ANSWER KEY

University of California, Davis
Department of Economics
Macroeconomics

Date: August 25, 2016
Time: 5 hours
Reading Time: 20 minutes

PRELIMINARY EXAMINATION FOR THE Ph.D. DEGREE

Directions: Answer all questions. Feel free to impose additional structure on the problems below, but please state your assumptions clearly. Point totals for each question are given in parentheses.

1. (10) Each period, an infinitely-lived agent divides his endowment of 1 unit of time between human capital production (a non-market activity) and work in order to maximize the present discounted value of lifetime earnings. Let h_t denote the human capital stock at time t and let $1 - l_t$ be time spent working. Income each period is given by $h_t (1 - l_t) w$ where w is the rental rate of human capital. New human capital (i.e. investment in human capital) is produced via the production function $(h_t l_t)^\alpha$; $\alpha \in (0, 1)$. Note that human capital depreciates at the rate δ and the real interest rate is given by the constant, r.

   (a) Express the agent's maximization problem as a dynamic programming problem and identify the states and controls.

   (b) Derive and interpret the Euler equations associated with this problem.

   (c) Assume that a steady-state exists so that $h_t = \bar{h}$ and $l_t = \bar{l}$. Solve for these steady-state values.

   (d) What is the impact of the two prices (w, r) on the steady-state values? Explain.

ANSWER: The state is represented by the beginning-of-period stock of human capital, h_t, and the control is the amount of time spent in human capital production, l_t. The dynamic programming problem is:

$$\begin{aligned} V(h_t) &= \max_{l_t} \left\{ h_t (1 - l_t) w + \frac{1}{1+r} V(h_{t+1}) \right\} \\ \textit{with } h_{t+1} &= h_t (1 - \delta) + (h_t l_t)^\alpha \end{aligned}$$

(Note: you could also set this up as a Lagrangian on the law of motion of human capital and then have h_{t+1} as an additional control. I solve this setup below.) The necessary conditions are:

$$l_t : h_t w = \frac{1}{1+r} V'(h_{t+1}) \left(\alpha (h_t l_t)^{\alpha - 1} h_t \right) \tag{1}$$

$$\textit{or } w = \frac{1}{1+r} V'(h_{t+1}) \left(\alpha (h_t l_t)^{\alpha - 1} \right) \tag{2}$$

The LHS represents the MC of producing more human capital (foregone income) while the right hand side represents the MP of labor in producing human capital weighted by the marginal lifetime income valuation of human capital. Applying the envelope theorem yields:

$$V'(h_t) = (1 - l_t) w + \frac{1}{1+r} V'(h_{t+1}) \left[(1 - \delta) + \alpha (h_t l_t)^{\alpha - 1} l_t \right] \tag{3}$$

This implies that the marginal lifetime income value of human capital is equal to the additional income produced at time t and the present value of the marginal lifetime income valuation of additional h_{t+1}next period (which is the sum of undepreciated h_t and the marginal product of h_t.) Using eq. (2) in eq. (3) yields:

$$\begin{aligned} V'(h_t) &= (1 - l_t) w + \frac{1}{1+r} V'(h_{t+1}) (1 - \delta) + w l_t \\ \textit{or } \; V'(h_t) &= w + \frac{1-\delta}{1+r} V'(h_{t+1}) \end{aligned} \tag{4}$$

Note that eq. (4) can be solved forward to yield:

$$V'(h_t) = w\sum_{j=0}^{\infty}\left(\frac{1-\delta}{1+r}\right)^j = w\left(\frac{1+r}{r+\delta}\right)$$

This makes sense: the marginal lifetime income valuation is the present discounted value of the stream of income produced. Alternatively, one could impose the steady-state condition on eq. (4) so that, in steady-state we have:

$$V'(\bar{h}) = w + \frac{1-\delta}{1+r}V'(\bar{h})$$

or

$$V'(\bar{h}) = w\left(\frac{1+r}{r+\delta}\right)$$

which is the same result (it better be!).

Use this result in eq.(2) (and imposing steady-state) to yield:

$$w = \frac{1}{1+r}w\left(\frac{1+r}{r+\delta}\right)\left(\alpha\left(\bar{h}\bar{l}\right)^{\alpha-1}\right)$$

or, upon rearranging:

$$\left(\bar{h}\bar{l}\right)^{1-\alpha} = \frac{\alpha}{r+\delta} \tag{5}$$

Note that we have immediately that the wage has no impact on the steady-state values. This is due to the fact that it is a constant and weights both the MC and MB of human capital production. That is, a change in w affects both proportionally so that it is not a factor. The other condition which solves the model is the steady-state law of motion for human capital:

$$\delta\bar{h} = \left(\bar{h}\bar{l}\right)^{\alpha} \tag{6}$$

Using eqs. (5) and (6) yields

$$\begin{aligned}\bar{h} &= \frac{1}{\delta}\left(\frac{\alpha}{r+\delta}\right)^{\frac{\alpha}{1-\alpha}}\\ \bar{l} &= \delta\left(\frac{\alpha}{r+\delta}\right)\end{aligned}$$

An increase in the real interest rate will result in a smaller stock of human capital because of the fall in the present discounted value of the income stream.

If one used a Lagrangian approach, the Bellman equation becomes:

$$V(h_t) = \max_{(l_t,h_{t+1})}\left\{\begin{array}{c} h_t(1-l_t)w + \frac{1}{1+r}V(h_{t+1}) + \\ \lambda_t\left[h_t(1-\delta) + (h_tl_t)^{\alpha} - h_{t+1}\right]\end{array}\right\}$$

and the necessary conditions are:

$$\begin{aligned}-h_tw + \lambda_t\alpha\left(h_tl_t\right)^{\alpha-1}h_t &= 0\\ \frac{1}{1+r}V'(h_{t+1}) - \lambda_t &= 0\end{aligned}$$

Note that the Lagrange multiplier is the "price" of human capital. Using the envelope theorem we have:

$$V'(h_t) = (1-l_t)w + \lambda_t\left[(1-\delta) + \alpha\left(h_tl_t\right)^{\alpha-1}l_t\right]$$

Updating this expression and replacing the Lagrange multiplier yields, upon simplification

$$\frac{1}{\alpha}(h_t l_t)^{1-\alpha} = \frac{1}{1+r}\left[1 + \frac{1}{\alpha}(h_{t+1}l_{t+1})^{1-\alpha}(1-\delta)\right]$$

Imposing steady-state and simplifying yields:

$$\left(\bar{h}\bar{l}\right)^{1-\alpha} = \frac{\alpha}{r+\delta}$$

which is the same as obtained in eq.(5).

2. (20) Consider a representative agent economy in which preferences are given by:

$$E_0\left(\sum_{t=0}^{\infty}\beta^t\left[\frac{c_t^{1-\gamma}}{1-\gamma} - \theta_t h_t\right]\right), \beta \in (0,1), \gamma > 0$$

where E denotes the expectations operator and θ_t is an *i.i.d.* random variable which affects the disutility of labor supply.

Output in the economy is produced by firms that use labor and an inelastically supplied unit of non-depreciating capital (owned by firms - you can think of this as land). The choice of labor is made to maximize profits each period:

$$\max_{h_t} \pi_t = y_t - w_t h_t$$
$$y_t = h_t^{1-\alpha},\ \alpha \in (0,1)$$

where y_t denotes output and w_t is the wage. The profits are returned to the households.

In addition to labor supply, agents also trade one period bonds (risk-free) that cost p_t at time t and return 1 unit of consumption in period $t+1$. Given this environment, do the following:

(a) Express the household's problem as a dynamic programming problem and derive the associated necessary conditions. Note that households take as given firm profits, the wage and the price of bonds (i.e. it is a standard competitive economy).

(b) Find the competitive equilibrium allocation by solving the social planner's problem for this economy.

(c) Solve for the policy functions which describe equilibrium consumption and labor. Provide an explanation for the implied behavior of these variables.

(d) Determine the solution for the equilibrium price of bonds. Explain how the preference shock, θ_t, affects the price of bonds.

ANSWER: The state is characterized by the beginning of period bonds, b_{t-1} and the taste shock, θ_t. (Firm profits could also be included as a state variable but, in equilibrium, these will be a function of the shock.) The agent's dynamic programming problem is:

$$V(b_{t-1},\theta_t) = \max_{c_t,h_t,b_t}\left\{\begin{array}{c}\left(\frac{c_t^{1-\gamma}}{1-\gamma} - \theta_t h_t\right) + \beta E\left[V(b_t,\theta_{t1})\right] + \\ \lambda_t\left(\pi_t + w_t h_t + b_{t-1} - c_t - p_t b_t\right)\end{array}\right\}$$

The associated necessary conditions, after applying the envelope theorem are:

$$1 = c_t^{-\gamma}\frac{w_t}{\theta_t} \quad (7)$$

$$p_t c_t^{-\gamma} = \beta E\left[c_{t+1}^{-\gamma}\right] \quad (8)$$

Both of these have standard interpretations: the first is the intra-temporal efficiency condition in which the MRS between labor and consumption is equal to the wage. I have written it to emphasize the role that the taste shock plays in the model. Specifically, an increase in the taste shock serves to effectively lower the real wage because of the increased disutility of labor. But, because the shock is a taste shock rather than a true productivity shock, we do not have the usual income and substitution effects; this will be shown below. The second is the Euler equation for one-period bonds that we have seen many times.

To solve the social planner problem, note that the lack of a capital decision makes the problem very simple: maximize utility given the production function each period. In other words, the social planner problem (written as a Lagrangian) is:

$$\max_{c_t, h_t} \frac{c_t^{1-\gamma}}{1-\gamma} - \theta_t h_t + \eta_t \left[h_t^{1-\alpha} - c_t\right]$$

The first-order condition is:

$$1 = c_t^{-\gamma} \frac{(1-\alpha)\, h_t^{-\alpha}}{\theta_t}$$

which is just eq. (7) where the wage is given by the marginal product of labor. Using the resource constraint $h_t^{1-\alpha} = c_t$ we have the equilibrium policy functions for labor and consumption:

$$h_t = \left[\frac{\theta_t}{1-\alpha}\right]^{\frac{1}{\alpha(\gamma-1)-\gamma}} = \left[\frac{\theta_t}{1-\alpha}\right]^{\frac{1}{\gamma(\alpha-1)-\alpha}}$$

$$c_t = \left[\frac{\theta_t}{1-\alpha}\right]^{\frac{1-\alpha}{\alpha(\gamma-1)-\gamma}} = \left[\frac{\theta_t}{1-\alpha}\right]^{\frac{1-\alpha}{\gamma(\alpha-1)-\alpha}}$$

Note that the term in the denominator of the exponent is always negative and increasing in absolute value in γ. Hence, an increase in the disutility of labor causes labor supply (and consumption) to fall. This demonstrates that we do not have the usual income effects in this model.

The equilibrium price of bonds is given by:

$$p_t = \left[\frac{\theta_t}{1-\alpha}\right]^{\frac{\gamma(1-\alpha)}{\gamma(\alpha-1)-\alpha}} \beta E\left\{\left[\frac{\theta_{t+1}}{1-\alpha}\right]^{\frac{-\gamma(1-\alpha)}{\gamma(\alpha-1)-\alpha}}\right\}$$

Given the assumption of *i.i.d.* shock, the expectations term is constant (no change in the forecast of next period's marginal utility of consumption) so that a fall in consumption caused by an increase in the disutility of working results in lower bond prices as agents try to smooth their consumption by selling bonds (i.e increase their borrowing). Given the representative agent assumption, the price of bonds must decrease to result in zero bond trade.

3. (20) Consider a standard optimal growth model in continuous time in which the aggregate production function is given by:

$$Y(t) = F[K(t), N(t)]$$

where $F(\cdot)$ has standard properties and $N(t)$ is growing at the rate $n > 0$.The depreciation rate of capital is given by $\delta > 0$.The single household inelastically supplies labor each period and then chooses consumption and savings in order to maximize:

$$\int_{t=0}^{\infty} e^{-\rho t} U(c(t))\, N(t)\, dt$$

where $c(t) = \frac{C(t)}{N(t)}$ is per-capita consumption and $U(\cdot)$ has the functional form:

$$U(c(t)) = \begin{cases} \frac{c(t)^{1-\theta}}{1-\theta};\theta \neq 1 \\ \ln c(t);\theta = 1 \end{cases}$$

It is assumed that all parameter values are such that a well-behaved equilibrium exists. In addition to output produced via the production function, output arrives exogenously every period at the rate of ϕ units per person. Given this environment, do the following:

(a) Express the social planner problem in intensive (i.e. per-capita) form - show your derivation.

(b) Write down the Hamiltonian for this problem and derive the necessary conditions; include the transversality condition.

(c) Derive the phase diagram for this economy - be sure to explain your derivation.

(d) Define the steady-state. What fraction of the exogenous output, ϕ, is consumed in steady-state? Why?

ANSWER: The law of motion for the level of the capital stock is given

$$\dot{K}(t) = F(K(t), N(t)) + \phi N(t) - C(t) - \delta K(t)$$

Hence the law of motion for the per-capita capital stock is given by taking the time derivative of $k(t) = \frac{K(t)}{N(t)}$ which yields:

$$\dot{k}(t) = \frac{\dot{K}(t)}{N(t)} - \delta k(t) = f(k(t)) + \phi - c(t) - (\delta + n) k(t) \tag{9}$$

where $f(k(t))$ is per-capita output (using the constant returns to scale of $F(\cdot)$). Note that this is identical to that studied in class (with no technological progress) except for the presence of the exogenously received output.

The objective function can be written in intensive form as (where I have normalized $N(0) = 1$ and, for notational convenience, ignore the log case):

$$\int_{t=0}^{\infty} e^{-(\rho-n)t} \frac{c(t)^{1-\theta}}{1-\theta} dt$$

Then the present-value Hamiltonian is:

$$H = e^{-(\rho-n)t} \frac{c(t)^{1-\theta}}{1-\theta} + \lambda(t) \left[f(k(t)) + \phi - c(t) - (\delta + n) k(t) \right]$$

The necessary conditions are:

$$\begin{aligned} c(t) &: \quad e^{-(\rho-n)t} c(t)^{-\theta} = \lambda(t) \\ \lambda(t) &: \quad f(k(t)) + \phi - c(t) - (\delta + n) k(t) = \dot{k}(t) \\ k(t) &: \quad \lambda(t) \left[f'(k(t)) - (\delta + n) \right] = -\dot{\lambda}(t) \end{aligned}$$

with the transversality condition: $\lim_{t\to\infty} e^{-(\rho-n)t} c(t)^{-\theta} k(t) = 0$. All of these are standard (studied in class) and taking the time derivative of the first expression and using it in the last yields the Keynes-Ramsey condition also derived there (for the case of no technological progress):

$$\frac{\dot{c}(t)}{c(t)} = \frac{f'(k(t)) - \delta - \rho}{\theta}$$

The necessary conditions for this economy are given by the two differential equations defined by the Keynes-Ramsey condition and the law of motion of the capital stock along with the transversality condition.

The construction of the phase diagram is identical to that studied in class. As can be seen in the Keynes-Ramsey condition, the exogenous income, ϕ, will not affect the steady-state level of the capital stock. Hence, the only change from the standard model is that the $\dot{k}$ locus is vertically displaced by the quantity ϕ. Hence this implies c^* (steady-state consumption) will increase 1-1 with ϕ.

MACROECONOMICS PRELIM, AUGUST 2016
ANSWER KEY FOR QUESTIONS 4-6

Question 4

a) Consider first System A. Under this taxation system, the typical households problem is:

$$V(k,K) = \max_{c,k'} \left\{ u(c) + \beta V(k',K') \right\}$$
$$s.t.\ \ c + k' = w + rk(1-\tau) + (1-\delta)k, \tag{1}$$
$$K' = H(K), \tag{2}$$
$$w = w(K) = F_2(K,1), \tag{3}$$
$$r = r(K) = F_1(K,1), \tag{4}$$

where k is the individual capital, and K is the aggregate capital. Moreover, (1) is the household's budget constraint, (2) is the aggregate law of motion of capital, (3) and (4) follow directly from market clearing.

Now cosinder System B. Under this taxation system, the typical households problem is:

$$V(k,K) = \max_{c,k'} \left\{ u(c) + \beta V(k',K') \right\}$$
$$s.t.\ \ c = w + [r + (1-\delta)(1+\tau)]k - (1+\tau)k', \tag{5}$$
$$K' = H(K), \tag{6}$$
$$w = w(K) = F_2(K,1), \tag{7}$$
$$r = r(K) = F_1(K,1), \tag{8}$$

which admits a similar interpretation as the problem under System A (although the budget constraint is very different, as I hinted in the question).

b) Consider first System A. Taking the FOC on the agent's problem yields the following Euler equation:

$$u'(c) = \beta[(1-\tau)F_1(K',1) + 1 - \delta]\, u'(c'),$$

which in steady state implies:

$$1 = \beta[(1-\tau)F_1(K,1) + 1 - \delta].$$

This gives us the answer to our question:

$$K_A^*(\tau) \equiv \left\{ K : 1 = \beta[(1-\tau)F_1(K,1) + 1 - \delta] \right\}. \tag{9}$$

c) Now consider System B. Taking the FOC on the agent's problem yields the following Euler equation:

$$u'(c)(1+\tau) = \beta[F_1(K',1) + (1-\delta)(1+\tau)]\, u'(c').$$

which in steady state implies:

$$1+\tau = \beta[F_1(K,1) + (1-\delta)(1+\tau)].$$

Therefore,

$$K_B^*(\tau) \equiv \Big\{K : 1+\tau = \beta[F_1(K,1) + (1-\delta)(1+\tau)]\Big\}. \tag{10}$$

d) With $F(K,N) = K^a N^{1-a}$, we have $F_1(K,1) = aK^{a-1}$. Then, all we need to do is substitute this expression into the definitions (9) and (10) and solve with respect to K. It is easy to show that:

$$K_A^*(\tau) = \left[\frac{a\beta(1-\tau)}{1-\beta(1-\delta)}\right]^{\frac{1}{1-a}}, \tag{11}$$

$$K_B^*(\tau) = \left[\frac{a\beta}{(1+\tau)[1-\beta(1-\delta)]}\right]^{\frac{1}{1-a}}. \tag{12}$$

e) At $\tau = 0$, we have $K_A^* = K_B^* = \left[\frac{a\beta}{1-\beta(1-\delta)}\right]^{\frac{1}{1-a}}$, which is a familiar expression, since this is the steady state capital stock chosen by the social planner (or by a society without distortionary taxes). Both K_A^* and K_B^* are decreasing in τ, but it is easy to prove that $K_A^* < K_B^*$ for any $\tau > 0$. This is because System A is more severe than B. More precisely, the tax on capital income is a tax on all the household's capital. The tax on investment is just a tax on newly created capital. It is also easy to verify that $K_A^*(1) = 0$, while $K_B^*(1) > 0$.

f) The total tax revenue under system B is

$$T_B = \tau i = \tau\delta K_B^*(\tau) = \tau\delta\left[\frac{a\beta}{(1+\tau)[1-\beta(1-\delta)]}\right]^{\frac{1}{1-a}}.$$

This is not surprising, since in the steady state investment is just enough to replace the depreciated capital. Notice that we can re-write

$$T_B = B\,\tau(1+\tau)^{\frac{1}{a-1}},$$

where $B \equiv \delta\left[\frac{a\beta}{1-\beta(1-\delta)}\right]^{\frac{1}{1-a}}$ is just a positive parameter. This makes differentiation of T_B with respect to τ very easy. We have:

$$\frac{\partial T_B}{\partial \tau} = B(1+\tau)^{\frac{1}{a-1}}\left[1 + \frac{\tau}{(a-1)(1+\tau)}\right].$$

The sign of $\partial T_B/\partial\tau$ depends only on the sign of the term inside the square bracket. More specifically, $\partial T_B/\partial\tau$ will be positive if and only if

$$\tau < \frac{1-a}{a}.$$

Notice that if $a \leq 1/2$, then $(1-a)/a \geq 1$. In this case, the inequality above will be satisfied for all a, τ, and, hence, the Laffer curve will be upward sloping for all a, τ.

But if $a > 1/2$, then $(1-a)/a < 1$. This means that for low τ (i.e., lower than $(1-a)/a$) the Laffer curve will be upward sloping, but for high τ (i.e., higher than $(1-a)/a$) it will be downward sloping.

Question 5

Notice that since $m = u^a v^{1-a}$, we have $q(\theta) = \theta^{-a}$ and $\theta q(\theta) = \theta^{1-a}$.

a) The Beveridge Curve is the standard one:

$$u = \frac{\lambda}{\lambda + \theta q(\theta)} = \frac{\lambda}{\lambda + \theta^{1-a}}. \tag{13}$$

b) The value functions for the firm are given by

$$rV = -pc + q(\theta)(J - V), \tag{14}$$
$$rJ = p - w + \lambda(V - J). \tag{15}$$

c) Due to free entry, we have $V = 0$. Given this fact, and combining (14) and (15), we arrive at the standard JC curve:

$$w = p - \frac{pc(r+\lambda)}{q(\theta)} = p - pc(r+\lambda)\theta^a. \tag{16}$$

d) The value functions for the worker are

$$rW = w - T + \lambda(U - W), \tag{17}$$
$$rU = z + \theta q(\theta)(W - U). \tag{18}$$

e) As is standard, the solution to the bargaining problem gives us

$$\beta J = (1-\beta)(W - U).$$

Then, following the usual steps, we can find that the WC is given by:

$$w = \beta p + (1-\beta)(z+T) + \beta pc\theta. \tag{19}$$

f) If the government's budget constraint is satisfied at all times, then the following condition is satisfied:

$$(1-u)T = uz \tag{20}$$

Substituting this into the wage curve yields:

$$w = \beta p + \beta pc\theta + \frac{(1-\beta)z}{1-u}.$$

g) We would like to get rid of u in WC, so that we can make the wage a function only of (θ, w). Using the Beveridge Curve we can replace u with θ, and re-write the WC as:

$$\begin{aligned} w &= \beta p + \beta pc\theta + \frac{(1-\beta)z(\theta q(\theta)+\lambda)}{\theta q(\theta)} \\ &= \beta p(1+c\theta) + (1-\beta)z + (1-\beta)z\lambda\theta^{a-1}. \end{aligned}$$

h) Now, we have our JC and WC curves containing only the variables (θ, w) (i.e., we have a self-containing block of equations), and we can analyze existence and uniqueness of equilibrium as usual. Notice that JC has the usual shape: it is decreasing in the (θ, w) space. What is new here is the shape of the WC. Notice that, through the WC, as $\theta \to 0$, $w \to \infty$. But also as $\theta \to \infty$, $w \to \infty$. Moreover,

$$\frac{\partial^2 w}{\partial \theta^2} = (1-\beta)z\lambda(a-1)(a-2)\theta^{a-3} > 0.$$

Hence, the WC curve here is a U shaped curve in the (θ, w) space. This means that equilibrium will either come in pairs, or will not exist at all (the case in which JC and WC just "touch", thus giving a unique equilibrium is non-generic).

The intuition behind the U shape of the WC is interesting: When θ is high, many firms enter the market and workers can easily switch to other job options. In order to keep the workers, firms need to pay them high wages. This is the force that is also present in the baseline model. Moreover, in this new model with taxes and UI, when θ is low too few firms enter the market, hence, unemployment is high, hence, the UI bill is high. To raise this money the authorities need to tax employed workers a lot, and, in order to compensate these workers for the high taxes, the wage must also be high. This gives WC its U-shape, which, in turn, gives rise to the multiplicity of equilibria.

i) In all the models we saw in class, an increase in z always led to a higher unemployment. This is not necessarily true here. As we saw, there are two equilibria, one with low θ (high u) and one with high θ (low u). What we are certain about is that an increase in z will shift the WC upwards. But what this means for θ, and, hence, for u, is unclear. If the equilibrium was the one associated with the high θ, then an increase in z would decrease θ and increase u, as in any other model. But if the relevant equilibrium was the one associated with the low θ, then an increase in z would increase θ and decrease u, which is a counter-intuitive result.

Question 6

Throughout this question variables with lower case letters stand for per-capita terms, as opposed to capital letters, which stand for aggregate variables. For any variable x_t also define the growth adjusted per-capita term $\tilde{x}_t = \frac{x_t}{(1+g)^t}$.
a) The resource constraint in aggregate terms is

$$(1+n)^t c_t + K_{t+1} = F(K_t, (1+n)^t(1+g)^t) + (1-\delta)K_t$$

For future reference, divide the constraint by the term $(1+n)^t(1+g)^t$, to express it in terms of the growth-adjusted per-capita variables. We obtain

$$\tilde{c}_t + (1+n)(1+g)\tilde{k}_{t+1} = F(\tilde{k}_t, 1) + (1-\delta)\tilde{k}_t \equiv f(\tilde{k}_t),$$

where the function f is defined exactly as in the lecture notes.
b) The Planner wishes to maximize expected utility

$$\sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma}}{1-\sigma},$$

which can be re-written as

$$\sum_{t=0}^{\infty} \tilde{\beta}^t \frac{\tilde{c}_t^{1-\sigma}}{1-\sigma},$$

where we have defined the new growth adjusted discount factor $\tilde{\beta} \equiv \beta(1+g)^{1-\sigma}$. The rest is a very standard problem, with the exception that, instead of working with the variables x_t, now we have to work with the variables $\tilde{x}_t$. The Euler equation for the Planner's problem is:

$$\begin{aligned}(1+n)(1+g)[f(\tilde{k}_t) - (1+n)(1+g)\tilde{k}_{t+1}]^{-\sigma} &= \\ = \tilde{\beta}[f(\tilde{k}_{t+1}) - (1+n)(1+g)\tilde{k}_{t+2}]^{-\sigma} f'(\tilde{k}_{t+1}).\end{aligned}$$

This problem, in terms of the $\tilde{x}$ variables, will have all the nice properties of the standard growth model. For instance, $\tilde{k}_t$ (and $\tilde{c}_t$) will have a unique globally stable steady state, which is given by

$$\tilde{k}^* \equiv \{k : (1+n)(1+g) = \tilde{\beta} f'(k)\}$$

c) Given that the terms $\tilde{k}_t$ (and $\tilde{c}_t$) will be in steady state in the long run, it is also easy to describe what will happen with per-capita consumption and total consumption in the long run. For instance,

$$c_t = (1+g)^t \tilde{c}_t = (1+g)^t \tilde{c}^*,$$

where $\tilde{c}^*$ denotes the steady state value of growth adjusted consumption. Hence, the growth rate of per-capita consumption is given by

$$g_c = \frac{c_{t+1} - c_t}{c_t} = \frac{(1+g)^{t+1}\tilde{c}_{t+1} - (1+g)^t \tilde{c}_t}{(1+g)^t \tilde{c}_t} = \frac{(1+g)^{t+1}\tilde{c}^* - (1+g)^t \tilde{c}^*}{(1+g)^t \tilde{c}^*} = g.$$

We conclude that per-capita consumption in this economy grows at the rate g.

Using a similar argument, one can verify that the aggregate consumption in this economy grows at the rate $g_C = (1+n)(1+g) - 1 \approx n + g$.