_t}(1-\alpha)\frac{Y_t}{N_t}$$

$$\frac{1-\beta h}{C_t} = \beta E_t\left[\frac{1-\beta h}{C_{t+1}}\alpha\frac{Y_{t+1}}{K_{t+1}}\right]$$

Guess $K_{t+1} = I_t = BY_t$ and $C_t = (1-B)Y_t$. Substitute these into the Euler equation above:

$$\beta E_t\left[\frac{1}{(1-B)Y_{t+1}}\alpha\frac{Y_{t+1}}{K_{t+1}}\right] = \frac{1}{(1-B)Y_t}$$
$$\Rightarrow \quad \alpha\beta E_t\left[\frac{1}{K_{t+1}}\right] = \frac{1}{Y_t}$$
$$\Rightarrow \quad \alpha\beta E_t\left[\frac{1}{BY_t}\right] = \frac{1}{Y_t}$$
$$\Rightarrow \quad B = \alpha\beta$$

Next, let's work on the FOC for labor supply:

$$N_t^{\eta} = \frac{1-\beta h}{(1-\beta)Y_t}(1-\alpha)\frac{Y_t}{N_t}$$

$$N_t^{1+\eta} = \frac{(1-\beta h)(1-\alpha)}{(1-\alpha\beta)}$$

$$N_t = \left(\frac{(1-\beta h)(1-\alpha)}{(1-\alpha\beta)}\right)^{1/(1+\eta)}$$

which shows that hours worked are constant.

Now, we can express all the choice variables as a function of state variables and parameters such that

$$N_t = \left(\frac{(1-\beta h)(1-\alpha)}{(1-\alpha\beta)}\right)^{1/(1+\eta)} \tag{5}$$

$$Y_t = Z_t K_t^{\alpha}\left[\left(\frac{(1-\beta h)(1-\alpha)}{(1-\alpha\beta)}\right)^{1/(1+\eta)}\right]^{1-\alpha} \tag{6}$$

$$C_t = (1-\alpha\beta)Z_t K_t^{\alpha}\left[\left(\frac{(1-\beta h)(1-\alpha)}{(1-\alpha\beta)}\right)^{1/(1+\eta)}\right]^{1-\alpha} \tag{7}$$

$$K_{t+1} = \alpha\beta Z_t K_t^{\alpha}\left[\left(\frac{(1-\beta h)(1-\alpha)}{(1-\alpha\beta)}\right)^{1/(1+\eta)}\right]^{1-\alpha} \tag{8}$$

Let's see what happens in this economy when a positive TFP shock occurs. From (5), we can see that labor supply does not respond to the TFP shock because the wealth effect and the substitution effect cancel out. Output, consumption and investment always increase and this is independent of the value of h (since $0 \leq h < 1$). Consumption increases because the TFP shock makes households richer. Households also choose to save for tomorrow which helps smooth consumption. In this simple model, this is reflected in the fact that consumption and investment are constant shares of output (income). This model can therefore explain the co-movement of output, consumption and investment that we see in the data. h does, however, affect the level of these variables. A higher h means that we don't want consumption to deviate too much from yesterday. In this specification, this affects the level of utility and, by reducing the marginal utility of consumption today, it lowers labor supply. But, note that h does not affect the *percentage* response to a TFP shock. This can be seen by linearizing the policy function for consumption: a 1% TFP shock has the same effect on consumption irrespective of the value of h. This is because, when $\sigma = 1$, preferences are additively separable and lagged consumption does not enter the marginal utility of consumption.

d) Now consider the possibility that $\sigma > 1$ (rather than $\sigma = 1$). By inspecting the relevant household equilibrium conditions from part (a), discuss how this might affect the dynamics of consumption following a TFP shock. You do not need to re-solve the model, just provide the relevant economic intuition based on the equilibrium conditions.

Let's compare the marginal utility of consumption when $\sigma > 1$:

$$\lambda_t = C_{t-1}^{-h}(C_t C_{t-1}^{-h})^{-\sigma} - h\beta E_t \left[C_{t+1} C_t^{-h-1} (C_{t+1} C_t^{-h})^{-\sigma} \right]$$

with the case where $\sigma = 1$

$$\lambda_t = \frac{1 - \beta h}{C_t}$$

When $\sigma = 1$, the marginal utility of consumption is actually very similar to the usual RBC model with log utility. As noted in part (c), a higher h affects the level of utility but lags and leads of consumption do not affect the marginal utility of consumption. By also using

$$\lambda_t = \beta E_t \left[\lambda_{t+1} r_{t+1}^k \right]$$

we can see that the consumption Euler equation will actually be the same as in the standard model without habits. Consumption growth is therefore not affected by habits when $\sigma = 1$. When $\sigma > 1$, this is no longer true and lags and leads of consumption affect the marginal utility of consumption and also enter the Euler equation. Consequently, when $\sigma > 1$ habit persistence is likely to have a much larger effect on the dynamics (persistence) of consumption.

c) Explain how you would solve this model using Value Function Iteration. Briefly mention any additional challenges that are specific to this model and how you might deal with them. For simplicity, assume labor supply is inelastic.

Answer

First, you will need to write the problem as a social planner's problem. This is an advantage of competitive economies that satisfy the welfare theorems (VFI for decentralized economies of this kind is much more complex). We can then generally apply the method we studied in class. But there are two challenges relative to the simple stochastic growth model: the inclusion of endogenous labor supply and the additional state variable H_t. In the question we are told to assume labor supply is inelastic, so we can ignore the complications associated with endogenous labor supply and set $N_t = 1$ (see Adda and Cooper (2003) for more discussion on how to use VFI with endogenous labor supply). The additional state variable will be discussed below.

In the computer we start by discretizing the state space, possibly centering things around the deterministic steady state. We then calibrate the parameters we need to. Some deep parameters can be matched using external information such as micro data. Note that this model has *two* endogenous state variables, K and H. The grid for H is simply all the possibly values of consumption which can be found using the resource constraint $C = ZK^{\alpha} - K'$. Having noticed this, we can still construct the value function as we did before, although the (vectorized) value function will need to have more rows because the state space is now (Z, K, H): for each K, Z pair the value of the problem will depend on yesterday's consumption choice.

Next we set up the value function as a vector of optimal utility values for each realization of the state variables (e.g. each (K, H, Z)).

Make an initial guess of the value function e.g. all zeros.

Plug this into the Bellman equation and find a new value function.

We then check if $|V_1 - V_0| < \varepsilon$. If it's close enough we stop, otherwise we go again.

Once we have the value function we can find the optimal choices given the states today, this calculation gives us the policy functions for the control variables.

Question 4 (*20 points*)

This question considers a variant of the standard New Keynesian model where the government can now purchase a basket of goods G_t, which is completely funded by lump sum taxes. Assume that G_t is not productive and does not provide utility.

The linearized conditions are given below. In percentage deviations from steady state: $\hat{c}_t$ is consumption, $\hat{w}_t$ is the real wage, $\hat{n}_t$ is hours worked, $\hat{y}_t$ is output, $\hat{\phi}_t$ is real marginal cost and $\hat{g}_t$ is government spending. In deviations from steady state: $\hat{i}_t$ is the nominal interest rate, $\hat{\pi}_t$ is inflation. $\tilde{y}_t$ is the output gap (relative to the model with flexible prices): $\tilde{y}_t = \hat{y}_t - \hat{y}_t^n$.

Households

$$E_t\hat{c}_{t+1} - \hat{c}_t = \frac{1}{\sigma}(\hat{i}_t - E_t\hat{\pi}_{t+1}) \tag{9}$$

$$\hat{w}_t = \sigma\hat{c}_t + \psi\hat{n}_t \tag{10}$$

Firms

$$\hat{y}_t = \hat{n}_t \tag{11}$$

$$\hat{w}_t = \hat{\phi}_t \tag{12}$$

$$\hat{\pi}_t = \beta E_t(\hat{\pi}_{t+1}) + \kappa\tilde{y}_t \tag{13}$$

σ and ψ come from household preferences. $1/\sigma$ is the elasticity of intertemporal substitution and ψ is the inverse of the Frisch elasticity. κ is inversely related to the degree of degree of price stickiness.[1] $0 < \beta < 1$.

Resource constraint

$$\hat{y}_t = \gamma_c\hat{c}_t + \gamma_g\hat{g}_t \tag{14}$$

γ_c is the steady state share of consumption in output and γ_g is the steady state share of government spending in output.

Policy:

$$\hat{i}_t = \phi_\pi\hat{\pi}_t \tag{15}$$

where $\phi_\pi > 1$. Government spending, $\hat{g}_t$, follows an AR(1) process

$$\hat{g}_t = \rho\hat{g}_{t-1} + e_t \tag{16}$$

where e_t is i.i.d. and $0 \leq \rho < 1$

[1] $\kappa = (\sigma + \psi)\frac{(1-\theta)(1-\beta\theta)}{\theta}$ where θ is the probability that a firm cannot adjust its price.

a) Show that, with flexible prices, the natural level of output can be written as

$$\hat{y}_t^n = \Gamma \gamma_g \hat{g}_t$$
$$\Gamma \equiv \frac{\sigma}{\sigma + \psi \gamma_c}$$

Briefly explain how and why a reduction in government spending causes a fall in output in this flexible price model. (**Hints:** start by combining equations 10, 11, 12 and 14. Also note that the real wage is constant in the flexible price model given the constant marginal product of labor).

Answer

Since this is a flexible price solution, we want to find the solution for the natural rate of output. Note that since $Y_t = N_t$, the marginal product of labor is always 1. In linearized form, the flexible price allocation therefore implies $\hat{\phi}_t = \hat{w}_t = 0$. Another way to think about this is that the dynamics of real marginal cost are driven by the time-varying markup that is generated by price stickiness. In the flexible price version of this model, there is no variation in the mark-up and $\hat{\phi}_t = 0$.

Using this result and by combining equations in the question we arrive at the following expression:

$$\hat{y}_t = -\gamma_c \frac{\psi}{\sigma} \hat{n} + \gamma_g \hat{g}_t$$

and, since this is a flexible price solution, the response of output is the response of the natural rate of output: $\hat{y}_t^n$. Rearranging yields:

$$\hat{y}_t^n = \frac{\sigma}{\sigma + \gamma_c \psi} \gamma_g \hat{g}_t$$

Consider a permanent decrease in government spending first. The fall in government spending implies a decrease in the lifetime tax burden. This creates a positive wealth effect on households. Since the labor supply curve depends on the marginal utility of consumption, higher consumption implies fewer hours worked (equivalently, leisure and consumption are normal goods, so the wealth effect leads to a rise in both). Consumption increases (government spending crowds in private spending). Since households work fewer hours, output falls. For a temporary decrease this wealth effect is smaller but there are also substitution effects as the fall in government demand pushes down the real interest rate. This also encourages consumption and discourages labor supply. From the solution, note that this effect on output is always less than 1 in this model.

b) The full sticky price model can be simplified to 3 equations (and equation (16)):

$$E_t \tilde{y}_{t+1} - \tilde{y}_t = \frac{\gamma_c}{\sigma}(\hat{i}_t - E_t \hat{\pi}_{t+1} - \frac{\gamma_g \sigma}{\gamma_c}(1 - \Gamma)(1 - \rho)\hat{g}_t) \quad (17)$$

$$\hat{\pi}_t = \beta E_t(\hat{\pi}_{t+1}) + \kappa \tilde{y}_t \tag{18}$$

$$\hat{i}_t = \phi_\pi \hat{\pi}_t \tag{19}$$

Using the method of undetermined coefficients, find the response of the output gap and inflation to an exogenous increase in $\hat{g}_t$ when prices are sticky and monetary policy follows the Taylor Rule above. Guess that the solution for each variable is a linear function of the shock $\hat{g}_t$. Is the fall in output larger or smaller in this sticky price model (than in part (a))? Explain.

Answer

Using the guesses into the Phillips Curve:

$$\Lambda_\pi \hat{g}_t = (\beta \Lambda_\pi \rho + \kappa \Lambda_y)\hat{g}_t$$

The solution must satisfy:

$$\Lambda_\pi = \frac{\kappa}{1-\beta\rho}\Lambda_y$$

Next, using the guesses, the AR(1) process and the policy rule in the dynamics IS curve:

$$\Lambda_y \hat{g}_t = \Lambda_y \rho \hat{g}_t - \frac{\gamma_c}{\sigma}(\phi_\pi \Lambda_\pi \hat{g}_t - \Lambda_\pi \rho \hat{g}_t) - \gamma_g(\Gamma - 1)(1-\rho)\hat{g}_t$$

Simplifying and substituting the result we found for Λ_π, the solution must satisfy:

$$\Lambda_y = \frac{(1-\rho)(1-\Gamma)\gamma_g}{1-\rho+\Psi} > 0$$

where

$$\Psi = \frac{\kappa \gamma_c (\phi - \rho)}{\sigma(1-\beta\rho)}$$

and, from above,

$$\Lambda_\pi = \frac{\kappa}{1-\beta\rho}\Lambda_y > 0$$

Let's inspect the solution. We know that stickier prices imply a flatter Phillips Curve. This means a lower value for κ. Ψ is smaller and Λ_y is larger. Since this is the response of the output gap — i.e. output relative to the model with flexible prices — sticker prices imply a larger effect of government spending on GDP. This is intuitive. Government spending shocks are demand shocks. When prices are flexible all firms adjust prices and output only falls because households want to supply less labor. But when prices are sticky, firms who don't adjust prices cut production following the reduction in demand. Labor demand therefore falls. In addition to the neoclassical effects in part (a), there is also a Keynesian-type demand effect. Inflation also falls with the demand shock (some firms adjust prices), although sticker prices will tend to limit the effect on inflation. Because the output gap is negative, output falls by more in the sticky price model. Part of the fall in output comes from the neoclassical channel and part of it comes from the Keynesian channel.

c) Instead of following the Taylor Rule above, monetary policy is now set optimally. Derive the optimal monetary policy rule under discretion. Assume the steady state is efficient. (**Hint:** As in class, assume that the loss function has quadratic terms for the output gap and inflation, with a relative weight ϑ on the output gap). What is the optimal path for *output* and inflation following the reduction in government spending?

Answer:

Optimal policy under discretion means the policy maker resets policy choices each period. We are told to assume an efficient steady state, so $\tilde{y}$ appears in the loss function. The policymaker therefore solves a static problem:

$$\min_{\tilde{y},\pi} \frac{1}{2} \left(\hat{\pi}_t + \vartheta \tilde{y}_t \right)$$

subject to the Phillips Curve. If we denote the Lagrange multiplier on the constraint as ξ_t, the first order conditions for inflation and the output gap are:

$$\hat{\pi}_t + \xi_t = 0$$

$$\vartheta \tilde{y}_t - \kappa \xi_t = 0$$

Combining these equations leads to a targeting rule that keeps the output gap proportional to inflation:

$$\hat{\pi}_t = -\frac{\vartheta}{\kappa} \tilde{y}_t$$

There are no trade-off shocks in the Phillips Curve so it is possible to close the output gap and the inflation gap at the same time. 0 is a solution to this policy rule. 0 also clearly minimizes the loss function and is consistent with the IS and PC equations. Optimal policy therefore completely offsets any demand effects from the fall in government spending. As a result, the effect on output equals the effect under flexible prices (the output gap is zero and output equals the natural rate of output from part (a)). Output therefore falls by $\frac{\sigma}{\sigma+\gamma_c\psi}\gamma_g$ % for a 1% fall in government spending. It is not optimal for monetary policy to offset the neoclassical effect.

d) Suppose the monetary policymaker wants to implement the optimal policy from part (c) using an interest rate rule for $\hat{i}_t$. The policymaker is considering a rule which sets $\hat{i}_t$ equal to $\frac{\gamma_g\sigma}{\gamma_c}(1-\Gamma)(1-\rho)\hat{g}_t$, the natural real interest rate in this model. Explain why this will not work. Furthermore, suggest a modification to the proposed rule that would successfully generate the outcomes in part (c). (**Hint:** you do not need to derive anything. You should be able to answer from your knowledge of the model)

Optimal policy under discretion in part (c) would *ex post* deliver zero inflation and output equal to the natural rate. This equilibrium is therefore characterized by

$$\hat{i}_t = \hat{r}_t^n$$

That said, using this as the *policy rule* will not work. A policy rule of this form is like an interest rate peg (because the natural rate is exogenous). It describes one equilibrium but does not rule out other possibilities. Without any feedback from inflation, this rule does not deliver a unique stable solution for inflation (or any endogenous variable in this model). Mathematically, this rule does not produce the right number of stable eigenvalues. To rule out multiple equilibria, the policy rule must specify how it handles deviations from the desired equilibrium. A rule including inflation will work:

$$\hat{i}_t = \hat{r}_t^n + \phi\hat{\pi}_t$$

as long as $\phi > 1$ and therefore satisfies the Taylor Principle. Ex-post, in this model, inflation and the output gap are zero.

e) Now consider two modifications to the model: (i) government spending provides utility where the utility function is: $\frac{C_t^{1-\sigma}}{1-\sigma} + \log G_t - \frac{N_t^{1+\psi}}{1+\psi}$ (ii) the increase in government spending is financed with debt, rather than lump sum taxes, in the short-run (the government eventually repays the debt). Discuss how these changes might affect your answers in parts (a) and (b). You do not need to derive anything, just explain the economic intuition.

Answer:

(i) Government spending is additively separable in the utility function. It affects the level of utility but not any of the marginal decisions. As a result, this does not affect any of the first-order conditions given in the question. The results in parts (a) and (b) will therefore be unaffected.

(ii) Ricardian equivalence holds in this model. Ricardian equivalence says that the timing of lump sum taxes is irrelevant. The results in (a) and (b) will be unchanged even if the government buys bonds today following the fall in g_t. The household regards any changes in government bond holdings as deferred taxation and their consumption plan is unaffected by the timing of taxes.

Prelim Retake 2019: Answer Key

Nicolas Caramp

Question 5 (*20 points*)

a) *Assume there are no financial markets available, so that individuals must simply invest on their own. Given that an individual has invested an amount I at time $t = 0$, what will be the optimal levels of consumption, c_1, c_2, if:*

(i) the individual receives a liquidity shock (i.e. is impatient);

(ii) the individual does not receive a liquidity shock (i.e. is patient).

Let $\hat{c}_1$ and $\hat{c}_2$ denote the consumption of an impatient individual in period 1 and of a patient individual in period 2, respectively.

Given I, agents have $1 - I$ under the mattress, RI if the projects is kept until period 2, and LI if the project is liquidated in period 1. Thus

(i) an impatient agent consumes $c_1 = 1 - I + LI$ and $c_2 = 0$

(ii) a patient agents consumes $c_1 = 0$ and $c_2 = 1 - I + RI$

From now on, we denote $\hat{c}_1 = 1 - (1 - L)I$ and $\hat{c}_2 = 1 + (R - 1)I$.

b) *What is the optimal level of investment when individuals have to invest on their own? Denote this level by $\hat{I}$. **Hint:** Show that there exists $\underline{L}, \overline{L} \in (0, 1)$ such that if $L \geq \overline{L}$, the optimal level of investment is equal to 1, and if $L \leq \underline{L}$, the optimal level of investment is zero.*

The problem of an individual in $t = 0$ is

$$\max_{c_1, c_2 \geq 0, I \in [0,1]} \pi \frac{c_1^{1-\sigma}}{1-\sigma} + (1-\pi) \frac{c_2^{1-\sigma}}{1-\sigma}$$

subject to

$$c_1 \leq 1 - (1 - L)I$$
$$c_2 \leq 1 + (R - 1)I$$

Replacing the constraints in the objective function, we get

$$\max_{I \in [0,1]} \pi \frac{[1 - (1 - L)I]^{1-\sigma}}{1-\sigma} + (1-\pi) \frac{[1 + (R - 1)I]^{1-\sigma}}{1-\sigma}$$

Taking FOC and imposing an interior solution

$$-\pi(1-L)[1-(1-L)I]^{-\sigma}+(1-\pi)(R-1)[1+(R-1)I]^{-\sigma}=0$$

Using that $\pi = 1/2$, and after some algebra, we get

$$I=\frac{1}{R-1}\frac{\left(\frac{R-1}{1-L}\right)^{\frac{1}{\sigma}}-1}{\left(\frac{R-1}{1-L}\right)^{\frac{1-\sigma}{\sigma}}+1} \tag{1}$$

Since $R-1>0$ and $1-L\geq 0$, it is immediate that (1) is positive if and only if

$$\left(\frac{R-1}{1-L}\right)^{\frac{1}{\sigma}}-1>0$$

or

$$L>2-R$$

Define $\underline{L}\equiv\max\{2-R,0\}$. On the other hand, (1) is less than 1 if and only if

$$\frac{1}{R-1}\frac{\left(\frac{R-1}{1-L}\right)^{\frac{1}{\sigma}}-1}{\left(\frac{R-1}{1-L}\right)^{\frac{1-\sigma}{\sigma}}+1}<1 \tag{2}$$

or

$$\left(\frac{R-1}{1-L}\right)^{\frac{1}{\sigma}}<\frac{R}{L}$$

We already know that if $L<\underline{L}$, (1) is negative. Moreover, if $L\to 1$, (2) doesn't hold. By continuity, there exists $\overline{L}$ such that, if $L<\overline{L}$ the inequality holds and $I<1$. Otherwise, $I=1$.

c) *Suppose that when types are realized in period* 1, *this information is publicly observable. Suppose there exists a social planner that individual's entrust all of their endowment to at time* 0. *The social planner will pay impatient individuals* c_1^* *in period* 1 *and patient individuals* c_2^* *in period* 2 *(and zero when they don't value consumption). Solving the social planner's problem, what is* c_1^* *and* c_2^**? How much does the social planner invest? That is, what is* I^**?*

The planner solves

$$\max_{c_1,c_2\geq 0,I\in[0,1]}\pi\frac{c_1^{1-\sigma}}{1-\sigma}+(1-\pi)\frac{c_2^{1-\sigma}}{1-\sigma}$$

subject to

$$\pi c_1\leq 1-I$$
$$(1-\pi)c_2\leq RI$$

Replacing the constraints in the objective function, we get

$$\max_{I\in[0,1]} \pi\frac{\left(\frac{1-I}{\pi}\right)^{1-\sigma}}{1-\sigma}+(1-\pi)\frac{\left(\frac{RI}{1-\pi}\right)^{1-\sigma}}{1-\sigma}$$

Taking FOC and imposing an interior solution, we get

$$-\left(\frac{1-I}{\pi}\right)^{-\sigma}+R\left(\frac{RI}{1-\pi}\right)^{-\sigma}=0$$

Using that $\pi = 1/2$ and after some algebra, we get

$$I^* = \frac{R^{\frac{1-\sigma}{\sigma}}}{R^{\frac{1-\sigma}{\sigma}}+1}$$

It is immediate to see that the expression is always between 0 and 1. Consumption of impatient and patient individuals is

$$c_1^* = \frac{1}{\pi}\frac{1}{R^{\frac{1-\sigma}{\sigma}}+1} = \frac{2}{R^{\frac{1-\sigma}{\sigma}}+1}$$

$$c_2^* = \frac{1}{1-\pi}\frac{R^{\frac{1}{\sigma}}}{R^{\frac{1-\sigma}{\sigma}}+1} = \frac{2R^{\frac{1}{\sigma}}}{R^{\frac{1-\sigma}{\sigma}}+1}$$

d) *Assume that $L = 1$. Show that $I^* < \hat{I}$, but that, if $\sigma < 1$, $c_1^* < \hat{c}_1$ and $c_2^* > \hat{c}_2$. In other words, show that the planner invests less than the individuals but it makes them face more risk.*

When $L = 1$, $\hat{I} = 1$. Thus, $I^* < \hat{I}$ if and only if

$$\frac{R^{\frac{1-\sigma}{\sigma}}}{R^{\frac{1-\sigma}{\sigma}}+1} < 1$$

which is clearly true.

Moreover, when $L = 1$, $\hat{c}_1 = 1$. Thus, $c_1^* < \hat{c}_1$ if and only if

$$\frac{2}{R^{\frac{1-\sigma}{\sigma}}+1} < 1$$

which is true if and only if $R^{\frac{1-\sigma}{\sigma}} > 1$ or $\sigma < 1$.

Thus, if $\sigma < 1$, the planner induces more variance in the consumption of the agents than the agents would get if they invested alone.

e) *Assume that $L \in [0,1]$ and $\sigma > 0$. Now suppose an agent's type is private information, and the social planner can only offer a contract contingent on an individual's announcement of her type at time 1. Furthermore, at time 1, she* *meets each agent only once, with the meeting order randomly determined. If individual's report honestly, can the social planner achieve the same allocation as in question c)? Is it optimal for an individual to report honestly when everyone else does?*

If individuals report their type, the planner can achieve the allocation in question c). The reason is straightforward: the allocation satisfies the resource constraints.

An impatient agent will always report to be impatient; she does not value consumption in period 2 so she will never want to wait. On the other hand, the patient agent might have incentives to lie. She might want to get the resources in period 1 and then store what they get in the mattress. However, if everyone else is reporting honestly, she always wants to report honestly. As mentioned before, since c_1^* and c_2^* satisfy the resource constraints, that's what agents get by reporting to be impatient or patient respectively. But since $c_2^* > c_1^*$ and the gross rate of return from keeping in the mattress is 1, she will choose to report honestly.

f) *Suppose that $L = 1$ and $\sigma < 1$. Suppose a fraction $1 - \varepsilon \in (0, 1 - \pi)$ of agents (all of whom are patient) fear a run. In particular, these agents believe that a fraction $\varepsilon > \pi$ are claiming to the planner that they are impatient. Is it optimal for these agents to lie as well? Given your answer to this question, are runs possible when $\sigma < 1$?*

Suppose a fraction $1 - \varepsilon$ of agents believe that a fraction $\varepsilon > \pi$ are running. Then, if they don't run, the planner will have for them in period 2

$$\frac{R(1 - \varepsilon c_1^*)}{1 - \varepsilon}$$

A sufficient condition for the agents to not run is that this amount is greater than $c_2^* = \frac{R(1-\pi c_1^*)}{1-\pi}$, which holds if and only if

$$\frac{R(1 - \varepsilon c_1^*)}{1 - \varepsilon} \geq \frac{R(1 - \pi c_1^*)}{1 - \pi}$$
$$1 - \pi - (1 - \pi)\varepsilon c_1^* \geq 1 - \varepsilon - (1 - \varepsilon)\pi c_1^*$$
$$\varepsilon - \pi \geq (\varepsilon - \pi)\pi c_1^*$$

and since $\varepsilon > \pi$, the inequality always holds when $\sigma < 1$. Therefore, the agents decide not to run even if they believe that the rest of the agents are running. Thus, we conclude that runs are not possible if $\sigma < 1$.

Question 6 *(20 points)*

a) *Find the optimal savings decision of the consumer born at time t, taking as given the prices w_t and r_{t+1}.*

The problem of an agent born at time t is

$$\max_{c_{t,t},c_{t,t+1},k_{t+1}\geq 0} \log(c_{t,t}) + \beta \log(c_{t,t+1})$$

subject to

$$c_{t,t} + k_{t+1} \leq w_t$$

and

$$c_{t,t+1} \leq r_{t+1}k_{t+1}$$

Replacing the constraints in the objective function, we get the following problem

$$\max_{k_{t-1}\geq 0} \log(w_t - k_{t+1}) + \beta \log(r_{t+1}k_{t+1})$$

The FOC is

$$-\frac{1}{w_t - k_{t+1}} + \beta\frac{1}{k_{t+1}} = 0$$

and hence

$$k_{t+1} = \frac{\beta}{1+\beta}w_t$$

b) *Solve the problem of the representative firm and use market clearing in the labor market to derive expressions for w_t and r_t as functions of k_t.*

The problem of the firm is standard, and is characterized by the following conditions:

$$r_t = \alpha k_t^{\alpha-1}l_t^{1-\alpha}$$

and

$$w_t = (1-\alpha)k^{\alpha}l_t^{-\alpha}$$

Since market clearing implies that $l_t = 1$, we have

$$r_t = \alpha k_t^{\alpha-1}$$

and

$$w_t = (1-\alpha)k_t^{\alpha}$$

c) *Obtain a law of motion for equilibrium k_{t+1}.*

Putting together the agent's and the firm's problem, we get

$$k_{t+1} = \frac{\beta}{1+\beta}(1-\alpha)k_t^{\alpha}$$

d) *Find a steady state with constant capital stock* $k_t = k_{SS}$. *Show that if*

$$\frac{\alpha}{1-\alpha}\frac{1+\beta}{\beta} < 1 \tag{3}$$

then $r_{SS} < 1$.

In SS,

$$k_{SS} = \left(\frac{\beta}{1+\beta}(1-\alpha)\right)^{\frac{1}{1-\alpha}}$$

and

$$r_{SS} = \frac{\alpha}{1-\alpha}\frac{1+\beta}{\beta}$$

Thus, if condition (3) holds, $r_{SS} < 1$.

e) *Let* $\overline{k}$ *be the level of capital that maximizes the per-period net output (that is, it maximizes* $y_t - k_t$*). Show that if* $r_{SS} < 1$, *then* $k_{SS} > \overline{k}$, *and hence a planner would be able to make all agents better off by reducing the capital stock in all periods.*

The resource constraint in steady state is

$$c_y + c_o + k_{SS} = k_{SS}^{\alpha} = \left(\frac{\beta}{1+\beta}(1-\alpha)\right)^{\frac{\alpha}{1-\alpha}}$$

where c_y is the consumption of the young and c_o is the consumption of the old.

Define

$$G(\varepsilon) = (k_{SS} - \varepsilon)^{\alpha} - (k_{SS} - \varepsilon)$$

$G(\cdot)$ denotes the total available resources for consumption in a steady state where capital is $k_{SS} - \varepsilon$. Let's take the derivative of G evaluated at $\varepsilon = 0$:

$$G'(0) = -\underbrace{\alpha k_{SS}^{\alpha}}_{=r_{SS}} + 1$$

Therefore, if condition (3) holds, $G'(0) > 0$, meaning that if we reduce the level of capital in steady state by a little, we have more resources available for consumption, so welfare will improve.

f) *Suppose now that the agents are allowed to trade a useless, non-reproducible asset in fixed unit supply, which trades at the price* p_t. *We call this asset a "bubble". Argue that if* $p_t > 0$ *and* $k_{t+1} > 0$ *the agent must be indifferent between holding capital and the bubble asset, and derive the associated arbitrage condition.*

The problem of an agent now is

$$\max_{c_{t,t}, c_{t,t+1}, k_{t+1}, m_t \geq 0} \log(c_{t,t}) + \beta \log(c_{t,t+1})$$

subject to

$$c_{t,t} + k_{t+1} + p_t m_t \leq w_t$$

and

$$c_{t,t+1} \leq r_{t+1} k_{t+1} + p_{t+1} m_t$$

where m_t denotes the holding of the bubble.

The FOC with respect to capital satisfied with equality is

$$\frac{1}{c_{t,t}} = r_{t+1} \beta \frac{1}{c_{t,t+1}}$$

and with respect to the bubble satisfied with equality is

$$\frac{1}{c_{t+1}} = \frac{p_{t+1}}{p_t} \beta \frac{1}{c_{t,t+1}}$$

Hence, if the agent is holding capital and bubble it must mean that the have the same return:

$$\frac{p_{t+1}}{p_t} = r_{t+1}$$

g) *Show that if* (3) *holds, there exists a steady state equilibrium with* $p_t = p_{SS} > 0$.

Let's guess that there exists a steady state equilibrium with positive price of the bubble. We will find conditions for that equilibrium to exist. First, note that such equilibrium will require $r_{SS} = 1$, hence $\alpha k_{SS}^{\alpha-1} = 1$ or

$$k_{SS} = \alpha^{\frac{1}{1-\alpha}}$$

Moreover,

$$w_{SS} = (1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}$$

From the FOCs of the agent, we have

$$\frac{1}{w_{SS} - k_{SS} - p_{SS}} = \beta \frac{1}{k_{SS} + p_{SS}}$$

hence

$$p_{SS} = \frac{\beta}{1+\beta} w_{SS} - k_{SS}$$

We need to check that $p_{SS} > 0$, which happens if

$$\frac{\beta}{1+\beta} w_{SS} > k_{SS} \iff \frac{\alpha}{1-\alpha} \frac{1+\beta}{\beta} < 1$$

which is exactly condition (3). Hence, only when condition (3) holds there exists a steady state equilibrium with positive price of the bubble.