Output Hysteresis 
and Optimal Monetary Policy *

Vaishali Garga     Sanjay R. Singh†

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(Online Appendix)

Abstract

We analyze the implications for monetary policy when deficient aggregate demand can cause a permanent loss in potential output, a phenomenon termed as output hysteresis. We incorporate Schumpeterian endogenous growth into a business cycle model with nominal rigidities. In the model, incomplete stabilization of a temporary shortfall in demand reduces the return to innovation, thus reducing R&D and producing a permanent loss in output. Output hysteresis arises under a standard Taylor rule, but not under a strict inflation targeting rule when the nominal interest rate is away from the zero lower bound (ZLB). In a calibrated medium-scale DSGE model, we find that a ZLB episode lasting six quarters permanently reduces output by 2.70% relative to the deterministic trend. At the ZLB, a central bank unable to commit to future policy actions suffers from hysteresis bias: it does not offset past losses in potential output. A new policy rule that targets zero output hysteresis approximates the optimal policy by keeping output at the first-best level. However, it is optimal to deviate from the deterministic trend when the economy is hit by both TFP and demand shocks.

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†Singh (Corresponding Author): sanjay_singh@brown.edu Garga: vaishali_garga@brown.edu. Department of Economics, 64 Waterman Street, Providence, RI 02912.
1 Introduction

"... a portion of the relatively weak productivity growth ... may be the result of the recession itself. ... In particular, investment in research and development has been relatively weak... Federal Reserve actions to strengthen the recovery may not only help bring our economy back to its productive potential, but it may also support the growth of productivity and living standards over the longer run."

Janet L. Yellen, Chair of the Board of Governors of the Federal Reserve System

In the aftermath of the Great Recession, the US economy has experienced its slowest post-recession recovery since World War II. Seven years in, real GDP is still approximately 15% below the pre-recession trend (Figure 1). Similar trajectories have been observed in other OECD countries as well (Figure 2). One of the primary drivers of this output shortfall has been slow productivity growth (Hall 2016, Stock and Watson 2016), the source of which has been a subject of extensive debate. Fernald (2014) and Cette, Fernald and Mojon (2016) show that total factor productivity (TFP) growth started slowing in 2004, three years before the recession started. Thus, they say slowed growth following the recession may not have been due to the recession itself. On the other hand, Decker, Haltiwanger, Jarmin and Miranda (2014) show that the recession accelerated the slowdown in startup entry, which is a significant channel for productivity growth. Similarly, investment in research and development (R&D), considered to be another important contributor to TFP growth, fell considerably during the recent recession (Figure 3).1 These facts underscore Chairwoman Yellen’s concerns as cited above.

The standard theoretical treatment of monetary policy is largely silent on the interaction of monetary policy with the productive potential of the economy.2 In this paper, we construct a model in which there is such an interaction. We embed a model of Schumpeterian Growth along the lines of Aghion and Howitt (1992) and Grossman and Helpman (1991) in a New Keynesian (NK) setting. A contraction in aggregate demand reduces the incentives for firms to invest in R&D, resulting in lower innovation. This leads to an endogenous slowdown in TFP growth, which results in a persistent output gap. Traditional NK models do not incorporate endogenous

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1 Barlevy (2007) shows that Research and Development (R&D) expenditure is pro-cyclical. There is a considerable literature on innovation and patents, starting with Griliches (1957), which argues that innovation expenditures significantly determine the TFP growth rate. See Hall, Mairesse and Mohnen (2010) and Syverson (2011) for detailed surveys. There is evidence that firms born during downturns start small and stay small (Moreira 2015, and Sedláček and Sterk 2016). Diez (2014) points to broad measures consistent with decline in entrepreneurship in the US during the Great Recession.

2 More recently, Yellen (2016) remarked, “Are there circumstances in which changes in aggregate demand can have an appreciable, persistent effect on aggregate supply? Prior to the Great Recession, most economists would probably have answered this question with a qualified “no.” ... This conclusion deserves to be reconsidered in light of the failure of the level of economic activity to return to its pre-recession trend in most advanced economies. This post-crisis experience suggests that changes in aggregate demand may have an appreciable, persistent effect on aggregate supply—that is, on potential output.” (October 14, 2016) There is a recent literature that explores these interactions, including Anzoategui et al. (2016), Bianchi, Kung and Morales (2016) and Benigno and Fornaro (2016). Ours is the first paper to analyze the interaction of optimal monetary policy at the ZLB, aggregate demand and endogenous growth. We discuss this later in related literature.
productivity and thus incorrectly predict a recovery to the pre-recession trend. Our model formalizes the possibility that following a recession, unemployment returns to its natural rate while output remains below the trend, what Barro (2016) refers to as the job-filled non-recovery. Using this framework, we ask whether monetary policy can affect the long-run potential output of the economy, and, if this is so, whether it is optimal for monetary policy to offset the permanent effects of temporary demand shocks on the level of GDP.

We use our framework to study an economy hit with temporary demand shortfalls. In this paper, we focus on liquidity demand shocks and monetary policy shocks. There are six main results. The first result is that a central bank following a standard Taylor (1993) rule, targeting inflation and unemployment, admits permanent output gaps. When demand is low, the Taylor rule does not completely stabilize the economy. As a result, there is an endogenous decline in productivity-enhancing investment which slows down TFP growth. As the shock abates, Taylor rule prescribes setting a path for interest rates that does not offset past shortfalls in TFP growth. Hence, output is permanently below the initial deterministic trend. We define these (deterministic) trend-based output gaps as output hystereses. Our model thus formalizes what Lawrence Summers refers to as Inverse Say’s law: a lack of demand distorts supply over time.

The second result is that output hysteresis is contingent on the monetary policy specification of the central bank. If the central bank strictly targets inflation and the nominal interest rate is away from the zero lower bound (ZLB), there is no output hysteresis. However, such a policy is unable to stabilize aggregate demand perfectly when the ZLB is binding. As a result, a strict inflation targeting or a Taylor rule admit output hysteresis after a ZLB episode. On the other hand, there exist policy rules which, if credibly communicated to the public, could prevent output hysteresis following recessions induced by shortage of aggregate demand, whether or not the ZLB is binding. One such rule is a policy of targeting zero output hysteresis, where the central bank commits to keeping interest rates lower until output is back at the initial trend. This rule signals ex-ante commitment by the central bank to running a high-pressure economy in the future when there is no slack in employment.

The third main result is that an optimizing policy-maker with ability to commit to future policy actions (optimal commitment policy) sets interest rates to offset the permanent output gap. When the economy is away from the ZLB, a central bank pursuing the objective of price-level-stability can achieve the first-best outcome of maintaining output at the pre-recession trend.

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3We will be more precise on what we refer to as demand shocks in Section 3. Broadly, we follow Blinder and Rudd (2012) in that aggregate “demand shocks are events that, on impact, move the price level and real output in the same direction”. Aggregate “supply shocks are events that, on impact, move the price level and real output in opposite directions”. The demand shocks we consider in our main analysis are liquidity demand shock and monetary policy shock, while stationary TFP shocks and wage markup shocks are the supply shocks. In Appendix G, we discuss the results for discount rate shocks.

4Speech titled “Fiscal Policy and Full Employment” at the Center on Budget and Policy Priorities, April 2014.
However, in the event that the central bank is unable to do so, due to constraints on policy such as the ZLB, these shocks can lead to output hysteresis. At the ZLB, the optimal policy response is to credibly commit to keeping future interest rates lower in order to incentivize recovery close to the pre-recession trend. We illustrate the potency of commitment policy at the ZLB using the two-state Markov Chain assumption of Eggertsson and Woodford (2003). In a parameterization of shocks with a 5% drop in aggregate output, a 1% drop in wage inflation, and the ZLB binding for 7 years, the standard Taylor rule leads to GDP being 1% lower relative to the initial deterministic trend. Optimal policy contains this gap to 0.12%. The key mechanism in this superior performance of commitment rules is the ‘forward guidance’ element of credibly committing to lower interest rates. This generates a temporary boom in output and inflation upon exit from the ZLB, which due to pro-cyclicality of research investment leads to an overshooting of TFP growth above its steady state and thus makes up for most of the past growth slowdowns. Furthermore, the anticipation of higher output in the future reduces the severity of the shock by inducing inflationary expectations during the binding ZLB period.

The fourth main result is that, at the ZLB, a policy-maker unable to commit to future policy actions (discretionary policy) faces a new dynamic inconsistency problem that we label as the *hysteresis bias*. This hysteresis bias implies that a central bank following a discretionary policy does not find in its interest to undo permanent output gaps. It complements our first finding that hysteresis is a consequence of a central bank’s policy constraints and not of inept or irrational behavior on part of the central bankers. More importantly, it implies that it is sub-optimal *ex post* for policy to be redesigned in order to offset the existing output hysteresis.

The fifth main result is that there emerges a new implementable rule for the central bank that closely approximates optimal policy for a range of calibrations. It has been recognized that an inertial policy rule is desirable in an economy faced with ZLB constraints. Our framework provides an additional rationale for using an inertial rule, that is to undo the permanent effects on the level of output of past contractions in aggregate demand. A zero output-hysteresis targeting policy rule, where the central bank commits to keeping interest rates at low levels as long as output is below its initial trend, can eliminate all the persistent effects, resulting from the constrained monetary policy, while also replicating the welfare gains achieved under optimal policy. This rule has the relative advantage in ease of communicating the central bank’s policy stance to the public, unlike optimal policy rules studied in the literature. However, we note the caveat that our rule is also subject to the informational problems as Taylor rules in obtaining reliable estimates of real, nominal, and potential GDP in real-time (Orphanides, 2003).

The sixth and final result is that optimal policy admits permanent output gaps in response

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5Reifschneider and Williams (2000), Eggertsson and Woodford (2003), among others.
to supply shocks (we consider stationary TFP shocks and wage markup shocks). This is because supply shocks reduce the amount of resources available for consumption and investment. As a result, it is indeed optimal to cut investment in R&D. Hence, a policy of hysteresis targeting is grossly sub-optimal to deal with supply shocks relative to a standard Taylor rule. Our model makes a case for monetary policy to play an active role in committing to low interest rates to neutralize the supply side effects of adverse demand shocks. However as argued earlier, the entire productivity slowdown may not be attributed to the contraction in demand and thus having a policy rule that targets to recover back to the pre-recession trend may be highly inflationary and undesirable (also for reasons such as financial stability, not modeled in the paper). This final finding suggests the need for identifying the source of business cycle fluctuations in determining optimal stabilization policy, given that temporary shocks may have long run-effects on output.

In order to explore the quantitative dimension of output hysteresis, we integrate the endogenous growth mechanism into a medium-scale DSGE model. The quantitative model includes investment adjustment costs, variable capital utilization, habit formation in consumption, price and wage rigidities with partial indexation to respective lagged values. We show that depending on the parametrization of innovation intensity elasticity, the permanent output gap ranges between 0.35-2.70%, following a period of ZLB binding for 6 quarters with a maximum output drop of 5% and deflation of 1% relative to the target. These estimates are significant given Hall (2016)’s estimated contribution of slower productivity growth to output gap of 4.4 p.p. following the Great Recession. In the paper, we argue that these estimates are somewhat conservative because of perfect-foresight solution concept used to solve the quantitative model, which precludes the possibility of long-lasting ZLB episode of a duration observed in the US recently (see Carlstrom, Fuerst and Paustian 2015, and Del Negro, Giannoni and Patterson 2015).

This paper also makes a methodological contribution to the optimal policy literature in that we derive a linear-quadratic approximation of the utility function of the representative agent under endogenous growth (see Benigno and Woodford 2004). This allows us to decompose objectives of the policy-maker into market distortions/wedges: the policymaker attempts to stabilize the labor wedge and the productivity growth rate at the efficient level. This latter consideration provides an additional rationale for stabilizing short-run fluctuations. We provide conditions under which the relative welfare weight on stabilizing fluctuations in the growth rate exceeds the corresponding weight on stabilizing the labor wedge and argue that these conditions are likely to hold in general. This has the implication that the optimal commitment policy equilibrium at the

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6The firm level R&D data do not make a distinction between knowledge creation and adoption. While it is more probable that observed fluctuations in R&D investment pertain to adoption decisions, our analysis is valid as long as the there is an effect on knowledge creation. Since we do not model technology adoption lags in our quantitative setting, robustness to different values of this elasticity underscores the plausibility of permanent output gaps. We take up this discussion in Section 7.
ZLB is more inflationary than the equivalent equilibrium in the exogenous productivity growth environment.

While we model changes in R&D investment as the mechanism for persistent effects of transitory shocks in the presence of monetary policy constraints, we view our model as an organizing framework consistent with other channels such as decline in business startups (Gourio, Messer and Siemer 2016, Mehrotra and Sergeyev 2015), depletion or stagnation of job-specific human capital (Davis and von Wachter 2011, Huckfeldt 2016) or other mechanisms through which demand contractions can affect the path of potential output.

1.1 Relation to the Existing Literature

Our paper is closely related to the recent work of Anzoategui, Comin, Gertler and Martinez (2016), Guerron-Quintana and Jinnai (2014), Garcia-Macia (2015) who integrate endogenous growth into a business cycle framework following the seminal work by Comin and Gertler (2006) on medium-term business cycle fluctuations. Bianchi, Kung and Morales (2016) analyze the business cycle implications of a Schumpeterian model (Peretto, 1999) with nominal rigidities. Benigno and Fornaro (2016) use a framework similar to our benchmark model to explore the possibility of multiple expectations equilibria - in particular they identify that ‘stagnation traps’ can be caused by expectations of low growth and thus the economy can be persistently away from trend. Queraltó (2015) integrates financial frictions into a Romer (1990) endogenous growth model to explain slow recoveries following financial crises in emerging economies. Our paper complements these analyses by providing a tractable framework within the relatively standard New Keynesian paradigm for understanding the policy implications in these environments. We analytically derive a quadratic approximation of the social welfare function, which helps in isolating the long-run objective of monetary policy and compare alternate policy rules.

In terms of implications for stabilization policy, our paper is related to DeLong and Summers (2012), and Fatás and Summers (2015) who argue that these permanent deviations can be avoided using appropriate policy tools. Further, Reifschneider, Wascher and Wilcox (2015) estimate that weaknesses in aggregate demand in the US had significant adverse effects on the supply side, and can account for 7% drop in the level of potential output. While the former two papers focus on fiscal policy as the appropriate mechanism to counteract the permanent negative effects, our analysis carves out a role for monetary policy, as suggested by Yellen (2016) recently. We provide an implementable policy rule for the central bank and show that it closely approximates the optimal policy.7

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7On fiscal policy, we show in Appendix F that investment tax credits are expansionary and in related work, it has been shown that debt-financed fiscal policy can be self-financing in hysteresis-prone environments (see Eggertsson, Mehrotra, Singh and Summers (2016)). However, our focus in the paper is on monetary policy. There has been
Persistent effects of cyclical shocks have also been studied by Galí (2016) who solves optimal policy in an insider-outsider model of labor markets (Blanchard and Summers, 1986). Erceg and Levin (2014) evaluate monetary policy rules in an environment where workers may exit the labor force to reconcile the lower labor force participation rates in the economy. Yagan (2016) provides evidence of hysteresis effects of Great Recession in labor markets. We complement these analyses by allowing contractions in demand to negatively affect long-term supply via endogenous productivity growth.

To our knowledge, ours is the first paper to analyze the issue of optimality of permanent gaps in output in the presence of severe demand shocks, particularly relevant once the ZLB is binding. Another set of papers related to our research are Annicchiarico and Rossi (2013), and Annicchiarico and Pelloni (2016) who study Ramsey policy under productivity shocks and government spending shocks in endogenous growth environments. This paper complements their analysis by providing a tractable framework to analyze the policy implications for output hysteresis, particularly at the ZLB.

We have already noted that our work is related to a large literature on monetary policy at the ZLB. Contributions include Fuhrer and Madigan (1997), Krugman (1998), Goodfriend (2000), Reifschneider and Williams (2000), Eggertsson and Woodford (2003), Jung, Teranishi and Watanabe (2005), Adam and Billi (2006) and Werning (2011) among many others. In these papers, the economy returns to the initial long-run trend once the shock is over. In our framework, contractionary demand shocks can cause the economy to be permanently below the trend. Moreover we emphasize that the discretionary policy suffers from the hysteresis bias, complementing Eggertsson (2006)’s deflation bias. This long-run consequence of policy constraints is, we argue, a reason for the policy-maker to pursue more aggressive stabilization policy through implementable rules during times of severe demand shortfalls.

Finally, our paper adds to the Hansen/Summers/Gordon secular stagnation literature. While our model does not generate permanent recessions (as in Eggertsson and Mehrotra 2015, and Auclert and Rognlie 2016) due to the representative agent setup, it formalizes one setting where demand-side and supply-side secular stagnation ideas are interrelated - severe contractions in demand can propagate through slowdown in TFP growth. We leave the investigation of this mechanism in heterogeneous agent setting to future work.

The paper is organized as follows: Section 2 introduces the benchmark economy. Section an extensive literature on persistent effects of short-run fluctuations - Stadler (1990), Fatás (2000), and Barlevy (2004) study the effect of fluctuations on the long-run path of the economy. More recently, Ball (2014), Blanchard, Cerutti and Summers (2015), and Martin, Munyan and Wilson (2015) provide empirical evidence that contractions in demand may indeed cause persistent damage to the economy’s long run path. Corra and Saxena (2008) document that permanent output gaps are a common phenomena following banking and financial crises in low and middle income countries. Blackburn and Pelloni (2005) is an early study examining the growth-volatility trade-offs faced by the central banks.
3 discusses the positive implications of our model for monetary policy under demand shocks. Section 4 analyzes the welfare implications of demand shocks. In Section 5, we summarize the optimality of permanent output gaps under supply shocks. Section 6 takes up a calibrated medium scale DSGE model to demonstrate quantitative relevance of the channel. Section 7 concludes and discusses extensions for future work.

2 A New Keynesian Model with Endogenous Growth

We integrate a textbook model of endogenous growth into a New Keynesian (NK) environment. Households set nominal wages in staggered contracts following Calvo (1983). On the production side, we use a discrete time version of the Schumpeterian growth model of Aghion and Howitt (1992), following Aghion and Howitt (2008, Ch. 4). There is a continuum of intermediate goods, each of which is produced by a sector-specific monopolist. Growth results from innovations that raise the productivity in the economy by improving the quality of products. These innovations are undertaken by profit-maximizing entrepreneurs in every sector, who spend final output in research. We assume that the central task of the monetary policy is to mitigate the effects of nominal rigidities, while fiscal policy is responsible for offsetting distortions associated with imperfect competition.

There are six main actors in our model - households, wage unions, firms, entrepreneurs, fiscal authority and the central bank - described below.

2.1 Households & Wage Setting

2.1.1 Households

There is a continuum of monopolistically competitive households (indexed on the unit interval), each of which supplies a differentiated labor service to the production sector. As is standard, we assume perfect risk sharing within the household. Household derives utility from consuming a final consumption good, dis-utility from supplying labor and utility from holding a risk-free bond.

$$E_t \Sigma_{s=0}^{\infty} \beta^s \left[ \log(C_{t+s}) - \frac{\omega}{1 + \nu} \int_0^1 L_{t+s}(j)^{1+\nu} dj + \xi_t \frac{B_{t+1}}{P_t} \right]$$

where $\nu > 0$ is the inverse Frisch elasticity of labor supply, $\omega > 0$ is a parameter that pins down the steady-state level of hours and the discount factor $\beta$ satisfies $0 < \beta < 1$.

We use this particular specification of the utility function augmented with taste for holding risk-free bonds in order to introduce the liquidity demand shock $\xi_t$. Fisher (2015) models this shock as a micro-foundation for the risk-premia shock considered by Smets and Wouters (2007).
The primary reason for our preference for this shock, as will become more clear in Section 3.2, is that $\xi_t$ allows us to maintain divine coincidence (Blanchard and Galí, 2007).\footnote{Anzoategui et al. (2016) also use the same specification for the demand shock because this shock induces a co-movement in investment and consumption. This is also a relevant feature for our setting. This shock also has a standard interpretation of a shock to the money in the utility function if the central bank paid interest on reserves. In Appendix G, we show the results for standard preference shocks to the household’s utility as employed in Eggertsson (2008). Alternately, we could have introduced these shocks through the budget constraint of the household. Amano and Shukayev (2012) show that such shocks are important ingredients for building models with binding ZLB. We prefer introducing them as shocks to the “wealth in the utility” function. Intrinsic desirability for wealth is not unconventional. See for instance Michaillat and Saez (2014) for detailed references. One insightful reference is to Keynes (1919, Chapter II): Keynes observed that among the capitalist class of Europe before the First World War, “the duty of saving became nine-tenths of virtue and the growth of the cake the object of true religion”, where the “cake” refers to the stock of capital in the economy. Keynes adds that while saving is a sign of virtue and thus a source of utility, consumption is viewed as a sin: “There grew round the non-consumption of the cake all those instincts of puritanism which in other ages has withdrawn itself from the world and has neglected the arts of production as well as those of enjoyment”. And while in principle saving could be used for consumption during retirement or consumption by offsprings, Keynes argues that this did not happen: “Saving was for old age or for your children; but this was only in theory—the virtue of the cake was that it was never to be consumed, neither by you nor by your children after you.” Keynes (1931, Chapter V) also comes back to this thesis.} That is, a monetary policy authority following optimal policy rule does not face a trade-off in completely stabilizing fluctuations in output and inflation arising from such shocks. This shock is an example of purely intertemporal shocks considered by Eggertsson (2008).

Labor income $W_t L_t$ is subsidized at a fixed rate $\tau^w$. Households own an equal share of all firms, and thus receive $\Gamma_t$ dividends from profits, and pay taxes $\tau^b$ on their incomes from riskfree bonds. Finally, each household receives a lump-sum government transfer $T_t$. Household’s budget constraint in period $t$ states that consumption expenditure plus asset accumulation must equal disposable income.

$$P_tC_t + B_{t+1} = (1 - \tau^b)B_t(1 + i_t) + (1 + \tau^w)W_tL_t + \Gamma_t + T_t$$

(1)

Utility maximization delivers the first order condition linking the inter-temporal consumption smoothing to the marginal utility of holding the riskfree bond

$$1 = \beta \mathbb{E}_t \left[ \frac{C_{t+1}}{C_t} (1 + i_t) \frac{P_t}{P_{t+1}} (1 - \tau^b) \right] + \xi_tC_t$$

(2)

The stochastic discount factor by which financial markets discount nominal income in period $t + 1$ is given by:

$$Q_{t,t+1} = \beta \frac{C_{t+1}}{C_t} \frac{P_t}{P_{t+1}}$$

The household does not choose hours directly. Rather each type of worker is represented by a wage union who sets wages on a staggered basis. Consequently the household supplies labor at the posted wages as demanded by firms.
2.1.2 Wage Setting

Wage Setting follows the modeling of Erceg, Henderson and Levin (2000). Perfectly competitive labor agencies combine \( j \) type labor services into a homogeneous labor composite \( L_t \) according to a Dixit-Stiglitz aggregation:

\[
L_t = \left[ \int_0^1 L_t(j)^{\frac{1}{\lambda_{w,t}}} \, dj \right]^{1+\lambda_{w,t}}
\]

where \( \lambda_{w,t} > 0 \) is the (time-varying) nominal wage markup. Labor unions representing workers of type \( j \) set wages (with indexation) on a staggered basis following Calvo (1983), taking given the demand for their specific labor input:

\[
L_t(j) = \left( \frac{W_t(j)}{W_t} \right)^{-\frac{1+\lambda_{w,t}}{\lambda_{w,t}}} L_t, \quad \text{where } W_t = \left[ \int_0^1 W_t(j)^{-\frac{1}{\lambda_{w,t}}} \, dj \right]^{-\lambda_{w,t}}
\]

In particular, with probability \( 1 - \theta_{w,t} \), the type-\( j \) union is allowed to re-optimize its wage contract and it chooses \( W_t^* \) to minimize the dis-utility of working for laborer of type \( j \), taking into account the probability that it will not get to reset wage in the future. If a union is not allowed to optimize its wage rate, it indexes wage to steady state wage inflation \( \bar{\Pi}_w \). In the medium-scale DSGE model, discussed in Section 7, we allow for partial wage indexation. Workers supply whatever labor is demanded at the posted wage. The first order condition for this problem is given by:

\[
E_t \sum_{s=t}^{\infty} (\beta \theta_{w,s})^{s-t} C_s^{-1} \left[ \frac{W_t^*(j)(\bar{\Pi}_w)^{s-t}}{P_s} - (1 + \lambda_{w,t}) \omega L_s(j) C_s \right] L_s(j) = 0 \tag{3}
\]

By the law of large numbers, the probability of the nominal wage resetting corresponds to the fraction of types who actually change their wage. Consequently, the nominal wage evolves according to:

\[
W_t^{\frac{1}{\lambda_{w,t}}} = (1 - \theta_{w}) W_t^{\frac{1}{\lambda_{w,t}}} + \theta_{w}(W_{t-1}^{\bar{\Pi}_w})^{\frac{1}{\lambda_{w,t}}} \tag{4}
\]

2.2 Firms

2.2.1 Final Good producer

Households consume the final good, which is produced by perfectly competitive firms who use identical production technology employing a homogeneous labor composite supplied by the
wage union and a CES composite of intermediate goods weighted by their productivity:

\[ Y_t^G = M_t^{1-\alpha}L_t^{1-\alpha} \int_0^1 A_t^{1-\alpha}x_{it}^\alpha di, \] (5)

where each \( x_{it} \) is the flow of intermediate product \( i \) used at time \( t \), the productivity parameter, \( A_{it} \) reflects the quality of that product and \( M_t \) is the stationary (aggregate) productivity shock.

The firms choose \( L_t \) and \( \{x_{it}\}_{i \in [0,1]} \) to maximize profits, taking as given both the wage index \( W_t \) and the prices of the intermediate goods \( \{p_{it}\}_{i \in [0,1]} \). The first-order conditions for profit maximization give inverse demands for labor composite and intermediate good \( i \).

\[ \frac{W_t}{P_t} = (1 - \alpha)M_t^{1-\alpha}L_t^{1-\alpha} \int_0^1 A_t^{1-\alpha}x_{it}^\alpha di \]
\[ \frac{p_{it}}{P_t} = \alpha M_t^{1-\alpha}L_t^{1-\alpha}A_t^{1-\alpha}x_{it}^{\alpha-1} \] (6)

### Intermediate goods producer

There is a continuum of intermediate products indexed by \( i \in [0,1] \), each of which is produced by a monopolist. The monopolist uses one unit of final good to produce one unit of his own good. As a result, every monopolist faces a marginal cost of \( P_t \). Each intermediate monopolist chooses prices flexibly in every period to maximize his profits, taking as given the final sector’s demand for its product. In particular, he solves

\[ \max_{p_{it}} (1 - \tau^p) p_{it}x_{it} - P_t x_{it} \quad \text{s.t. inverse demand in eq 6} \] (7)

where \( \tau^p \) is a sales tax/subsidy imposed on the monopoly price. Further, we assume that there is a competitive fringe in every sector who produce the intermediate good with identical quality \( A_{it} \) but face a higher marginal cost of \( \chi P_t \), where \( \chi \in (1, \frac{1}{\alpha}) \). As a result, the intermediate monopolist cannot charge a price higher than \( p_{it} = \chi P_t \). In equilibrium, the monopolist charges a price given by:

\[ p_{it} = \zeta P_t \equiv \min \left( \chi, \frac{1}{(1 - \tau^p)\alpha} \right) P_t \]

Because of the assumption of linearity in the use of rival goods in the final goods’ production function, the solution to the monopolist’s problem yields profits which are linear in labor employed in the final good production and own productivity. Higher own productivity enables the

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\(^{10}\)We denote gross output by \( Y_t^G \), to keep it distinct from \( Y_t \) (defined shortly after), which we refer to as the GDP analog of our model.
firm to capture a larger share of the demand for the final good. Profits are given by

\[ \Gamma_t(A_{it}) = \chi^m P_t M_t L_t A_{it} \]

where \( \chi^m = (\zeta - 1) \left( \frac{\alpha}{\zeta} \right)^{\frac{1}{\alpha-1}} \)  

(8)

2.2.3 Entrepreneurs

There is a single entrepreneur in each sector who invests \( RD(z_{it})A_{it} \) of final good in research and development in period \( t \), where \( RD' > 0, RD'' > 0 \). The dependence on productivity \( A_{it} \) is assumed for stationarity. With probability \( z_{it} \) she is successful, the productivity in that sector goes up by a factor of \( \gamma > 1 \) (step size of innovation) and she gets the monopoly rights (patent) over production of the intermediate good in the following period. If she fails, then the incumbent monopolist continues to produce with productivity \( A_{it} \) until replaced by a successful innovation. Hence,

\[ A_{it+1} = \begin{cases} \gamma A_{it} \text{ with probability } z_{it} \\ A_{it} \text{ otherwise} \end{cases} \]

(9)

The cost of research is increasing in the innovation intensity chosen by the entrepreneur and the existing level of technology in the intermediate sector in which the entrepreneur operates. Specifically we assume that \( RD(z_{it}) = \delta z_{it}^q \), where \( \delta > 0 \) and and \( q > 1 \) is the inverse elasticity of innovation intensity to R&D expenses. \( \tau' \) is a research subsidy provided by the government to the entrepreneur. The entrepreneur in every sector chooses \( z_{it} \) to maximize her expected discounted profits (from the patent):

\[ \max_{z_{it} \in [0,1]} \{ z_{it} \mathbb{E}_t Q_{t,t+1} V_{t+1}(\gamma A_{it}) - (1 - \tau') P_t RD(z_{it}) A_{it} \} \]

(10)

where value of the patent is given by:

\[ V_t = \Gamma_t + (1 - z_{it}) \mathbb{E}_t Q_{t,t+1} V_{t+1} \]

The value function is linear in productivity due to the linearity in the production function (see Appendix A). Writing the normalized value function as \( \tilde{V}_t \equiv \frac{V_t}{P_t A_{it}} \) and focusing on the symmetric

---

\[ ^{11} \text{We follow Aghion, Akcigit and Howitt (2014) in this discrete time analog of their classic Schumpeterian model, but extend it to allow for a general specification for decreasing returns to R&D. Benigno and Fornaro (2016) also use a similar model but with } RD'' = 0. \text{ Assuming } RD'' > 0 \text{ introduces decreasing returns to innovation, which is a feature stressed regularly in the innovation literature. See Acemoglu and Akcigit (2012), Kortum (1993) and Hall, Griliches and Hausman (1986) for evidence and further references.} \]
equilibrium, we solve for interior solution (where \( z_t > 0 \)):

\[
\varrho \gamma_i^{c-1} = \beta \mathbb{E}_t \left( \frac{C_{t+1}}{C_t} \right)^{-1} \frac{\gamma \tilde{V}_{t+1}}{(1 - \tau_i) \delta}
\]  

(11)

According to equation (11), the entrepreneur chooses innovation intensity so that the discounted marginal revenue of an additional unit of innovation intensity is equal to the marginal cost of this unit. Increase in demand for final good increases the value of obtaining the patent. This is because of the market size effect - for a given cross-sectional distribution of productivities, increase in demand for final good requires higher quantities of intermediate goods to fulfill that demand. Since a monopolist’s profits are increasing in the quality of its product, she can capture higher share of the increased market with a successful innovation.

2.3 Aggregation & Market clearing

The aggregate behavior of the economy depends on the aggregate (which also corresponds to the average in this case) productivity index defined as:

\[
A_t = \int_0^1 A_{it} \text{di}
\]  

(12)

Because of the linearity assumption in the production function, we can aggregate the firm-level variables to form aggregate composites. Specifically \( RD_t = \int RD_{it} \text{di} \) is the total R&D expenditure and \( X_t = \int X_{it} \text{di} \) is the aggregate intermediate good produced in the economy. We can rewrite the aggregate output and nominal wage purely in the form of aggregates:

\[
Y_t^G = \left( \frac{\alpha}{\tau} \right)^{\frac{1}{1-\alpha}} M_t L_t A_t
\]  

(13)

\[
W_t = (1 - \alpha) \left( \frac{\alpha}{\tau} \right)^{\frac{1}{1-\alpha}} M_t A_t P_t
\]  

(14)

The growth rate of output in the economy is equal to the growth rate of aggregate productivity:

\[
g_{t+1} = \frac{A_{t+1} - A_t}{A_t}
\]  

(15)

In any period, innovations occur in \( z_t \) sectors and \( 1 - z_t \) sectors use previous period’s production technology. Aggregating across all the sectors, we get the following equation governing the dynamics of aggregate productivity:

\[
A_{t+1} = \int_0^1 [z_t \gamma A_{it} + (1 - z_t) A_{it}] \text{di} = A_t + z_t (\gamma - 1) A_t
\]  

(16)
This means that the growth rate of the economy in period $t + 1$ is pre-determined in period $t$ and equal to the number of innovating sectors times the step-size of innovation:

$$g_{t+1} = z_t (\gamma - 1)$$

(17)

The final output produced in the economy is used for consumption, research and production of intermediate goods:

$$Y_t^G = C_t + RD_t + X_t$$

(18)

Henceforth, we define $Y_t^G - X_t = (1 - \frac{\alpha}{\zeta})Y_t^G \equiv Y_t$ as GDP. Lastly, we assume that the risk-free bonds are in zero net supply. Thus the market clearing for the risk-free bond yields:

$$B_t = 0$$

2.4 Fiscal & Monetary policy

The government’s budget is balanced every period, so total lump-sum transfers are equal to intermediate-good, wage and research subsidies.

$$P_t T_t = \tau p \int_0^1 p_{it} x_{it} di + \tau^r P_t RD_t + \tau^w \int_0^1 W_t(h) L_t(h) dh$$

(19)

An independent central bank follows a Taylor rule in setting the nominal interest rate in the economy:

$$1 + i_t = \max \left( 1, (1 + i_{ss}) \left( \frac{\Pi_{W,t}}{\Pi_{W}} \right) ^{\phi_{\pi}} \left( \frac{L_t}{L} \right) ^{\phi_y} \varepsilon^i_t \right) ; \quad \phi_{\pi} > 1, \phi_y \geq 0$$

(20)

The nominal interest rate is set in order to target deviations of wage inflation and employment at respective steady state targets, as long as the implied nominal interest rate is non-negative. $\varepsilon^i_t$ is assumed monetary policy shock.

2.5 Equilibrium

We formally define the competitive equilibrium of the economy in Appendix A. In order to get a stationary system of equations, we normalize the equilibrium equations by dividing the non-stationary variables such as consumption, output, real wage by the level of productivity. We define $c_t = \frac{C_t}{A_t}$ as the normalized (productivity adjusted) consumption and so forth. This allows us to solve for the steady state.\textsuperscript{12} We log-linearize the competitive equilibrium around the steady

\textsuperscript{12}We find the steady state by imposing restrictions on the parameters such that the steady state satisfies a) $z \in (0, 1)$, b) consumption is non-negative and c) nominal interest rates are non-negative. It is possible to find a closed form solution for steady state when $\rho = 1$ (scenario with linear research costs). For values higher than 1, we rely on
state and define the following approximate equilibrium:

**Definition 2.1** (Approximate Equilibrium). The approximate competitive equilibrium in this economy with endogenous growth is defined as a sequence of variables \{\hat{\mathcal{A}}_t^{\nu}, \hat{c}_t, \hat{y}_t, \hat{g}_t, \hat{\lambda}_t, \hat{L}_t, \hat{\omega}_t, \hat{\pi}_t, \hat{\nu}_t\} which satisfy the following equations, for a given sequence of exogenous shocks \{\hat{s}_t, \hat{M}_t, \hat{\epsilon}_t, \hat{\lambda}_t\}.\textsuperscript{13}

Aggregate Demand:

\[
- (\mathbb{E}_t \hat{c}_{t+1} - \hat{c}_t + \hat{g}_{t+1}) + \hat{\tau}_t - \mathbb{E}_t \hat{\mathcal{A}}_{t+1} + \hat{\xi}_t = 0
\]  
(21)

Endogenous Growth equations:

\[
(q - 1) \eta_g \hat{g}_{t+1} = - (\mathbb{E}_t \hat{c}_{t+1} - \hat{c}_t + \hat{g}_{t+1}) + \mathbb{E}_t \hat{\nu}_{t+1}
\]  
(22)

\[
\hat{\nu}_t = \eta_y \hat{y}_t - \eta_z \hat{g}_{t+1} - \eta_q (\mathbb{E}_t \hat{c}_{t+1} - \hat{c}_t + \hat{g}_{t+1}) + \eta_q \mathbb{E}_t \hat{\nu}_{t+1}
\]  
(23)

where \(\eta_g = \frac{1+g}{g} > 1\), \(\eta_y = 1 - \frac{(1-z)\beta}{1+g} > 0\), \(\eta_z = \frac{\beta}{\gamma - 1} > 0\), \(\eta_q = \frac{(1-z)\beta}{1+g} > 0\)

Market clearing:

\[
\frac{c}{y} \hat{c}_t + \mathbb{E}_t \eta_g \hat{g}_{t+1} = \hat{y}_t
\]  
(24)

\[
\hat{y}_t = \hat{M}_t + \hat{L}_t
\]  
(25)

Wage setting:

\[
\hat{\mathcal{A}}_t^{\nu} = \beta \mathbb{E}_t \hat{\mathcal{A}}_{t+1}^{\nu} + \kappa_w \left[ \hat{c}_t + \nu \hat{L}_t - \hat{\omega}_t \right] + \kappa_w \hat{\lambda}_{wt}
\]  
(26)

\[
\hat{\omega}_t = \hat{M}_t
\]  
(27)

\[
\hat{\mathcal{A}}_t^{\nu} = \hat{\omega}_t - \hat{\omega}_{t-1} + \hat{\pi}_t + \hat{\xi}_t
\]  
(28)

where \(\kappa_w \equiv \frac{(1-\theta_w)(1-\beta\theta_w)}{\theta_w(1+\nu(1+1/w))} > 0\)

Monetary policy rule:

\[
\hat{i}_t = \max \left( - \frac{\hat{t}}{1+\gamma}, \phi_i \hat{\mathcal{A}}_t^{\nu} + \phi_y \hat{L}_t + \hat{\xi}_t \right)
\]  
(29)

The Aggregate Demand, Market Clearing and Wage Phillips Curve are relatively standard in the NK setting. The new ingredient is the endogenous growth block (eqns 22-23). It is a log-linear transformation of profit-maximization condition of the entrepreneur.

The endogenous growth condition (eq 22) states that the entrepreneur makes her R&D investment decision based on the expected present discounted value of the future profits. Thus numerical solutions and verify that the model is locally determinate around the steady state.

\textsuperscript{13}For any variable \(x, \hat{x} = \log \left( \frac{x}{\bar{x}} \right)\), where \(\bar{x}\) is the efficient/non-distortionary steady state. With few exceptions: \(\hat{g}_{t+1}\) is the deviation of gross growth rate from the steady state value that is \(\hat{g}_{t+1} \equiv \log \left( \frac{1+g_{t+1}}{1+\bar{g}} \right)\). For liquidity demand shock \(\hat{\xi}_t \equiv cA_t \hat{\xi}_t\) since the steady state value of the shock is 0.
there are two forces governing her decision: the rate at which she discounts the future, and the expected value of future profits. In our model, firms are owned by households. Therefore, the rate at which firms discount the future is given by the stochastic discount factor of the household. A higher stochastic discount factor increases the entrepreneur’s incentive to innovate (discounting effect) because of lower discounting of future profits. Second, higher expected future output increases her incentive to invest in innovation because of the market size effect, discussed above. Furthermore, a percentage change in innovation investment translates into \( \frac{1}{\varrho} \) p.p. change in productivity (gross) growth rate, where \( \frac{1}{\varrho} \) is the elasticity of innovation intensity, and \( \varrho \) is assumed to be greater than 1 following the innovation literature (see Acemoglu and Akcigit 2012). This implies decreasing returns to investment in innovation: a higher value of \( \varrho \) signifies lower responsiveness of innovation success (and productivity growth rate as a result) to innovation investment.

**Calibration and Impulse Responses**

To illustrate the dynamics, we calibrate the model with parameters reported in Table 1.\(^{14}\) Time is quarterly. There are nine parameters: we calibrate five of these using values standard in the New Keynesian literature. The discount factor \( \beta \) equals 0.99. Labor share \( 1 - \alpha \) is set to 0.67. Preferences are logarithmic in consumption and the inverse Frisch elasticity \( \nu \) is set at 2. The wage adjustment probability is set such that wages are reset once every 4 quarters and the steady state wage markup is 10%. In order to calibrate the remaining four innovation parameters - step size of innovation \( \gamma \), the (inverse of the) innovation elasticity \( \varrho \), cost parameter in R&D investment \( \delta \) and the (real) marginal cost for fringe competitors \( \chi \), we target four moments in the US economy - a) growth rate of US real GDP per capita of 2% per annum over 1947-2015, b) creative destruction rate\(^{15} \) of 3.6% (Howitt, 2000), c) R&D to GDP ratio of 2.38% (NIPA 1953-2007), and d) Quarterly Corporate Profits (after Tax) to GDP ratio of 6.2% (BEA, 1947-2007).

Figures 5 and 6 plot the impulse responses for normalized output, wage inflation, real interest rate and productivity growth rate for a positive shock to liquidity demand \( \xi_t \) and a contractionary monetary policy shock \( \epsilon_t \), each following an AR(1) process with persistence 0.90 and 0.92.

\(^{14}\)We verify the local determinacy of the steady state. In the Appendix, we show the necessary condition for determinacy for the model with perfect wage rigidity. In order to solve the model, we use a standard rational expectations solution with an additional constraint that research intensity is non-negative, that is \( z_t \geq 0 \).

\(^{15}\)Howitt (2000) selects 3.6% from Caballero and Jaffe (1993) who find that the average U.S. company that does not innovate loses value at a 3.6-percent annual rate. Alternatively, we used time before an innovation occurs within a sector, as reported by Acemoglu and Akcigit (2012). They use an average of 3 years as the time between two innovations. Other measures of creative destruction used are annual job turnover rate of 26% (Aghion et al., 2016) and Garcia-Macia, Hsieh and Klenow (2016)’s estimated unconditional probability of creative destruction from incumbents and entrants of 32.5% (annual). Qualitative results do not change. Further, the estimate of \( \varrho \) lies in the range of point estimates (1 and 5) reported by Acemoglu and Akcigit (2012) in the innovation and patents literature. We show robustness to extreme values of this parameter at the end of Section 3.
A positive liquidity demand shock corresponds to a fall in annualized natural interest rate of 1%. It increases the desire for saving in the risk-free bond and thus diverts the resources away from consumption. Consequently, lower anticipated aggregate demand results in lower investment in R&D by entrepreneurs, exerting a drag on productivity growth. Further, a positive liquidity demand shock reduces household’s stochastic discount factor, for a given nominal interest rate, which is equivalent to an increase in the “borrowing cost” for investment in innovation for the entrepreneur. Collectively, these two forces reduce investment in innovation. Hence, the productivity growth rate is lower following a contraction in demand induced by the liquidity demand shock.

Similarly, a surprise contractionary monetary policy shock (annualized 68 basis points) implies a 12 basis points increase in the real interest rate and tends to lower the nominal wage. Due to the stickiness of nominal wages, aggregate demand adjusts downwards. The equilibrium increase in the real interest rate combined with expectations of a lower future aggregate demand leads to a reduction in investment in R&D and, therefore, in TFP growth following a monetary policy shock.

### 3 Demand Shocks and Output Hysteresis: Positive Implications

Standard New Keynesian models do not make a distinction between output $Y_t$ and normalized output $y_t$ because productivity is modeled as an exogenous process. Because of endogeneity of productivity, the relevant variables of interest in this endogenous growth environment are the level of output $Y_t$ and the level of consumption $C_t$. While the normalized output and normalized consumption regain the pre-shock values, it is not obvious whether output and consumption recover back to the pre-shock trajectory. Is it the case that the level of output and consumption return to the counterfactual level that would have prevailed in the absence of demand shocks? Do shocks always lead to permanent displacement of these variables from their trajectory?

Here we examine the three key implications of our model: (i) under a Taylor rule (eq 29), negative demand shocks displace the economy to a permanently lower output path, (ii) strict inflation targeting mitigates these permanent effects. But, (iii) even under a policy of strict inflation targeting these permanent gaps can be sizable when the ZLB is binding. However an inertial policy rule can aid the economy in recovery to the pre-shock trend. This section thus formalizes

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16 We choose the persistence and standard deviation of the liquidity demand shock from Anzoategui et al. (2016). For the monetary policy shock, we follow Rudebusch (2002) estimates.

17 Natural interest rate is defined as the interest rate consistent with inflation stabilized at the target rate. We follow Neiss and Nelson (2003), Adolphson et al. (2011) in defining flexible wage economy as the one where wages are assumed to be flexible since the beginning of time $t = 0$. This is referred to as the unconditionally flexible economy in a setting with pre-determined state variables.
the argument that the potential output can contract following adverse demand shocks, but appropriately designed monetary policy rules can avoid such phenomena. We defer the discussion on optimal policy to the following section.

Let superscript \( e \) denote the counterfactual path of productivity and output in the absence of nominal wage rigidities. When we refer to flexible wages, we assume wages have been flexible since the beginning of time \( t = 0 \). A more formal discussion of flexibility is supplied in Section 4. Under the Taylor rule, a permanent gap emerges in the level of productivity and output relative to this counterfactual path. This follows intuitively from the observation that transitory shocks affect investment spending in research, which in turn determines productivity growth. If the policy rule is not a strict targeting (i.e. \( \phi_\pi \not\to \infty \cup \phi_y \not\to \infty \)) rule, presence of a positive real interest rate gap causes output and investment in innovation to decline. Since the policy rule has no inertia, the nominal interest rate recovers to steady state as the shock abates, leaving the economy on a parallel but lower trajectory relative to the counterfactual path.

Using the method of undetermined coefficients, given local determinacy, we can derive the deviations in level of productivity and output from the respective levels under flexible wages as:

\[
\log A_t - \log A_t^e = \sum_{s=0}^{t-1} \psi_s^i \epsilon_s^i, \quad \log Y_t - \log Y_t^e = \hat{y}_t + \sum_{s=0}^{t-1} \psi_s^y \epsilon_s^y
\]

where \( \psi_i^y > 0 \) (detailed expression in the appendix) and \( \epsilon_i^y \) is the liquidity demand shock or the monetary policy shock at time \( t \). We refer to the permanent deviation in output from the flexible wage benchmark as the output hysteresis (or alternately as permanent gap). We summarize the first implication as follows:

**Proposition 1 (Output hysteresis).** Given the monetary policy rule (eq 29) and in the absence of a zero lower bound constraint on the nominal interest rate, transitory (modeled as AR(1) process) liquidity demand shocks or monetary policy shocks induce a permanent gap in the time series of output from the counterfactual (flexible wage-) level of output if and only if monetary policy is not a strict targeting rule i.e.

\[
Y_T \neq Y_T^e \iff \{ \phi_\pi, \phi_y > 0 : \phi_\pi \not\to \infty \cup \phi_y \not\to \infty \}
\]

where \( 1 < T < \infty \) such that \( y_T = y \) (steady state value) and \( y_T \equiv \frac{Y_T}{A_T} \) is the normalized (or stochastically detrended) output.

**Proof.** See Appendix D

Intuitively, as long as there is incomplete stabilization of normalized output i.e. \( \hat{y}_t \neq 0 \ \forall t \), permanent gaps emerge in this economy. This is a consequence of a standard monetary policy specification assumed in eq 29. Normalized output (and the growth rate of productivity) exhibits
a monotonic response to the shocks which approaches zero as the shocks die out. Thus, the sum of the productivity growth rate deviations from the steady state cumulate to the output hysteresis denoted henceforth by \( h_t \equiv \sum_{s=1}^{t} \hat{\gamma}_s \approx \hat{\gamma}_t + h_{t-1} \) as we now clarify with numerical example using parameters from last section.

Panel A in Figures 7 and 9 plots the simulated path of output in response to liquidity demand and monetary policy shocks, respectively. Since entrepreneurs are forward looking, expectations of low future demand depresses investment in innovation. This causes a slowdown in productivity growth, which is not offset by the monetary policy rule. Hence, potential output is permanently lower relative to the flexible wage economy (dashed line). As inflation and employment approach the steady state, output tends to this permanently lower level of potential output. Had the monetary policy followed a strict inflation targeting rule, these permanent effects would not have emerged. Note that under the considered demand shocks, the property of divine coincidence (Blanchard and Galí, 2007) holds. This implies that the central bank faces no trade-off in stabilizing output and inflation. Setting nominal interest rate so as to track the natural interest rate leads to perfect stabilization of the economy, and therefore there are no long-lasting supply effects from the demand shocks. Inability of the central bank to track the natural interest rate perfectly gives rise to permanent supply side deviations following demand shocks. This is the second key implication of our framework.

In our calibration, the implicit output elasticity of growth rate in our calibration is 0.0113 (inverse quarters). This implies that output is permanently lower by 0.113 % following a shock (with persistence 0.9) which causes a drop of 1% (on impact) in normalized output. DeLong and Summers (2012) calibrate this elasticity in the range of 0.05-0.2. We define this parameter to be the hysteresis elasticity - which measures the long-term effect on output of a given percentage-point quarters of slack in the economy. In Section 6, we discuss how our model implied hysteresis elasticity is amplified when various frictions are appended to the model.

In the IRFs shown in Figures 5-10, the demand shocks induce a drop in output (on impact) of about 0.6% and 0.42% respectively, corresponding to common calibrations used in the New Keynesian literature on explaining business cycle fluctuations (see for example Justiniano, Primiceri and Tambalotti 2013). These result in an output hysteresis cumulating to 0.07% and 0.05% respectively as the shock abates. A policy of strict inflation targeting would neutralize these permanent effects.

An alternative to the strict inflation targeting policy is an inertial policy rule. Instead of strict inflation (or output) targeting, the central bank can target past deviations of productivity growth rate from the counterfactual path, which we refer to as zero output hysteresis targeting rule.
Specifically if the central bank followed an augmented Taylor rule of the form:

$$\hat{i}_t = \max \left( -\frac{\hat{r} - \bar{r}_{\Pi w}}{1 + i}, \phi_{\Pi} \hat{r}_{\Pi w} + \phi_y \hat{L}_t + \phi_h h_t + \hat{\epsilon}_t \right),$$  \hspace{1cm} (30)

which incorporates a cumulative sum of past deviations in productivity growth rate $h_t$ as a target, it could avoid the permanent gaps by committing to maintaining a path of interest rates until output is restored to the counterfactual path of output. Under this policy, the central bank credibly signals the willingness to tolerate excess wage inflation, in turn allowing real interest rate to fall so as to close the gap from the natural interest rate. A policy rule is defined as a strict output hysteresis targeting rule if $\phi_h \to \infty$ in equation 30. Such a policy may be especially desirable under circumstances such as the binding ZLB constraint. We turn to this scenario next.

**Binding Zero Lower Bound Constraint**

In our calibrated exercise, we showed that a non-inertial policy of incomplete-stabilization (the Taylor rule in equation 29) causes permanent negative effects on the economy’s potential output. A strict targeting rule, while non-inertial, could mitigate the permanent effects by tracking the natural interest rate. However, at the ZLB such a policy that tracks natural interest rate perfectly may not be implementable. This is because the nominal interest rate consistent with complete stabilization of (normalized) output is negative. Thus monetary policy may be constrained by a lower bound on the nominal interest rate resulting in episodes of output hystereses.

To illustrate, we follow Eggertsson and Woodford (2003) in setting up a two-state Markov Chain for the natural interest rate $\hat{r}_{nt}$ in the endogenous growth economy.\(^\text{18}\) Structurally, a negative shock to the natural interest rate is an increase in the demand for risk-free bonds representing the flight to safety aspects of the financial crisis of 2007-09 (Krishnamurthy and Vissing-Jorgensen, 2012) as shown in Figure 4 by the spread between the AAA-bond and the 20-year Treasury bond.\(^\text{19}\) We assume that the economy hits the ZLB unexpectedly in period 1, that is the nominal interest rate consistent with the stable inflation target breaches a policy lower bound constraint $r_{nt} < i_{LB}$ (assume $i_{LB} = 0$).

\[ A1a \quad \hat{r}_{nt} = \hat{r}_S < 0 \quad \forall 1 \leq t < T^c \]  \hspace{1cm} (31)

With probability $\mu$, it continues to stay in the low state and with complementary probability, the shock returns to the steady state. We assume that the economy is back at the no-deflation steady

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\(^{18}\) Natural or neutral interest rate is defined as the real interest rate consistent with nominal wage inflation at the target rate $\Pi^e$. In the notation of our framework, $\hat{r}_{nt} = -\hat{z}_t + (1 - \beta)$. $\hat{z} > 1 - \beta$ makes the ZLB binding.

\(^{19}\) Krishnamurthy and Vissing-Jorgensen (2012) argue that variation in the AAA-20 year Treasury spread captures the liquidity demand variation. Alternatively, one could plot the “on-the-run” vs “off-the-run” spreads between different vintages of otherwise similar 10 year Treasury bonds.
state after a stochastic but finite time $T^e < \infty$.

$$\hat{r}_t^n = (1 - \beta) > 0 \quad \forall \ t \geq T^e$$ (32)

Further, we assume restrictions on parameters such that the equilibrium is locally determinate around the no-deflation steady state (Assumption A2).\(^{20}\) We calibrate the expected duration of ZLB at 4.6 quarters (14 months approx.) and the natural interest rate is set to -3% (annual). This calibration implies a drop of 5% in (normalized) output and 1% in nominal wage inflation relative to the target. The Central Bank is assumed to follow the Taylor rule (eq 29), targeting inflation and employment deviations from steady state.

**Proposition 2** (Output Hysteresis at the ZLB). *Given the monetary policy rule (eq 29), a positive shock to liquidity demand such that the zero lower bound is binding for finite time $T^e$ results in a permanent gap in output from the flexible wage counterfactual.*

**Proof.** See Appendix D

This result follows from the fact that a) when the ZLB ($t < T^e$) is binding, there is wage deflation and low output along equilibrium path, and b) after time $t \geq T^e$ (when the ZLB is non-binding), monetary authority raises the nominal interest rate to the level consistent with wage inflation and (normalized) output at the respective steady state. While the economic indicators of employment and wage inflation return back to full capacity levels, the productive potential of the economy is permanently lower relative to the counterfactual path, in which the ZLB is not binding. This results from deficiency in demand during the binding ZLB episode. Such losses in potential output can be sizable for reasonable durations of binding ZLB constraint.

In Figure 11, we plot output (solid line) when ZLB is binding for 28 quarters. Output falls on impact by 5% and in the subsequent periods productivity continues to grow at a rate slower than its (annual) steady state growth rate of 2% because investment in R&D is reduced during the recessionary period. The hysteresis elasticity in this calibrated experiment is 0.0128 - which implies that when the interest rates are maintained at the ZLB for a period of 7 years, as in US economy, the economy loses 1.8 percent of potential output relative to the counterfactual path. Hall (2016) provides a decomposition of the trend-based output gap in the US based on deviations of Okun’s law and attributes 4.4 percentage points of the 11.7% gap (as of 2014) to slower productivity growth since the Great Recession. Our simple calibrated model is able to capture a sizable (approx 40%) component of the estimated long-term scarring effect.

The output hysteresis is a key implication of assuming a standard specification for the monetary policy rule, which prescribes raising interest rates as soon as deflationary pressures subside.

\(^{20}\) In the appendix, we show the determinacy condition similar to that derived by Eggertsson (2011).
and employment is back to full capacity. Thus the long-lasting supply effects of demand shocks in our framework suggest a role for policy based on an inertial rule. Reifschneider and Williams (2000), Eggertsson and Woodford (2003) and others have shown that optimal policy at the ZLB involves some form of history dependence. The key new result is that an inertial rule is needed in order to offset negative supply side effects at the ZLB. The dashed line in figure 11 tracks the level of output under the hysteresis-augmented Taylor rule. This is an inertial policy which signals commitment by the central bank to maintain a path of nominal interest rates consistent with reversing past policy constraints/mistakes. A positive liquidity demand shock results in a drop in (normalized) output and wage inflation. However, since the central bank is committed to undoing any permanent gaps in output, it is willing to tolerate excess wage inflation (Figure 11, panel B). This reduces the real interest rate gap, which results in lower growth rate deviations on impact, and allows subsequent growth rate overshooting to undo past constraints on policy. Thus the hysteresis targeting policy embeds a forward guidance mechanism, credibly signaling the intention to tolerate excess inflation.

This channel of forward guidance offers a relatively clear description of the central bank’s intentions to the public. So long as the potential output is lower than the counterfactual trend, the central bank maintains a low interest rate. It initiates gradual tightening of interest rates when actual output has sufficiently overshot the trend so as to make up for the losses in economy’s productive potential. Hence this policy is equivalent to targeting what is referred to in the literature as the unconditional output gap. Adolfson et al. (2011), following Neiss and Nelson (2003), define the unconditional potential output as the level of output that would prevail in the economy had prices and wages been flexible since the beginning of time. Consequently, the difference of output from this unconditional potential output is the unconditional output gap. Previously, the New Keynesian literature has explored implications of unconditional output gap targeting in models with exogenous productivity, where the potential level of output is exogenous. As a result the economy always finds its way back to the (unconditional) potential output. In an endogenous productivity setting, potential output is responsive to fluctuations in demand because productivity growth results from pro-cyclical investment decisions of entrepreneurs. When the ZLB is binding, due to severe deficiency in aggregate demand, it is thus possible to have long-lasting losses in economy’s potential relative to the flexible wage counterpart.

Parameter sensitivity
We close our positive analysis by showing robustness of key results to alternate parameter calibration and model specifications. Specifically, we show that (i) away from ZLB, the output hysteresis may be quantitatively small; however (ii) during episodes of binding ZLB, output
hysteresis remains quantitatively significant.

As discussed earlier, a policy-invariant measure of output hysteresis is the elasticity of productivity growth rate to cyclical slack in the economy (measured by deviations of normalized output from steady state), \( e_{gy} = \frac{d\ln(1+g)}{d\ln y} \). This sufficient statistic which we refer to as the hysteresis elasticity provides information on the percentage shortfall in TFP growth rate following a quarterly employment slack of 1%. Within our framework, this statistic is closely governed by the innovation intensity parameter \( \varrho \), which creates a link between changes in output and TFP growth rate through incentives to undertake innovation investment by entrepreneurs. Since investment spending generates decreasing returns in innovation success, a marginal increase in R&D spending increases the probability of success by more if \( \varrho \) is lower. Thus \( \varrho \) captures the sensitivity of R&D spending to changes in aggregate conditions.

In our baseline calibration we calibrated \( \varrho = 1.32 \), which provided an implicit output elasticity of productivity growth at approximately 0.013 (depending on the persistence of the shock). Griliches (1990) surveys the evidence on this elasticity. Cross-sectional data on patents and R&D expenditure by large firms points to the estimates of this elasticity lying in the range of (1, 1.67), while the time-series evidence favors estimates in the range of 1.6 and 2.5. The subsequent literature has calibrated models using a wide range of parameters. Benigno and Fornaro (2016) calibrate \( \varrho = 1 \), while Acemoglu and Akcigit (2012) use a value of 2.8. As a result, due to lack of a clear guide on this crucial parameter, we report sensitivity to using alternate values. In Table 3, we report the permanent output hysteresis for a grid of values for \( \varrho \). Row 1 reports the output hysteresis for a shock that causes output to drop by 1% on impact and has persistence of 0.9. In row 2, we report the output hysteresis for a ZLB shock binding for 28 quarters with an output drop of 5% and 1% wage deflation. An implication of this sensitivity analysis is that the permanent output gaps following regular business cycle shocks are somewhat small. But at the ZLB, since shocks are relatively severe, the resulting output hysteresis is likely to be significant even with high \( \varrho \). With a value of 2.8, the output hysteresis is 0.38% of GDP. On the other extreme, a value of 1.05 for \( \varrho \) implies permanent output gap of 2.60%. Hence the simple benchmark model can generate sizable losses in output at the ZLB, given the 4.4% estimate (Hall, 2016) of contribution of slow productivity growth to trend-based output gap in the US. In Section 6, we show that this mechanism is quantitatively sizable in a medium-scale DSGE model for a relatively modest shock.

Our framework thus formalizes the hypothesis that the productive potential of the economy is endogenous to the monetary policy rule implemented by the central bank. We find that sizable

\[^{21}\text{We calibrate all parameters again so that growth rate in the steady state is 2\%, creative destruction rate is 3.6\%, R&D to GDP ratio is 2.38\%, and profits-gdp ratio is approximately 6\%. Full list of parameters for the sensitivity analysis is available on request.}\]
permanent output gaps emerge in response to transitory shocks under standard Taylor rules for monetary policy. While appropriately designed monetary policy rules can offset these permanent gaps, the question arises - is it optimal for the policy maker to undo these long-run effects of transitory shocks? To answer this question, we next study the normative implications for the conduct of monetary policy in our endogenous growth framework.

4 Normative Implications for Conduct of Monetary Policy

We derive the normative implications for the conduct of monetary policy when the economy is subject to liquidity demand and monetary policy shocks. We highlight three main results. One, we show that the equilibrium under optimal policy does not involve permanent shifts in output. Away from the ZLB, this policy is equivalent to a strict inflation targeting rule discussed in the previous section. Two, at the ZLB, optimal policy commits to keeping interest rates lower in the future. Such a policy keeps the economy close to the pre-shock trend. Three, we show that a discretionary policy at the ZLB involves excessive output hysteresis relative to commitment policy. We label this as the hysteresis bias of discretionary policy. Numerically, we show that the strict hysteresis targeting policy implies significant welfare gains over a standard monetary policy rule. These welfare gains approximate the welfare gains achieved under optimal policy.

We begin by defining key concepts that are necessary for the normative analysis:

Steady State Efficiency

We define the efficient steady state as the one in which the welfare of the representative household is maximized subject to the production technology of consumption good (eq 13), the law of motion of aggregate productivity (eq 16), and economy’s resource constraint (eq 18) for a given level of initial productivity. The complete system of equations is provided in Appendix C.

We show in Proposition 3 that the steady state of the competitive equilibrium allocation is inefficient because of the presence of three distortions in our setup: (i) monopoly power in each intermediate goods sector, (ii) monopolistic competition in the labor market and (iii) inter-temporal research externalities. Whereas the first two distortions are relatively standard in the NK literature, the third is peculiar to the endogenous growth setup. The entrepreneur is unable to reap all the benefits of her technology advancement because she gets replaced with positive probability by a new entrant (surplus appropriability effect). This makes her under invest in R&D. On the other hand, an entrant replaces the incumbent to profit from the full step size of innovation $\gamma$ rather than the incremental gain in knowledge $\gamma - 1$. This business stealing effect (Aghion and Howitt, 1992) incentivizes the entrepreneur to over-invest in R&D. As a result of these two
opposing forces, private investment in research can be higher or lower than the first-best.

We assume that the fiscal authority has access to lump-sum taxes, and so the first best allocation in the steady state can be implemented by a set of constant taxes elaborated in the following proposition:

**Proposition 3 (Steady State Efficiency).** Assuming the policy maker has access to non-distortionary lump-sum taxes, the steady state of the competitive equilibrium can be made efficient using the following three fiscal tools:

1. **Sales subsidy** \( \tau^p = 1 - \frac{1}{\alpha} \)
2. **Wage tax cut** \( \tau^w = \frac{1 - \lambda_w}{\lambda_w} \), and
3. **Research tax /subsidy** \( \tau^r = 1 - \left[ \left( \frac{\gamma l^*(1 - \alpha) a^{\frac{\alpha}{1 - \alpha}}}{1 - \beta(1 - z^*)} \right) \left( \frac{1 - \beta}{(\gamma - 1)c^*} \right) \right] \), where terms with \(^*\) denote the efficient steady state values.

**Proof.** See Appendix D

It is commonly argued in the endogenous growth literature that the private sector underinvests in R&D (Jones and Williams 1998), and therefore growth rate is higher in the efficient steady state. These distortions would imply that in the absence of relevant fiscal instruments, monetary policy could affect the growth rate of output in the long-run. We follow the monetary economics literature (Rotemberg and Woodford 1998, Woodford 2003) and suppose that the average productivity growth rate is optimal and independent of monetary policy. The idea is to disassociate the welfare losses from fluctuations in growth rate from those arising from suboptimal growth solely due to monopoly distortions and research externalities. We make the following assumption in this section:

**Assumption 1.** The fiscal authority provides the set of constant subsidies described in Proposition 1 such that the competitive equilibrium is efficient in the steady state.

The crucial difference to note from the earlier monetary economics literature is that monetary policy in our setting has a bearing on the long-run level of output even though we do not allow monetary policy to influence the steady state distortions. Only role of monetary policy in our economy is to offset short-run nominal distortions.

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Under the parameter calibration assumed in Section 2, the corresponding efficient steady state is characterized by R&D investment- GDP ratio of 15 times the decentralized economy’s steady state, implying 2.5 times the productivity growth rate. This is comparable to Jones and Williams (1998)’s finding of heavy under-investment in R&D by the decentralized economy. We recalibrate the parameters (Table 4) such that the productivity growth rate is 2 percent (annual) in the efficient steady state. In Section 6 we show the sensitivity of our results to alternate parameter configurations.
Equilibrium Concepts and Policy instruments

We assume that the (normalized) economy is in the efficient steady state at beginning of time $t = 0$.

We now define first-best allocation and natural rate allocation concepts, as we will refer to them shortly. The first-best allocation is the competitive equilibrium allocation under flexible wages such that the fiscal authority utilizes (non-distortionary) time-varying taxes in order to maximize the representative agent’s welfare. The natural-rate allocation (or interchangeably flexible-wage allocation) is the competitive equilibrium allocation under flexible wages such that the fiscal authority provides (non-distortionary) constant tax instruments outlined in Proposition 1.

There are two concepts of flexibility in the presence of a pre-determined state variable. One is the Neiss and Nelson (2003) definition of flexible wages, under which wages have been set flexibly since time 0 and remain flexible indefinitely. Wages set under this concept are called time-0 flexible wages. Second concept of flexibility is the Woodford (2003)’s definition where wages are set flexibly in the current and future periods taking as given the evolution of state variable. Wages set under this concept are called time-t flexible wages. Based on two concepts of flexible wages, there are time-0 first best, time-0 natural rate, time-t first best and time-t natural rate allocations.

To avoid clutter of notation, we will use first best allocation for time-0 first best allocation and natural rate for time-0 natural rate allocation whenever possible without ambiguity. What is usually referred to as the potential output in the literature coincides with the time-t first-best allocation in our setting. In the normative analysis we make the distinction between these concepts when referring to permanent output gaps - which pertain to the difference in the level of output under the time-0 first-best and time-t first best allocations. For the ease of exposition, we refer to time-0 flexible wages as flexible wages.

Sticky wage allocation is the equilibrium allocation under staggered (nominal) wages such that the fiscal authority provides (non-distortionary) constant tax instruments outlined in Proposition 1. We refer the reader to Appendices C.9.1, C.9.2 and C.9.3 for the mathematical definition of these equilibria concepts.

We will focus in this section only on the liquidity demand and monetary policy shocks. This is because under these shocks the natural-rate and the first-best allocations coincide. Specifically, it is the case that these shocks do not affect the flexible-wage equilibrium output, consumption and investment in R&D. The economy stays on the initial balanced growth path (BGP), which implies that any drop in output that occurs in the sticky-wage economy is due to nominal frictions. This helps isolate the role of monetary policy. Such shocks are sometimes referred to as purely inter-
temporal shocks in the literature (Eggertsson, 2008) since these only affect the Aggregate Demand block. The conventional distinction between aggregate demand and aggregate supply shocks is somewhat blurred under endogenous productivity growth setting where shocks to demand can have effects on supply in the long-run.

**Proposition 4.** The (time-0) natural rate allocation coincides with the (time-0) first-best allocation under liquidity demand and monetary policy shocks.

**Proof.** See Appendix D

Proposition 4 implies that the representative agent’s welfare is maximized if the policy maker could replicate the natural rate allocation. This outcome is always possible if the policymaker has access to time-varying tax instruments (see for example Correia, Nicolini and Teles 2008, and Correia, Farhi, Nicolini and Teles 2013). In Appendix C.6, we illustrate how the first-best can be implemented by appropriate state-contingent fiscal instruments even at the ZLB. Henceforth, we assume that the policy maker does not have access to these time varying fiscal instruments: fiscal authority satisfies Assumption 1, and adjusts lump-sum taxes every period to balance the budget. The Central Bank sets the nominal interest rate $i_t$ on the risk-free (nominal) bond $B_t$ subject to the ZLB constraint:

$$i_t \geq 0 \quad \forall t.$$  

(33)

This is the bank’s only policy instrument.

**Quadratic approximation of Welfare** We next derive a quadratic approximation of the lifetime utility of representative agent:

**Proposition 5.** Assume that the economy is at the efficient steady state at time $t = 0$, with given productivity level $A_0$. Under the sticky wage allocation, quadratic approximation of representative agent’s lifetime utility function $W_0$ around the non-stochastic efficient steady state is given by

$$W_0 - W_0^* \over U_{cy_y} = -\frac{1}{2} \sum_{t=0}^{\infty} \beta^t \left[ \lambda_y \left( \hat{y}_t - \beta \frac{1}{1 - \beta} \left( v + \frac{\beta}{c} \hat{y}_{t+1} \right) \right)^2 + \lambda_g \left( \hat{g}_{t+1} \right)^2 + \lambda_\pi \left( \hat{\pi}_t \right)^2 \right] + O(||\hat{\xi}_t, \hat{\epsilon}_t||^3) + t.i.p. \quad (34)$$

where $\lambda_y = (v + \frac{v}{c}) > 0$, $\lambda_g = \frac{v}{(1 - \beta)} \left[ \frac{\beta}{v + \frac{\beta}{c}} \frac{\beta}{1 - \beta} + [(q - 1) \eta_g + 1] \right] > 0$, $\lambda_\pi = \frac{1 + \lambda_w}{\lambda_w} \frac{1}{\kappa_w}$, $\kappa_w > 0$, $\kappa_w \equiv \frac{(1 - \theta_w)(1 - \beta \theta_w)}{\theta_w (1 + \theta_w (1 + \frac{\beta}{c})} > 0$, $\eta_g = \frac{1 + \eta_g}{\sigma} > 1$ and $t.i.p.$ stands for “terms independent of policy”. $W_0^*$ denotes
welfare under the (time-0) first-best allocation. The approximation is scaled by the constant $U_{css} y_{ss} = \frac{y_{ss}}{y_{ss}}$ (evaluated at the efficient steady state).

Proof. See Appendix D

This approximation is composed of three gaps/wedges - (i) labor efficiency gap, (ii) productivity growth rate gap, and (iii) wage inflation gap. These are the three stabilization goals for a planner maximizing social welfare.

Labor efficiency gap is the difference between the marginal product of labor and the marginal rate of substitution between consumption and leisure for the representative household.

$$(i) = m_{rs} - mpn_t$$

where these terms denote deviations from the respective steady state values. Since we do not model price setting frictions in this simple benchmark model, and do not consider price-markup shocks, $mpn_t$ corresponds to the (productivity-adjusted) real wage. Thus the labor efficiency gap captures the time-varying wedge in the disutility of the household from supplying labor at a pre-set nominal wage.

The third term (wage inflation gap) describes the loss in efficiency resulting from dispersion in wages across the members of the household. Wage dispersion, similar to price dispersion in standard New Keynesian models, is costly because firms hire different number of hours from various members of the household, causing marginal disutility of labor to vary within the household. Under flexible wages, both labor inefficiency gap and the wage inflation gap go to zero.

The new component - productivity growth rate gap - is a key ingredient of the endogenous growth model. Investment in R&D in a given period contributes to increase in productivity which persists into the indefinite future. These inter-temporal spillovers of R&D investment may not be internalized by the private agents and may result in too high or low responsiveness of investment relative to the first-best. Starting from a productivity level $A_0$, the growth rate gap in eq. (34) captures the sub-optimality of deviations from the first-best level of productivity given by $A_t^* = A_0 (1 + g_{ss})^t$ at all times $t > 0$. Under nominal rigidities, as discussed in last section, demand shocks may induce this permanent gap, thus leaving the agent permanently worse off.

In Corollary 1 we show the conditions under which the welfare loss resulting from these productivity growth rate deviations impose larger welfare loss than the changes in the labor

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23 As discussed above, under the first-best allocation, the growth rate, output, consumption and labor do not deviate from the steady state when liquidity demand or monetary policy shocks hit the economy. Hence the quadratic approximation is expressed solely in the form of sticky-wage allocation variables. Further we have substituted out (normalized) consumption and labor in terms of productivity growth rate and (normalized) output deviations, hence the labor efficiency gap here assumes this particular form.
efficiency gap. The proposition provides a sufficient condition for the growth rate gap to be of higher importance for stabilization than the labor efficiency wedge. We argue below that this condition is likely to be satisfied generally for a wide range of parameters.

**Corollary 1** (Importance of Growth Stabilization). The relative weight on growth rate gap is higher than the relative weight on labor efficiency wedge if

\[
\frac{\beta}{1 - \beta} > \frac{y}{c} \left( v + \frac{y}{c} \right)
\]  

(35)

*Proof.* See Appendix D

Common calibration values of discount rate \( \beta \) at quarterly frequency lie in the range of \([0.98, 1)\). This implies a lower bound on the left hand side of the condition (35) at 49. We bound the right hand side as follows: Consumption Output ratio in the US has fluctuated between 0.54 and 0.66 from 1960 -2014 (BEA). Estimates of Frisch elasticity of labor \( 1/\eta \) in the micro literature lie between 0.1 and 0.5 (Chetty et al. 2016) while the macro literature uses the estimates in the range of (2,4). Using value of 0.1 for \( \eta^{-1} \) and 0.54 for \( c/y \) ratio, this implies an upper bound on the Right Hand Side at 22. Hence for a wide range of parameter calibration and estimates used in the macroeconomics literature, the welfare loss from a given growth rate deviations is likely to be higher than the welfare loss from a similar change in labor efficiency gap. This may not be surprising in light of the results from Section 3. A given deviation in growth rate from steady state may matter for the path of productivity and hence output into the infinite future depending on the policy rule. On the other hand, fluctuations in the labor efficiency pertain to welfare losses only in the period these are encountered. This corollary thus clarifies the importance of stabilizing the productivity growth rate around the first-best allocation.

**Optimal Policy away from ZLB**

We now turn to investigating the implications for the conduct of monetary policy in our model and show the main results outlined at the beginning of this section. First, we show that optimal policy involves setting the nominal interest rate in order to perfectly stabilize output and productivity along the first-best allocation.

**Proposition 6** (Optimal Policy away from ZLB). Given a process for liquidity demand and monetary policy shocks, optimal policy under sticky wage allocation tracks the natural rate of interest when the Zero Lower Bound constraint is slack.

*Proof.* See Appendix D
From Proposition 4, we know that the flexible wage allocation coincides with the first-best allocation. Under a sticky wage allocation, setting the nominal interest rate to track the natural interest rate implements the flexible wage allocation, thereby replicating the first-best allocation. This implies that the output follows a trend stationary process (see Hayashi 2000) since the normalized output and productivity growth rate are always at the steady state. Hence, the following corollary follows;

**Corollary 2.** *When the ZLB is slack, the time series of output under optimal policy is a trend stationary process (integrated of order zero), that is,*

\[ \log Y_t = a + b \cdot t \]

*where* \( a = \log Y_0 \) *is the initial level of output, and* \( b = \log(1 + g_{ss}) \) *is the steady state productivity growth rate.*

**Proof.** See Appendix D

Figure 7 plots the impulse response of output under the optimal policy (dashed line) and under a non-strict Taylor rule specification (solid line) in response to a positive liquidity demand shock in period 1. Output under a strict inflation targeting rule \( (\phi_\pi \to \infty \text{ in eq 29}) \) follows the optimal policy path. It is a standard result in the monetary economics literature (Woodford 2003) that when the monetary policy does not face any tradeoff in stabilizing output and inflation, strict inflation targeting implements the optimal policy. However, a fairly conventional monetary policy rule (eq 29) admits a permanent output gap, and thus lower welfare.

We established that permanent output gaps are an undesirable feature of the endogenous growth economy in response to temporary demand shocks. The optimal policy does not allow for these hysteresis effects. But, as discussed in Section 3, it may not be possible to implement the optimal policy due to a binding ZLB constraint. As a result, under standard monetary policy rule, temporary contractions in aggregate demand result in permanent downward shifts in output. Should monetary policy offset these hysteresis effects at the ZLB? We take up this question next. Our analysis at the ZLB retains the assumptions (A1 and A2) regarding the natural interest rate and the expected duration of the shock we made in Section 3.

**Optimal Policy at the ZLB**

We now turn to solving the optimal commitment policy, when the central bank can credibly commit to future state-contingent policy actions. At the ZLB, the economy is characterized by deflation and drop in output. By committing to pursuing accommodative policy in the future, the central bank manages expectations of private agents regarding the future path of inflation.
Commitment policy achieves two objectives - (i) it reduces the severity of economic contraction during the ZLB, and (ii) the pursued commitment policy allows aggregate demand to overshoot the steady state level after the the ZLB stops binding. While the first (forward guidance) channel reduces the drops in output from the trend through reduced contraction in demand, the second channel tends to reverse the past drops in output that occurred during the ZLB. The key takeaway from this analysis is that the optimal policy returns the economy close to the pre-recession trend. While Taylor rule admits a permanent output gap of one percent, the optimal policy involves a permanent gap of only 0.12 percent.

The policy maker maximizes the lifetime utility of the household subject to assumption 1 and the competitive equilibrium conditions: (i) Euler Equation (eq 21), (ii) Wage Setting Block (eqns 26-28), (iii) Endogenous growth block (eqns 22-23), (iv) resource constraints and market clearing conditions (eqns 24-25), and (v) the lower bound on the nominal interest rate (eq 33). The simplified problem is as follows:

$$\max_{\hat{c}_t, \hat{y}_t, \hat{g}_t, \hat{V}_t; \hat{\pi}_t} \frac{1}{2} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[ \lambda_y (\hat{y}_t - \chi \hat{g}_{t+1})^2 + \lambda_g \hat{g}_{t+1}^2 + \lambda_{\pi} (\hat{\pi}_t)^2 \right]$$

subject to following constraints (see appendix D.1.1 for coefficients and derivation):

- **Euler equation**
  $$\hat{c}_t = \hat{c}_{t+1} - (i_t - \hat{\pi}_t + \hat{\pi}_w)$$

- **Wage Phillips Curve**
  $$\hat{\pi}_t = \beta \hat{\pi}_{t+1} + \kappa_w (\hat{c}_t + \nu \hat{y}_t)$$

- **Resource Constraint**
  $$(\epsilon - 1) \eta \hat{g}_t = -(\mathbb{E}_t \hat{c}_{t+1} - \hat{c}_t + \hat{g}_{t+1}) + \mathbb{E}_t \hat{V}_{t+1}$$

- **Endogenous Growth eq 1**
  $$\hat{V}_t = \eta_y \hat{y}_t - \eta_c \hat{c}_{t+1} - \eta_q (\mathbb{E}_t \hat{c}_{t+1} - \hat{c}_t + \hat{g}_{t+1}) + \eta_v \mathbb{E}_t \hat{V}_{t+1}$$

- **Zero Lower bound**
  $$i_t \geq 0$$

Since the first order conditions involve a complementary slackness condition, the solution to the optimal policy problem does not have a closed form. We solve it numerically for each state contingent realization of the shock. We provide the first order conditions in the appendix. For details on the solution method, we refer the reader to Eggertsson and Woodford (2003).

Figure 12 shows the equilibrium output, inflation and nominal interest rate under a realization of the shock binding for 28 quarters. A central bank with the ability to credibly commit offsets the permanent output gap by promising to keep interest rates lower after the ZLB stops binding. Under optimal policy, the central bank minimizes total losses in welfare by trading welfare losses during the ZLB against the welfare losses from policy that arise after the ZLB stops binding. By committing to keeping interest rates lower upon exit from the ZLB, the central bank
creates anticipation of a boom, which lowers the real interest rate during the ZLB. This has the effect of reducing the impact of the shock relative to a discretionary policy. On impact, the drop in wage inflation and output are only 0.05% and 1.35% respectively.

Upon exit from the ZLB, the central bank keeps interest rate lower for an additional quarter to follow through with its promise and thus creating a boom in output and inflation. Because of pro-cyclicality of investment in innovation, the boom in output allows for growth rate to overshoot its target. Hence the permanent output gap is reduced substantially on account of two reasons a) the forward guidance channel of optimal policy and (b) the accommodation of excess wage inflation upon exit from the ZLB. In the steady state, output is only 0.12 percent below the (time -0) efficient path of output (solid line).

Note that this is the optimal policy subject to the binding ZLB constraint. If the policymaker had access to time-varying proportional tax instruments such that it could replicate the flexible wage allocation, then the first-best allocation can be implemented (as shown in Appendix C.6). However, the optimal policy trades off welfare losses during the ZLB episode against welfare losses in the future in the absence of appropriate time-varying tax instruments. It is possible to avoid the permanent output gap altogether by a commitment to accommodating even higher inflation post the ZLB. Such a policy would be optimal had the social planner put higher weight on growth rate stabilization relative to the “true” welfare weight in eq (34) (as shown in row 3 of Table 5). However, under the “true” welfare weights, the policy maker allows some permanent output gap because perfectly neutralizing the permanent output gap comes at the expense of higher wage dispersion inefficiency upon exit from ZLB. Thus the ZLB introduces a short-run versus long-run tradeoff for the central bank even when we have assumed away initial steady state distortions (by Assumption 1).

How does this optimal policy compared to the policy when the central bank does not internalize that it can influence productivity growth rate? That is a policy-maker solves the optimal policy problem as in eq 36 except she does not choose productivity growth rate. The solution to optimal policy under this setting does not emphasize the permanent output gap enough. As a result, the optimal policy under this non-internalizing scenario does not allow the central bank to accommodate as much inflation as would the fully optimal policy. Consequently, the permanent output gap is somewhat larger. Figure 13 shows the optimal policy under this “misspecified” setting and compare it to the optimal policy when the central bank internalizes the consequence of its actions on TFP growth rate. Quantitatively, this difference in the optimal policies is negligible. This is because the key problem in this economy is deficiency of aggregate demand. Since the R&D sector has been calibrated to hold smaller share of the output relative to consumption, stabilizing consumption demand approximately stabilizes output. The main implication of this
analysis is that while optimal policy prescriptions are not quantitatively different under the two environments, the cost of not adhering to optimal rules is elevated because of permanent output gaps.

**Markov-Perfect Policy at the ZLB**

Next we analyze the optimal policy when the policy maker is unable to commit to policy actions announced in the future. Such a policy is referred to as the discretionary policy and the resulting equilibrium as the *Markov Perfect equilibrium* (MPE, formally defined in Maskin and Tirole 2001). The key result here is that the discretionary policy is characterized by a new dynamic inconsistency (Kydland and Prescott 1977) problem that we label as the *hysteresis bias*: once the ZLB stops binding, the nominal interest rate is set without any intent to offset the long-run effects of past contractions in aggregate demand. Hence, a policy of committing to lower future interest rates is not time-consistent because the central bank would increase the interest rates as soon as employment recovers back to full employment. The discretionary policy-maker treats past productivity losses as bygones.

The policy maker sets the current short-term nominal interest rate in order to maximize the quadratic approximation of the welfare function (eq 34) subject to assumption 1 and the constraints: (i) Euler Equation (eq 21), (ii) Wage Setting Block (eqns 26-28), (iii) Endogenous growth block (eqns 22-23), (iv) resource constraints and market clearing conditions (eqns 24-25), and (v) the lower bound on the nominal interest rate (eq 33). The problem is similar to the optimal commitment problem, except the policy maker cannot commit to future policy actions.

**Proposition 7** (Optimal Discretionary Policy at the ZLB). If Assumptions A1 and A2 hold and for a given level of productivity at time 0, A₀, the Markov equilibrium is characterized by:

\[
\log A_t = \log A_0 + \log(1 + g_{st})
\]

for \(0 < t < T^e\)

\[
\hat{y}_t = \psi_y r^n_S < 0; \hat{n}t = \psi_p r^n_S < 0; \hat{g}_t = \psi_s r^n_S < 0
\]

\[
\log A_{t+1} = \log A_t + \psi_s r^n_S
\]

and when \(t \geq T^e\)

\[
\hat{y}_t = \hat{n}t = \hat{g}_t = 0
\]

\[
\log A_{t+1} = \log A_{t+1}^* + (T^e - 1)\psi_s r^n_S < \log A_{t+1}^*
\]

where \(\psi_y = \frac{(1-\beta)(1-\mu)(1-\mu)}{(1-\beta)(1-\mu) - \kappa_w (1+\eta_g)} > 0, \psi_p = \frac{\kappa_w (1+\eta_g)}{1-\mu} \psi_y > 0, \text{ and } \psi_s = \frac{1-\gamma}{\eta_g} \psi_y > 0. A_{t+1}^* \text{ is the (time-0) first-best output at time } t + 1.
Proof. See Appendix D.1.2

Since the policymaker is unable to commit to future actions, optimal policy involves setting interest rates such that the economy returns to the (normalized) steady state as soon as the shock abates. This leads to excessive deflation during the ZLB relative to the commitment policy that involves $\hat{\pi}_{wT} > 0$. This dynamic inconsistency problem identified as the *deflation bias* by Eggertsson (2006) is present in our setup. The new feature is that when the ZLB stops binding at stochastic time $T^e$, the discretionary policy maker does not offset the difference in level of productivity from the first-best. MPE thus admits a unit root in the time-series of productivity and hence output. This is the *hysteresis bias* we identify.

Under discretionary policy, the policymaker re-optimizes every period, hence past deviations in growth rate from the steady state are no longer under the influence of a policy-maker at time $T^e$ onwards. In order to bring the output back to the first-best output, the policy maker needs to incentivize excess investment in R&D after the economy has recovered back to full employment. Such an allocation is not desirable from the perspective of policymaker from time $T^e$ onwards. This can be seen by directly looking at the first-order conditions of discretionary equilibrium. Once the shock to the natural interest rate is over, the policy-maker sets interest rate equal to the natural interest rate implying zero slack in the economy. Intuitively this happens because of the following: even though the level of productivity is an endogenous state variable, it only affects the absolute level of the stochastically-trending variables. The efficiency of resource allocation in the normalized economy is independent of the level of productivity. As soon as the central bank is able to set the normalized variables to their steady state values, it would do so. Past deviations of growth rate enter the welfare-loss as additive inefficiencies that cannot be influenced by policymaker optimizing at time $t \geq T^e$. In other words, what is relevant for the stabilization at time $t$ is the gap from the time-$t$ first-best allocation. Once the ZLB has stopped binding, setting interest rates such that employment is back to the efficient steady state implements the time-$t$ first-best allocation.

Figure 14 plots the path of output under MPE. There is an unanticipated shock at time $t = 1$. The output falls by 5% and continues to grow at a slower pace. When the shock stops binding in period $T^e = 28$, the economy is permanently at a lower output trajectory. This also corresponds to the policy under a standard monetary policy specification we discussed in Section 3. The output in the new steady state is permanently lower by 1 percent. Figure 15 compares the equilibrium evolution of variables under discretionary policy to that under optimal commitment policy. The discretionary policy leads to excessive deflation and slack in the economy during the ZLB. Since the discretionary policy does not offset output hysteresis, it also leads to a larger permanent output gap.
This hysteresis bias of discretionary policy thus strengthens the result from Section 3 that output hysteresis is an artifact of policy-constraints faced by the central bank and does not arise because of irrational or inept behavior on part of the central bank. Further implication of this finding of hysteresis bias is that it is sub-optimal for the central bank to redesign policy ex-post, in order to offset past output hystereses. Hence, if the central bank could *credibly commit to being irresponsible* as suggested by Krugman (1998), it could not only reduce the deflation during the ZLB but also minimize the permanent output gaps. This raises the stakes for optimal commitment policy that the central bank must credibly communicate to the public ex-ante.

**Alternative Policy Rules at the ZLB**

Eggertsson and Woodford (2003) have underscored the complex nature of the optimal commitment policy in that it may not be feasible to properly communicate the policy stance to the public even if full credibility can be achieved. On the other hand, we showed that the discretionary policy-maker suffers from *hysteresis bias* and does not offset past inefficiencies. In this regard, alternate simple policy rules that have built-in commitment to reverse past policy mistakes assume importance. Such policy rules are presumably easier to communicate to the public such as commitment to keep interest rates low until the permanent output gap is filled. We illustrated the potency of this strict output hysteresis targeting rule in Section 3, given by:

\[ h_t = \sum_{s=1}^{t} \hat{g}_s = 0 \]

The central bank ex-ante announces to set interest rates in order to completely cut down the permanent losses in output. Such a rule is fully optimal in the absence of the zero lower bound. At the ZLB, though not fully optimal this rule may have a relative advantage in ease of communication to the public. Figure 16 plots nominal interest rate, output and wage inflation under such a rule contrasting with the realized paths of these variables under optimal commitment rule. The central bank keeps the interest rates low for an additional quarter as in the optimal policy. The forward guidance element through anticipation of higher inflation leads to a reduction in the real interest rate, which implies a lower drop in inflation and normalized output (on impact). In the calibrated experiment, output drops by 1.2% on impact. The commitment to this simple rule implies that the central bank accommodates excess wage inflation up to 0.3% before it starts to raise interest rates gradually. Such a policy is relatively more accommodative than the optimal policy. Rows 2 and 5 in table 5 show that the hysteresis targeting policy achieves most of the welfare gains under optimal policy relative to a strict inflation target (or a discretionary) policy, conditional on ZLB being binding in period 1. An optimal commitment policy with higher
weight on output gap can also close the output gap (as shown in row 3 of Table 5), resulting in similar welfare losses as the strict hysteresis targeting rule.

Contrast this policy with the policy of nominal wage level targeting (analogue of a simple price level targeting rule), where the central bank ex-ante announces its intention to set interest rates in order to attain a particular level $w^*$ for the normalized output $y_t$ adjusted nominal wages $w_t^n$:

$$w_t^n + \lambda y_t = w^*; \quad \text{where } \lambda \equiv \frac{1 + \lambda_w}{\lambda_w}$$

Figure 17 shows the realized paths of output inflation and nominal interest rate under wage level targeting against those obtained under optimal commitment policy. This simple policy also approximates the welfare gains achieved under optimal commitment policy (as seen in row 6 of Table 5) relative to the discretion policy, but results in a permanent output gap of 0.4 percent given that it is not as accommodative as the optimal policy.

Compared with wage level targeting, the hysteresis targeting rule requires the central bank to be more tolerant of higher inflation upon exit from ZLB. But it may have an advantage in communication and operationalization over a policy of wage-level targeting. A central bank’s commitment to keep interest rate lower until output has been restored to pre-shock trend is more readily observable and such a policy may presumably be easier to communicate, assuming that achieving credibility is not a constraint for the central bank. Such a policy of hysteresis targeting is equivalent to a real GDP targeting rule because:

$$\log Y_t - \log Y_t^c = h_t$$

where $Y_t^c$ denotes the counterfactual path of output under time-0 flexible-wage allocation.

A third operational rule is what is referred to as the Nominal GDP (NGDP) targeting rule (see Woodford (2012) and references therein). Since our benchmark model features only nominal wage-frictions, a comparison with conventional NGDP targeting rule may not be a useful comparison.\footnote{We have carried out the analysis with the conventional NGDP targeting in this benchmark framework. As expected, the NGDP targeting performs poorly both in terms of reducing permanent output gap and achieving welfare gains relative to optimal commitment policy. In Section 6 we introduce price rigidities and show the path of output under a conventional NGDP targeting rule.} The analogue of NGDP targeting in this simple framework is the $W \times Y$ rule:

$$W_t \times Y_t = W_t^c \times Y_t^c$$

where $W_t^c$ is the counterfactual path of nominal wages under time-0 flexible-wage allocation. The central bank commits to adjusting interest rates in order to achieve this target relationship whenever possible. As shown in row 6 of table 5, this $W \times Y$ rule also implies significant welfare
gains over the discretionary policy and resulting in permanent output gaps close to those under optimal commitment policy. This is expected given that the calibrated nominal wage rigidity parameter is 0.75, which implies considerable nominal wage rigidity. As nominal wages are made more flexible by increasing the probability of wage adjustment, the $W \times Y$ rule approximates the strict wage-level targeting rule.

Table 6 compares permanent output gaps and welfare losses in these three operational rules against the optimal commitment policy for a range of innovation elasticity parameters. Hysteresis targeting policy closely approximates optimal policy for this range of parameters. This analysis primarily highlights that a new operational rule that approximates optimal policy is available for implementation in our framework. Since the standard NK models feature exogenous productivity, this rule is not available to the policy-maker in those environments.

5 Supply Shocks and Potential Output

In order to isolate the implications of negative demand shocks, we have so far abstracted from supply shocks. Now we turn to complete the analysis by examining the implications for monetary policy in the presence of stationary productivity shocks $\hat{M}_t$ and wage markup shocks $\hat{\lambda}_{wt}$.

We underscore two main findings: (i) Taylor rule admits a unit root in output under temporary aggregate productivity shocks and wage markup shocks, (ii) permanent output gaps also emerge under optimal monetary policy. However the reason for optimality of permanent output gaps is different for productivity and wage markup shocks. Under productivity shocks, time-0 first best allocation is characterized by output following a unit root process. Because optimal policy closely approximates the first-best, permanent output gaps are optimal under productivity shocks. Under wage markup shock, in contrast, the first-best allocation does not entertain permanent gaps. But optimal policy still features permanent gaps.

Supply shocks and Taylor rule

We first show that under a Taylor rule, output is permanently below its pre-shock trend following a negative productivity shock or an exogenous increase in wage markups. The rationale for permanent deviations is similar to that discussed in Section 3. Under a Taylor rule, the endogenous (normalized) variables return to their steady state values as soon as the shock subsides. Since the transition to the steady state in the approximate equilibrium is monotonic, any effect of the shocks on the productivity growth rate stays on as hysteresis.

Figures 18 and 19 plot the impulse response of output to a productivity shock and wage markup shock. Shocks are parameterized such that output falls by 1 percent on impact. A nega-
tive productivity shock reduces the resources available for consumption and investment. Because of contraction in the size of the pie, there is excess inflation and low output. Since less resources are available for investment in R&D, productivity growth rate shrinks leaving output permanently below the pre-shock trend. Similarly, an increase in wage markups reduces demand for labor and hence output contracts. Because of lack of inertia in the Taylor rule, the effect of shocks on the the level of output is permanent.

Supply shocks and first-best allocation
Now we show the result that the (time-0) first best allocation in the presence of productivity shocks allows a unit root in aggregate productivity and hence output. This rationale is as follows: A negative (temporary) productivity shock $M_t$ reduces the resources available for consumption and investment into R&D. As a result, the growth rate of productivity declines and output is on a lower trajectory.

Proposition 8. Given i.i.d. aggregate productivity shocks, output follows a unit root under the (time-0) first best allocation.

Proof. See Appendix D.2

Further, note that the (time-0) first-best allocation is defined as the equilibrium allocation where the policymaker has access to time-varying taxes. In such an equilibrium, wage-markup shocks can be neutralized by time-varying labor market subsidies. Hence, output under a first-best allocation is a trend-stationary process.

The solid line in Figures 18 and 19 plot the first-best allocation in response to productivity and markup shocks. Output under Taylor rule seems to approximate the first-best allocation following a productivity shock. However, under a markup shock the Taylor rule keeps the economy further from the efficient path. We next show why allowing for a permanent output gap is an optimal policy than targeting to offset the permanent output gaps.

Optimal Policy
Optimal Commitment Policy under supply shocks interestingly admits large permanent output gaps. As the analysis of first-best allocation suggests, such a conclusion is expected in the case of productivity shocks because the (time-0) first best allocation admits permanent output gaps. However, the first-best allocation does not admit unit root under markup shocks. We now argue that optimal commitment policy (in the absence of time-varying tax instruments) allows permanent output gaps.

Consider first the flexible wage allocation. The policy maker can replicate this allocation by
committing to a strict inflation targeting rule. However, this implies accommodating a severe loss in output today as shown in Figure 19 (dashed brown line). Positive wage markup shock implies an increase in wages, which reduces the demand for labor and hence output. The central bank can reduce this excess wage inflation by creating deflationary expectations. If the central bank could “credibly” commit to maintaining a negative output gap in the future, wage setters do not raise the nominal wages by the full amount of the wage markup shock. This negative output gap implies less resources are invested into innovation, which lower the growth rate of productivity. Thus in a bid to reduce current wage inflation, the central bank promises to keep output permanently below time-0 efficient.

The policy maker maximizes the lifetime utility of the household subject to the decentralized equilibrium conditions: (i) Euler Equation (eq 21), (ii) Wage Setting Block (eqns 26-28), (iii) Endogenous growth block (eqns 22-23), (iv) resource constraints and market clearing conditions (eq 24-25). The simplified problem is as follows:

$$\max_{\hat{c}_t, \hat{y}_t, \hat{\pi}_t, \hat{g}_t, \hat{V}_t} \frac{1}{2} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[ \lambda_y (\hat{y}_t - \hat{y}^*_t - \tilde{\chi} (\hat{g}_t + 1 - \hat{g}^*_t + 1)) + \lambda_g (\hat{g}_t + 1 - \hat{g}^*_t + 1) + \lambda_\pi (\hat{\pi}_t)^2 \right]$$

subject to following constraints:

- **Wage Phillips Curve**
  $$\hat{\pi}_t = \beta \hat{\pi}_{t+1} + \kappa_w (\hat{c}_t + \nu \hat{y}_t - (1 + \nu) \hat{M}_t) + \hat{\lambda}_{wt}$$

- **Resource Constraint**
  $$\frac{c}{y} \hat{c}_t + \frac{R}{y} \eta_g \hat{g}_{t+1} = \hat{y}_t$$

- **Endogenous Growth eq 1**
  $$(q - 1) \eta_g \hat{g}_{t+1} = -(\mathbb{E}_t \hat{c}_{t+1} - \hat{c}_t + \hat{g}_{t+1} + \mathbb{E}_t \hat{y}_{t+1})$$

- **Endogenous Growth eq 2**
  $$\hat{V}_t = \eta_y \hat{y}_t - \eta_g \hat{g}_{t+1} - \eta_q (\mathbb{E}_t \hat{c}_{t+1} - \hat{c}_t + \hat{g}_{t+1}) + \eta_v \mathbb{E}_t \hat{V}_{t+1}$$

where the terms with asterisk denote corresponding values under the first-best allocation. We provide the first order conditions in the appendix D.2.1.

Figures 18 and 19 show output under optimal commitment policy (solid line with diamonds). The optimal policy allows a significant permanent output gap relative to the first best under markup shocks. This is because the policy maker by managing expectations of wage setters is able to improve the tradeoff introduces by markup shocks. Figure 21 shows the impulse responses of normalized output gap ($\hat{y} - \hat{y}^*$), wage inflation and real interest rate gap following a wage markup shock. Circles denote the allocation under Taylor rule, while diamonds denote the allocation under optimal policy. By committing to a lower output gap in the future, central bank is able to tame the hike in wage inflation under optimal policy.

Under productivity shocks, the optimal policy closely follows the first best allocation (figure 18) and results in permanent output gap. Unlike the standard New Keynesian models, optimal
policy under productivity shocks does not coincide perfectly with the (time-0) efficient allocation. This is because of the presence of dynamic inefficiencies in this model. Nuño (2011) shows in a real business cycle economy version of Aghion-Howitt model that the natural rate allocation is inefficient because of different objectives of the entrepreneur and representative agent in undertaking innovation investment.\textsuperscript{25} This generates a tradeoff for the central bank in stabilizing output and inflation. In our calibration, however, the difference between the natural rate allocation and the first-best allocation is quantitatively small, which is why the optimal policy and efficient allocation almost overlap in Figure 18. We leave the quantitative investigation of these endogenous tradeoffs for future work.\textsuperscript{26}

**Welfare Evaluation of Policy rules**

We now compare the welfare under various policies using the quadratic approximation derived earlier. Specifically, we compare Taylor rule (eq 29), strict nominal GDP targeting, strict hysteresis targeting, strict inflation targeting and optimal commitment policy. Table 7 reports the consumption equivalent welfare loss under various policies and shocks. Optimal policy minimizes the welfare loss, but also leads to a significant permanent output gap under supply shocks as shown in Figures 22 and 23. A simple Taylor rule policy outperforms the strict hysteresis targeting rule under productivity and markup shocks - demonstrating the non-optimality of closing the permanent gap. A nominal GDP targeting rule closely approximates the optimal policy.

In Sections 3 and 4, we underscored the importance of closing the permanent output gap and found that the hysteresis targeting rule approximates the optimal policy better than the nominal GDP targeting or a price level targeting rule. The contrasting findings under supply shocks suggest to the importance of carefully identifying the source of business cycle fluctuations.

## 6 Medium-scale DSGE model with Schumpeterian growth

Finally, we show that the endogenous growth channel can be easily appended to a medium-scale Dynamic Stochastic General Equilibrium (DSGE) model. We scale up the simple benchmark model in order to assess the quantitative relevance of permanent output gaps. We underscore two

\textsuperscript{25}Profit maximizing entrepreneurs internalize the probability that if successful they will be replaced by another entrant. As a result, these entrepreneurs discount future more heavily relative to the representative agent. This creates an externality in a creative destruction model.

\textsuperscript{26}We can show that persistent productivity shocks introduce a time-varying wedge between time-0 first-best and time-0 natural rate allocation, which we refer to as dynamic research externalities. As a result, the central bank faces a tradeoff in stabilizing output and inflation at the first-best allocation in the presence of persistent productivity shocks. This finding that productivity shocks introduce inefficient “wedges” implies that the set of shocks that maintain “divine coincidence” is possibly narrower under endogenous growth and that monetary policy faces short-run tradeoffs beyond the exogenous markup shocks. Note that this is merely a qualitative result. These complement the findings of Bilbiie, Ghironi and Melitz (2008), Bilbiie, Fujiwara and Ghironi (2014)
main findings: (i) for a reasonable realization of binding ZLB of 6 quarters, the output hysteresis generated can be significant, and (ii) there is substantial amplification in the medium-scale model relative to the benchmark model - the elasticity of productivity growth rate to quarterly slack in the economy is about 4 times as high.

We introduce capital in the production of intermediate good into the benchmark model of Section 2, following Howitt and Aghion (1998). Households own and accumulate capital subject to investment adjustment costs and rent it out to intermediate good monopolists. The specification for investment adjustment costs follows the NK literature (Christiano, Eichenbaum and Evans, 2005). We append price-rigidity by introducing a retail sector that sells the final good produced by the perfectly competitive producer. Monopolistically competitive retailers set prices on a staggered basis following Calvo (1983). Further, we allow for variable capital utilization, (internal) habits in consumption, and partial indexation of prices and wages to the respective lagged inflation rates. We leave the model discussion to appendix H.

The model is calibrated to match the steady state moments corresponding to the US economy. We calibrated the innovation parameters to match the R&D GDP ratio of 2.38% (NIPA 1947-2008), Gross Private Fixed (Non-Residential) Investment to GDP ratio of 11.9% (NIPA 1947-2008), BGP productivity growth rate of 2% (BEA), creative destruction rate of 3.6% (Howitt, 2000), and Profits to GDP ratio of 6.2% (NIPA 1947-2008). Remaining parameters are relatively standard and are taken from the range of estimated coefficients of Smets and Wouters (2007), Justiniano, Primiceri and Tambalotti (2010), and Justiniano, Primiceri and Tambalotti (2013) (JPT13 henceforth). Table 8 reports the calibrated parameters.

We calibrate the liquidity demand shock so as to generate a maximum drop in output of 5% and 1% drop in inflation relative to the inflation target of 2.48 percent (JPT13 calibration). Further the shock is disciplined so as to generate an increase of 3 p.p. in the liquidity spread for the duration of binding ZLB, as shown in the AAA-Treasury 20 year bond spread in Figure 4. The ZLB is binding for 6 quarters, chosen to broadly coincide with the NBER dating of length of the Great Recession. We find that the ZLB episode is followed by a 1.2 percent permanent drop in output relative to the pre-recession trend. The solid line in Figure 24 plots output, inflation and nominal interest rate. As discussed in Sections 3 and 4, a strict hysteresis targeting policy can not only minimize the extent of the recession, but also keep the economy at the pre-recession trend.

The circled (blue) line plots the path for the economy following a strict hysteresis targeting rule. However such a policy also requires that the Fed tolerates excess inflation of 0.25% relative to the target. The key difference relative to the simple benchmark model considered in Section 2 is that the endogenous capital accumulation along with other standard frictions amplifies the
effect of a ZLB shock on incentives to undertake R&D. In the model without capital, a shock of similar magnitude induces 0.013 percent change in permanent output in response to one percent drop in (normalized) output, which measures the slack in the economy. In contrast, the corresponding hysteresis elasticity\(^{27}\) in the DSGE model is 0.044. Thus, the endogenous capital accumulation makes the investment decisions of the entrepreneurs sufficiently responsive to business cycle fluctuations. DeLong and Summers (2012) calibrate this parameter to 0.05 in their main specification.

The key elasticity determining the quantitative plausibility of permanent output gaps is the elasticity of innovation intensity to innovation expenditures \(\varrho^{-1}\). It determines the quantitative relevance of permanent gaps as an outcome of contractionary shocks. As discussed before, the innovation and patents literature estimates for \(\varrho\) typically lie in the range of 1-3. However these estimates are based on annual or even lower frequency data. Since we do not have a clear guide on this crucial parameter, we show robustness by calibrating the steady state with either of the extreme values of this elasticity. Table 9 shows sensitivity to calibrating different innovation elasticity \(\varrho\). We re-calibrated other parameters (reported in Table 10) so that the model matches steady state moments discussed above and shocks are set so as to produce a 5% maximum drop in output and 1% drop in inflation at a ZLB binding for 6 quarters. Under a strict inflation targeting Taylor rule, the permanent output gap ranges between 0.35% and 2.68% for parameterizations of \(\varrho\) in the range of \((1.1, 2.9)\).

This permanent drop in output under a strict inflation targeting rule is likely to be understated because of the solution concept employed (see Carlstrom, Fuerst and Paustian 2015, Del Negro, Giannoni and Patterson 2015, and McKay, Nakamura and Steinsson 2016). In contrast to our earlier analysis at the ZLB, we use a perfect foresight solution method which suffers from the caveat that the actual realization must equal the expected duration of binding ZLB. Financial market surveys such as Blue Chip economic indicators expected the 3-month Treasury yield to remain below 30 basis points for only 3 quarters both in 2009:Q1 and 2010:Q2. Gust, López-Salido and Smith (2013) infer similar forecasts from Eurodollar quotes and three-month forward rate swaps. In this scenario, the two-state Markov chain analysis is useful in that it allows for an arbitrarily long realization of zero lower bound episode while also approximating the market expectations. Under the two-state benchmark model when the expectations and realizations of duration of ZLB coincide at 4.76 quarters, the permanent output gap is only 0.26%, while under an actual realization of 28 quarters, the permanent output gap is 1.79%. It is perhaps not too unrealistic to extrapolate from this analysis that the perfect foresight analysis yields a lower bound for the permanent output gap.

\(^{27}\)These elasticities were calculated as ratio of average changes in permanent output and normalized output.
The DSGE analysis thus points to the quantitative relevance of the endogenous growth mechanism at extreme events such as binding ZLB.

7 Conclusion and Discussion

This paper undertakes optimal monetary policy analysis in an environment where the long-run potential output of the economy is endogenous to short-run fluctuations in demand. An optimizing policy maker at the ZLB commits to keeping interest rates lower in order to offset the long-run effects of contraction in aggregate demand. However, a policymaker unable to commit to future interest rates does not offset permanent output gaps following a ZLB episode. This is the hysteresis bias of discretionary policy that we highlight. We compare alternate monetary policy rules and show that a policy of hysteresis targeting closely approximates the optimal policy at the ZLB. Finally in a calibrated medium-scale DSGE model, we show that the resulting permanent gaps may be sizable in recessions of the magnitude of the Great Recession of 2007-09.

Recent literature has argued that loss in potential output could result from decline in startups (Gourio, Messer and Siemer 2016), decline in labor force participation (Erceg and Levin 2014), decline in capital investment by existing firms, or reduced R&D investment (this paper). While we have modeled only the R&D channel following the endogenous growth literature, our model is consistent with these other hypotheses of slowdown of potential output in response to demand shocks. There are however certain shortcomings in our analysis that we now highlight.

Our modeling assumption in the paper is that a new innovation gets adopted with certainty in the following period. This is clearly unrealistic. Comin and Hobijn (2010) and others have found that firms adopt new technology with average lags of up to 7 years. As long as contraction in demand results in lower investment in knowledge creation, the model of output hysteresis presented in this paper has insights for the conduct of monetary policy. The key elasticity determining the long-run effect of sub-optimal monetary policy is the elasticity of innovation to R&D expenditure. We have discussed robustness to calibrating various parameterizations of this elasticity. However we leave the investigation of optimal monetary policy in a richer model with implementation lags and technology diffusion (see for example Andolfatto and MacDonald 1998, Anzoategui et al. 2016) for future work.\footnote{An alternative interpretation of our modeling assumption is that R&D entrepreneurs invest in order to implement knowledge from a pool of existing technologies. This requires that the distance of the set of implemented technologies from the most-sophisticated non-implemented idea does not influence adoption decisions of the firms. In fact, the general purpose technology literature points to very slow adoption of IT innovations (25-30 years) and thus there being large time-lags between available and implemented ideas due to absence of complementary structures necessary for implementation. Our baseline calibration of $z = 0.036$ implies that time between successive technology upgradations is about 7 years consistent with Comin and Hobijn (2010) findings. As long as the distance from the counterfactual frontier of ideas and patents is not a major determinant of incentives to invest in technology adoption, this alternate interpretation of our modeling assumption seems plausible.}
In this paper we focused primarily on short-term nominal interest rate as the instrument for stabilization policy. Aghion, Angeletos, Banerjee and Manova (2010) provide evidence that firms undertaking R&D activities face liquidity frictions. An extension of our framework with financial frictions following Del Negro, Eggertsson, Ferrero and Kiyotaki (2016) can shed light on the potency of unconventional monetary policy such as Fed’s short-term liquidity facilities.

While the empirical evidence on the interaction between monetary policy and long-term investment in research is still scant, there is a large literature emphasizing the potency of tax credits for spurring R&D growth (Aghion, Hemous and Kharroubi 2014, Atkeson and Burstein 2015, Dechezleprêtre, Einiö, Martin, Nguyen and Reenen 2016 among others). While time-varying fiscal instruments in the presence of non-distortionary lump-sum taxation can replicate the first-best outcome in our framework, we limited our focus in this paper to time-varying use of monetary policy instrument. We leave the analysis of optimal fiscal policy to future research.

References


interpretation is consistent with our model.


Figures

Figure 1: Real GDP

Quarterly Real GDP data from St. Louis FRED database. CBO Potential Output 2015 and CBO Potential Output 2007 estimates are taken from the Congressional Budget Office February 2016 releases. The trend line until 2007Q4 is estimated on quarterly data from 1947 Q1: 2007 Q4 using Hodrick-Prescott filter with a smoothing parameter of 1600. The solid black line with circles is constructed using 2% annual growth rate starting from 2009. The shaded areas represent the recessions dated by NBER.

Figure 2: Real GDP in the OECD

Figure 3: Procyclicality of Private R&D

Source: Annual R&D data comes from Compustat Database. This comprises of publicly traded firms in the US that undertake R&D in a given year. Annual GDP growth rate is taken from St. Louis FRED database. Original Graph by Barlevy (2007) used data until 2004. The correlation in the two series is 0.34. Y-axis plots growth rates in percentage points. The shaded areas represent the recessions dated by NBER.

Figure 4: Private R&D Investment and Borrowing Spreads

Figure 5: IRFs: liquidity demand shock

- $\frac{y}{A}$
- $\pi_w$

Policy Rule targets employment and wage inflation

Figure 6: IRFs: contractionary MP shock

- $y_t = y_{t-1} + \beta \pi_{t+1} + r^h_t$
- $\gamma_t$
- $\pi_{t+1}$
- $g_t$

Policy Rule targets employment and wage inflation
Figure 7: Output: liquidity demand shock

Policy Rule targets employment and wage inflation

Figure 8: Output: liquidity demand shock

Policy Rule targets employment and wage inflation
Figure 9: Output: contractionary MP shock

Policy Rule targets employment and wage inflation

Figure 10: Output: contractionary MP shock

Policy Rule targets employment and wage inflation
Figure 11: Strict Targeting Policy at ZLB
Figure 12: Optimal Policy at the ZLB
EW2003 labels the optimal policy when the policy maker does not internalize the effects on productivity growth. Optimal rule labels the optimal policy when the policy maker internalizes these effects.
Figure 14: Discretion Policy

Figure 15: Discretion and Optimal Commitment Policy
Figure 16: Hysteresis targeting and Optimal Commitment Policy

Figure 17: Nominal Wage Level (NWLT) targeting and Optimal Commitment Policy
Figure 18: Output: contractionary TFP shock

Figure 19: Output: Wage markup shock
Figure 20: IRFs : contractionary TFP shock

Policy Rule targets employment and wage inflation

Figure 21: IRFs : Wage markup shock

Policy Rule targets employment and wage inflation
Figure 22: Output under Alternative Rules - contractionary TFP shock

Figure 23: Output under Alternative Rules: Wage markup shock
Figure 24: Medium Scale Endogenous Growth NK Model at the ZLB

Figure 25: Investment at the ZLB
Figure 26: Demand Shocks and Potential Output

Policy Rule strictly targets price inflation. ZLB binding for 6 quarters. Maximum output drop = 5 %. Permanent Gap = -1.19 p.p. Hatched area denotes the loss in potential output. Speckled area denotes the contraction in demand due to the positive liquidity demand shock. Parameters reported in Table 8
Tables

**Table 1:** Parameters for the Simple Benchmark Model

<table>
<thead>
<tr>
<th>Standard Parameters</th>
<th>Formula</th>
<th>Value</th>
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</thead>
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<tr>
<td>Labor share</td>
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<td>Discount rate</td>
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<td>1. Innovation Step Size</td>
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<td>3. Innovation Cost parameter</td>
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<td>4. Real marginal cost of fringe</td>
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**Table 2:** Targeted Moments

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<td>1. Growth Rate</td>
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<td>2. Creative Destruction Rate (Howitt, 2000)</td>
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<td>3. R&amp;D to GDP (NIPA 1953-2007)</td>
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<td>4. Profits GDP ratio (BEA: 1947-2008)</td>
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**Table 3:** Permanent output gap for different values of innovation elasticity $\rho$

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<th>Innovation Intensity $\rho$</th>
<th>1.05</th>
<th>1.10</th>
<th>1.20</th>
<th>1.32</th>
<th>1.50</th>
<th>1.71</th>
<th>2.8</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>AR(1) process</td>
<td>0.146</td>
<td>0.137</td>
<td>0.121</td>
<td><strong>0.113</strong></td>
<td>0.080</td>
<td>0.060</td>
<td>0.026</td>
<td>0.024</td>
<td>0.016</td>
<td>0.013</td>
</tr>
<tr>
<td>ZLB shock</td>
<td>2.61</td>
<td>2.39</td>
<td>2.02</td>
<td><strong>1.79</strong></td>
<td>1.22</td>
<td>0.89</td>
<td>0.38</td>
<td>0.34</td>
<td>0.27</td>
<td>0.18</td>
</tr>
</tbody>
</table>

Notes: Row 1 reports the permanent output gap for an AR(1) shock with persistence 0.9 and drop in output of 1% on impact. Row 2 reports the permanent output gap for the ZLB scenario where the ZLB binds for 28 quarters. At the ZLB, the drop in output is 5% and wage deflation is 1%. Bolded numbers refer to the benchmark scenario parametrized in Table 1. Detailed list of parameters is provided in the Online Appendix.
Table 4: Parameters for Welfare Analysis in the Simple Benchmark Model

<table>
<thead>
<tr>
<th>Standard Parameters</th>
<th>Formula</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Labor share</td>
<td>$1 - \alpha$</td>
<td>0.67</td>
</tr>
<tr>
<td>Discount rate</td>
<td>$\beta$</td>
<td>0.99</td>
</tr>
<tr>
<td>Steady State Wage Markup</td>
<td>$\lambda_w$</td>
<td>0.10</td>
</tr>
<tr>
<td>Calvo probability of wage adjustment</td>
<td>$(1 - \theta_w)$</td>
<td>$1 - 0.75$</td>
</tr>
<tr>
<td>Inverse Frisch Elasticity</td>
<td>$\nu$</td>
<td>1.60</td>
</tr>
<tr>
<td>Innovation Step Size</td>
<td>$\gamma$</td>
<td>1.10</td>
</tr>
<tr>
<td>Inverse Innovation Elasticity</td>
<td>$\varrho$</td>
<td>1.34</td>
</tr>
<tr>
<td>Innovation Cost parameter</td>
<td>$\delta$</td>
<td>4.9</td>
</tr>
</tbody>
</table>

Table 5: Policy Rules at the ZLB: Welfare Comparison

<table>
<thead>
<tr>
<th>Policy Rule</th>
<th>Welfare Loss</th>
<th>Permanent Output Gap</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Optimal rules</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Discretion (MPE)</td>
<td>100%</td>
<td>-1.00%</td>
</tr>
<tr>
<td>Commitment</td>
<td>4.71%</td>
<td>-0.12%</td>
</tr>
<tr>
<td>Commitment with higher wt on $\hat{g}_t$</td>
<td>10.90%</td>
<td>0</td>
</tr>
<tr>
<td><strong>Simple rules</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Taylor rule eq 27</td>
<td>100%</td>
<td>-1.00%</td>
</tr>
<tr>
<td>Hysteresis Targeting</td>
<td>6.88%</td>
<td>0</td>
</tr>
<tr>
<td>Wage Level Targeting</td>
<td>23.92%</td>
<td>-0.39%</td>
</tr>
<tr>
<td>$W \times Y$ targeting</td>
<td>7.26%</td>
<td>-0.13%</td>
</tr>
</tbody>
</table>

Notes: Values report the conditional welfare loss starting from an efficient steady state. Loss is expressed in consumption equivalent units relative to discretionary rule. Computation details in the Appendix. The true relative weight on growth rate gap is 3.7. Under a weight of 154.2, the permanent output gap is 0.
Table 6: Policy Rules at the ZLB: Welfare Comparison for Range of $\varrho$

<table>
<thead>
<tr>
<th>Innovation Intensity $\varrho$</th>
<th>1.02</th>
<th>1.09</th>
<th>1.20</th>
<th>Bench</th>
<th>1.32</th>
<th>1.50</th>
<th>1.71</th>
<th>2.90</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Permanent Output Gap</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Discretion (MPE)</td>
<td>-1.57%</td>
<td>-1.45%</td>
<td>-1.20%</td>
<td>-1.00%</td>
<td>-0.83%</td>
<td>-0.68%</td>
<td>-0.34%</td>
<td></td>
</tr>
<tr>
<td>Commitment</td>
<td>-0.25%</td>
<td>-0.19%</td>
<td>-0.16%</td>
<td>-0.12%</td>
<td>-0.10%</td>
<td>-0.08%</td>
<td>-0.04%</td>
<td></td>
</tr>
<tr>
<td>Hysteresis Targeting</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>Wage Level Targeting</td>
<td>-0.44%</td>
<td>-0.49%</td>
<td>-0.45%</td>
<td>-0.39%</td>
<td>-0.33%</td>
<td>-0.27%</td>
<td>-0.14%</td>
<td></td>
</tr>
<tr>
<td>$W \times Y$ targeting</td>
<td>-0.37%</td>
<td>-0.21%</td>
<td>-0.21%</td>
<td>-0.13%</td>
<td>-0.18%</td>
<td>-0.15%</td>
<td>-0.09%</td>
<td></td>
</tr>
<tr>
<td><strong>Welfare Loss</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Commitment</td>
<td>3.46%</td>
<td>4.55%</td>
<td>4.36%</td>
<td>4.71%</td>
<td>5.15%</td>
<td>5.58%</td>
<td>6.54%</td>
<td></td>
</tr>
<tr>
<td>Hysteresis Targeting</td>
<td>20.83%</td>
<td>7.11%</td>
<td>6.35%</td>
<td>6.88%</td>
<td>7.40%</td>
<td>7.82%</td>
<td>8.53%</td>
<td></td>
</tr>
<tr>
<td>Wage Level Targeting</td>
<td>8.34%</td>
<td>13.99%</td>
<td>19.40%</td>
<td>23.92%</td>
<td>27.42%</td>
<td>30.47%</td>
<td>36.49%</td>
<td></td>
</tr>
<tr>
<td>$W \times Y$ targeting</td>
<td>12.13%</td>
<td>4.99%</td>
<td>5.65%</td>
<td>7.26%</td>
<td>8.87%</td>
<td>10.42%</td>
<td>14.18%</td>
<td></td>
</tr>
</tbody>
</table>

Notes: Values report the conditional welfare loss starting from an efficient steady state. Loss is expressed in consumption equivalent units relative to discretionary rule. Only two parameters are adjusted. Innovation Intensity elasticity ($1/\varrho$) and research cost $\delta$ to target 2% annual growth rate.

Table 7: Policy Rules : Welfare Comparison

<table>
<thead>
<tr>
<th>Policy Rule</th>
<th>Markup shock</th>
<th>Productivity Shock</th>
<th>Liq Demand Shock</th>
<th>MP shock</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Optimal rules</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Commitment</td>
<td>0.17%</td>
<td>0.0002%</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Discretion</td>
<td>0.715%</td>
<td>0.0008%</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td><strong>Simple rules</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Taylor rule eq 27</td>
<td>2.011%</td>
<td>0.001%</td>
<td>0.024%</td>
<td>0.020%</td>
</tr>
<tr>
<td>Hysteresis Targeting</td>
<td>2.81%</td>
<td>0.436%</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Wage Level Targeting</td>
<td>0.265%</td>
<td>0.0003%</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Nominal GDP growth targeting</td>
<td>0.434%</td>
<td>0.0004%</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Notes: Values report the conditional welfare loss starting from an efficient steady state. Welfare losses are computed as an average over 10,000 simulations, each starting at the same efficient steady state. Loss is expressed in consumption equivalent units. Computation details in the Appendix.
### Table 8: Parameters for the Medium Scale DSGE Model

<table>
<thead>
<tr>
<th>Standard Parameters</th>
<th>Formula</th>
<th>Value</th>
<th>Source</th>
</tr>
</thead>
<tbody>
<tr>
<td>Steady State Price Markup</td>
<td>$\lambda_p$</td>
<td>0.10</td>
<td>JPT, SW</td>
</tr>
<tr>
<td>Steady State Wage Markup</td>
<td>$\lambda_w$</td>
<td>0.10</td>
<td>JPT, SW</td>
</tr>
<tr>
<td>Calvo probability of price adjustment</td>
<td>$(1 - \theta_p)$</td>
<td>1 - 0.84</td>
<td>JPT, SW</td>
</tr>
<tr>
<td>Calvo probability of wage adjustment</td>
<td>$(1 - \theta_w)$</td>
<td>1 - 0.72</td>
<td>JPT, SW</td>
</tr>
<tr>
<td>Price Indexation</td>
<td>$\tau_p$</td>
<td>0.16</td>
<td>JPT, SW</td>
</tr>
<tr>
<td>Wage Indexation</td>
<td>$\tau_w$</td>
<td>0.05</td>
<td>JPT, SW</td>
</tr>
<tr>
<td>Steady State Inflation (annual)</td>
<td>$400(\pi_{ss} - 1)$</td>
<td>2.48%</td>
<td>JPT, SW</td>
</tr>
<tr>
<td>Labor share</td>
<td>$1 - \alpha$</td>
<td>0.278</td>
<td>JPT, SW</td>
</tr>
<tr>
<td>Discount rate</td>
<td>$\beta$</td>
<td>0.994</td>
<td>JPT, SW</td>
</tr>
<tr>
<td>Consumption habit parameter</td>
<td>$h$</td>
<td>0.65</td>
<td>JPT, SW</td>
</tr>
<tr>
<td>Capital Utilization cost</td>
<td>$\frac{\sigma(1)}{\pi(1)}$</td>
<td>4.95</td>
<td>JPT, SW</td>
</tr>
<tr>
<td>Investment Adjustment cost</td>
<td>$S''(1)$</td>
<td>3.64</td>
<td>JPT, SW</td>
</tr>
<tr>
<td>Steady State Government Spending</td>
<td>$1 - \frac{1}{\lambda_g}$</td>
<td>0.22</td>
<td>JPT, SW</td>
</tr>
<tr>
<td>Inverse Frisch Elasticity</td>
<td>$\nu$</td>
<td>1.85</td>
<td>JPT, SW</td>
</tr>
<tr>
<td>Capital Depreciation Rate</td>
<td>$\delta_k$</td>
<td>0.0265</td>
<td>Calibrated</td>
</tr>
<tr>
<td>Innovation Step Size</td>
<td>$\gamma$</td>
<td>1.56</td>
<td>Calibrated</td>
</tr>
<tr>
<td>Inverse Innovation Elasticity</td>
<td>$\theta_i$</td>
<td>1.344</td>
<td>Calibrated</td>
</tr>
<tr>
<td>Innovation Cost parameter</td>
<td>$\delta$</td>
<td>2.84</td>
<td>Calibrated</td>
</tr>
<tr>
<td>Real marginal cost of fringe</td>
<td>$\chi$</td>
<td>1.168</td>
<td>Calibrated</td>
</tr>
</tbody>
</table>

Notes: JPT, SW implies the calibration is in the range of parameters considered by Smets and Wouters (2007), Justiniano, Primiceri and Tambalotti (2010), and Justiniano, Primiceri and Tambalotti (2013). Calibrated implies parameters are chosen to hit following steady state target moments: Annual Productivity Growth Rate of 2%, R&D to GDP ratio of 2.38%, Investment to GDP ratio of 11.9%, Creative Destruction Rate of 3.6% and Profits to GDP ratio of 6.2%. Details in the main text.

### Table 9: Permanent output gap for different values of innovation elasticity $\varphi$

<table>
<thead>
<tr>
<th>Innovation Intensity $\varphi$</th>
<th>1.09</th>
<th>1.34</th>
<th>1.639</th>
<th>2.90</th>
<th>7.66</th>
</tr>
</thead>
<tbody>
<tr>
<td>Output hysteresis (in %)</td>
<td>2.68</td>
<td>1.19</td>
<td>0.96</td>
<td>0.35</td>
<td>0.16</td>
</tr>
</tbody>
</table>

Notes: Table reports the permanent output gap for the ZLB scenario where the ZLB binds for 7 quarters. The maximum drop in output is 5% and drop in inflation is 1%. Detailed list of parameters is provided in the Table 10.
Table 10: Range of Parameters for the Medium Scale DSGE Model

<table>
<thead>
<tr>
<th>Standard Parameters</th>
<th>Formula</th>
<th>Cal 1</th>
<th>Main</th>
<th>Cal 2</th>
<th>Cal 3</th>
<th>Cal 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Labor share</td>
<td>$1 - \alpha$</td>
<td>0.257</td>
<td>0.278</td>
<td>0.276</td>
<td>0.234</td>
<td>0.343</td>
</tr>
<tr>
<td>Discount rate</td>
<td>$\beta$</td>
<td>0.9897</td>
<td>0.9940</td>
<td>0.9940</td>
<td>0.99</td>
<td>0.995</td>
</tr>
<tr>
<td>Inverse Frisch Elasticity</td>
<td>$\nu$</td>
<td>1.60</td>
<td>1.85</td>
<td>1.646</td>
<td>1.603</td>
<td>1.66</td>
</tr>
<tr>
<td>Capital Depreciation Rate</td>
<td>$\delta_k$</td>
<td>0.062</td>
<td>0.0265</td>
<td>0.053</td>
<td>0.15</td>
<td>0.059</td>
</tr>
<tr>
<td>Innovation Step Size</td>
<td>$\gamma$</td>
<td>1.55</td>
<td>1.56</td>
<td>1.17</td>
<td>1.14</td>
<td>1.452</td>
</tr>
<tr>
<td>Inverse Innovation Elasticity</td>
<td>$\varphi$</td>
<td>1.127</td>
<td>1.344</td>
<td>1.639</td>
<td>2.90</td>
<td>7.66</td>
</tr>
<tr>
<td>Innovation Cost parameter</td>
<td>$\delta$</td>
<td>0.48</td>
<td>2.84</td>
<td>10.31</td>
<td>238.22</td>
<td>$e^{15.23}$</td>
</tr>
<tr>
<td>Real marginal cost of fringe</td>
<td>$\chi$</td>
<td>1.177</td>
<td>1.168</td>
<td>1.26</td>
<td>1.68</td>
<td>2.34</td>
</tr>
</tbody>
</table>

Notes: Each column reports the set of parameters calibrated to hit following steady state target moments as closely as possible: Annual Productivity Growth Rate of 2%, R&D to GDP ratio of 2.38%, Investment to GDP ratio of 11.9%, Creative Destruction Rate of 3.6%. Rows 1, 2, and 3 do not significantly influence the calibration. However some adjustments are made to match the moments more closely. Remaining parameters are same as reported in Table 8.
Output Hysteresis and Optimal Monetary Policy

Online Appendix

Vaishali Garga and Sanjay R. Singh†

November 15, 2016
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†Singh: sanjay_singh@brown.edu Garga: vaishali_garga@brown.edu Department of Economics, 64 Waterman Street, Providence, RI 02912.
A Competitive Equilibrium

Definition A.1 (Competitive Equilibrium). The competitive equilibrium is defined as a sequence of 9 quantities \{C_t, z_t, V_t, \Gamma_t, Y_t^C, Y_t, RD_t, L_t, A_t\} and 7 prices \{i_t, Q_{t,t+1}, P_t, W_t, K_t, F_t, \pi_{W_t}\} which satisfy the following 16 equations, for a given sequence of exogenous shocks \{\epsilon_t, \xi_t, M_t, \lambda_{w,t}\} and exogenously specified policy variables \{\tau^b_t, \tau^r_t, \tau^\pi_t, \tau^\mu_t\}.

1. Euler Equation and Stochastic Discount Factor

\[
1 = \beta E_t \left[ \frac{C^{t+1}}{C_t} (1 + i_t) \frac{P_t}{P_{t+1}} (1 - \tau^b_t) \right] + \xi_t C_t
\]

\[
Q_{t,t+1} = \beta \frac{C^{t+1}}{C_t} \frac{P_t}{P_{t+1}}
\]

2. Endogenous Growth Block

\[
(1 - \tau^r_t) P_t \delta Q_t^\omega^{-1} = E_t Q_{t,t+1} V_{t+1}(A_{t+1})
\]

\[
V_t(A_t) = \Gamma_t + (1 - z_t) E_t Q_{t,t+1} V_{t+1}(A_t)
\]

\[
\Gamma_t = (\bar{\zeta} - 1) \left( \frac{\alpha}{\bar{\zeta}} \right) P_t M_t L_t A_t
\]

where \(\bar{\zeta} \equiv \min \left( \chi, \frac{1}{1-\phi^{(1)}_m} \right)\), and \(\chi \geq 1\).

3. Wage Setting frictions

\[
K_t = \frac{F_t}{\bar{F}_t} = \left( \frac{1 - \theta_w(\tau^\pi_t)^{\gamma - 1}}{1 - \theta_w} \right)^{-\lambda_{w,t} + (1 + \lambda_{w,t}) \nu}
\]

\[
K_t = \omega (1 + \lambda_{w,t}) \bar{L}_t^{1+\nu} + \theta_w \beta \Pi_{W_t}^{-\frac{1}{\gamma} - \nu} \left( \frac{\Pi_{W_{t+1}}}{\Pi_{W_t}} \right)^{\gamma - 1} K_{t+1}
\]

\[
F_t = (1 + \tau^\pi_t) \bar{L}_t \frac{W_t}{\Pi_{W_t}} + \theta_w \beta \Pi_{W_{t+1}^{\gamma - 1}} \left( \frac{\Pi_{W_{t+1}}}{\Pi_{W_t}} \right) F_{t+1}
\]

\[
\Pi_{W,t} = \frac{W_t}{W_{t-1}}
\]

4. Law of motion of productivity

\[
A_t = A_{t-1} + z_{t-1}(\gamma - 1) A_{t-1}
\]

5. Market Clearing Conditions and Production Technologies

\[
Y_t^C = \left( \frac{\alpha}{\bar{\zeta}} \right) \bar{M}_t L_t A_t
\]

\[
RD_t = \delta Y_t^C A_t
\]

\[
Y_t = C_t + RD_t
\]

\[
W_t = (1 - \alpha) \left( \frac{\alpha}{\bar{\zeta}} \right) \bar{M}_t A_t P_t
\]

6. Monetary Policy Rule

\[
1 + i_t = \max \left( 1, (1 + i_{ss}) \left( \frac{\pi_{W,t}}{\pi_{W}} \right)^{\phi_{\pi}} \left( \frac{L_t}{L} \right)^{\phi_{L}} \epsilon_t \right) ; \quad \phi_{\pi} > 1, \phi_{L} > 0
\]

Stationarizing the System

The competitive equilibrium defined above is non-stationary. Specifically, consumption, output, nominal wage, are
co-integrated with TFP level $A_t$. We normalize the variables as follows:

$$c_t \equiv \frac{C_t}{A_t}, y_t \equiv \frac{Y_t}{A_t}, r_t^G \equiv \frac{Y^G_t}{A_t}, r_d_t \equiv \frac{RD_t}{A_t}, \Gamma_t \equiv \frac{\Gamma_t}{P_t A_t}, w_t \equiv \frac{W_t}{P_t A_t}$$

Further note that because of the linearity assumption in the production of final goods, the Value function is a linear function in productivity with which an entrepreneur enters the sector:

$$\tilde{\mathcal{V}}_t = \frac{V_t}{A_t} = \Gamma_t + (1 - z_t) E_t Q_{t+1} \tilde{\mathcal{V}}_{t+1}$$

where $\tilde{\mathcal{V}}$ is normalized by the productivity with which the entrepreneur enters the sector. Finally the growth rate of productivity, determined in period $t$, is given by

$$(1 + \gamma_{t+1}) = 1 + z_t (\gamma - 1)$$

Remaining variables are stationary. Hence

**Definition A.2 (Normalized Competitive Equilibrium).** The normalized competitive equilibrium is defined as a sequence of 9 stationary quantities $\{c_t, \tilde{\mathcal{V}}_t, \Gamma_t, y^G_t, y_t, r_d_t, L_t, \gamma_{t+1}, z_t\}$ and 6 stationary prices $\{t_t, w_t, K_t, F_t, \pi_{w,t}, \Pi_t\}$ which satisfy the following 15 equations, for a given sequence of exogenous shocks $\{\varepsilon_t, \xi_t, M_t, \lambda_{w,t}\}$ and exogenously specified policy variables $\{r^p_t, \eta_t, \tau_t^p, \tau_t^w\}$.

1. Euler Equation and Stochastic Discount Factor

$$1 = \beta E_t \left[ \frac{c_{t+1} (1 + \gamma_{t+1})}{c_t} \frac{1 + \bar{r}_t}{\bar{r}_t+1} \right] + \xi_t c_t$$

where $\xi_t = \xi_t A_t$

2. Endogenous Growth Block

$$1 - \bar{r}_t \delta \varepsilon_t^{\gamma - 1} = \mathbb{E}_t \left[ \frac{c_{t+1} (1 + \gamma_{t+1})}{c_t} \right] \gamma \tilde{\mathcal{V}}_{t+1}$$

$$\tilde{\mathcal{V}}_t = \tilde{\mathcal{V}}_t + (1 - z_t) E_t \left[ \frac{c_{t+1} (1 + \gamma_{t+1})}{c_t} \right] \tilde{\mathcal{V}}_{t+1}$$

$$\bar{\Gamma}_t = (\bar{\xi} - 1) \left( \frac{1}{\bar{\xi}} \right) M_t L_t$$

where $\xi = \min \left( \chi, \frac{1}{1 - \bar{\xi} \gamma} \right)$, and $\chi \geq 1$.

3. Wage Setting frictions

$$K_t \equiv \frac{F_t}{\lambda_{w,t}} \left( 1 - \frac{\theta_w (\Pi_{w,t}^{\gamma_{t+1}})}{1 - \theta_w} \right)^{-\lambda_{w,t} + (1 + \lambda_{w,t}) \gamma_{t+1}}$$

$$K_t = \omega (1 + \lambda_{w,t}) L_t^{1 + \gamma_{t+1}} + \theta_w \beta \Pi_{W_t}^{\gamma_{t+1}} \Pi_{w_{t+1}}^{\gamma_{t+1}} K_{t+1}$$

$$F_t = (1 + \tau_t^w) L_t c_t^{\gamma_{t+1}} w_t + \theta_w \beta \Pi_{W_t}^{\gamma_{t+1}} \Pi_{w_{t+1}}^{\gamma_{t+1}} F_{t+1}$$

$$\pi_{w,t} = \frac{w_t}{w_{t-1}} (1 + \gamma_{t+1})$$

4. Productivity growth rate

$$(1 + \gamma_{t+1}) = 1 + z_t (\gamma - 1)$$

5. Market Clearing Conditions and Production Technologies

$$y^G_t = \left( \frac{1}{\bar{\xi}} \right) \frac{\alpha}{M_t L_t}$$

$$r_d_t = \delta \varepsilon_t^\gamma$$

A.2
\( y_t = c_t + rd_t \)
\( w_t = (1 - \alpha) \left( \frac{a}{\zeta} \right)^{\frac{1}{\alpha}} M_t \)

6. Monetary Policy Rule

\[ 1 + i_t = \max \left( 1, (1 + i_{ss}) \left( \frac{\pi W_t}{\tilde{\pi} W_t} \right)^{\phi_y} \left( \frac{L_t}{L} \right)^{\phi_r} \right) ; \quad \phi_r > 1; \phi_y > 0 \]

**Steady State**

Six variables \( z, g, V, L, C, Y \) solve the following six equations

1. Endogenous Growth Equation

\[ (1 - \tau')q e^{e - 1} = \frac{\beta}{1 + g} \frac{\gamma \hat{V}}{\delta} \]

2. Value Function

\[ \hat{V} = (\zeta - 1) \left( \frac{q}{\zeta} \right)^{\frac{1}{\alpha}} L \]

3. Intra-temporal Labor Supply condition

\[ \omega L^c = (1 - a) \left( \frac{a}{\zeta} \right)^{\frac{1}{\alpha}} L \]

4. Aggregate Production Function

\[ y = \left( 1 - \frac{a}{\zeta} \right) \left( \frac{a}{\zeta} \right)^{\frac{1}{\alpha}} L \]

5. Resource Constraint

\[ c + \delta z^0 = y \]

6. Growth equation (law of motion of productivity)

\[ g = z(\gamma - 1) \]

Other steady state variables can be backed out after solving this system. We look for steady state such that \( z \in (0, 1) \) and \( c \geq 0 \). 8 Parameters to be calibrated are \( \phi, \beta, \gamma, \delta, \chi, a, v, \omega \). Labor disutility parameter \( \omega \) is a normalizing constant. Choose \( \beta = 0.99, v = 2, \alpha = 0.33 \) (standard in the literature). Remaining 5 parameters are chosen to hit the following targets:

1. Growth rate of output of 2% (annualized).
2. Creative Destruction rate of 0.036 for \( z \) (Howitt 2000)
3. R&D to GDP ratio of 2.38% (NIPA 1953-2007)
4. Profits to GDP ratio of 6.2% (BEA, Quarterly Corporate Profits (after tax) to GDP ratio 1947-2007)
5. Steady State Labor normalized to 1.

Table 1 shows the calibrated parameters. We log-linearize the variables around the steady state as follows: for any variable \( x, \tilde{x}_t = \log \left( \frac{\bar{x}}{x} \right) \), where \( \bar{x} \) is the steady state. \( \hat{g}_{t+1} \) is the deviation of gross growth rate from the steady state value that is \( \hat{g}_{t+1} \equiv \log \left( \frac{1 + \hat{g}_{t+1}}{1 + \bar{g}} \right) \). For liquidity demand shock \( \hat{\xi}_t \equiv \epsilon_{A_t} \xi_t \) since the steady state value of the shock is 0.

**Definition A.3 (Approximate Equilibrium).** An approximate competitive equilibrium in this economy with endogenous growth is defined as a sequence of variables \( \{ \hat{\pi}_{W_t}, \hat{c}_t, \hat{g}_t, \hat{y}_t, \hat{L}_t, \hat{w}_t, \hat{N}_t, \hat{V}_t \} \) which satisfy the following equations, for a given sequence of exogenous shocks \( \{ \hat{\xi}_t, \hat{M}_t, \hat{\epsilon}_{t}, \hat{\lambda}_{aw} \} \).

Aggregate Demand:

\[ -(E_t \hat{\epsilon}_{t+1} + \hat{\epsilon}_t + \hat{\xi}_{t+1}) - \hat{i}_t - (E_t \hat{\pi}_{W_t} + \hat{\pi}_t) = 0 \] (37)
Endogenous growth equations:

\[(q - 1)\eta_{g} \delta t_{t+1} = - (E_{t} \delta_{t+1} - \delta_{t} + \delta_{t+1}) + E_{t} \delta_{t+1} \tag{38}\]

\[\dot{V}_{t} = \eta_{y} \delta_{t} - \eta_{g} \delta_{t+1} - \eta_{q} (E_{t} \delta_{t+1} - \delta_{t} + \delta_{t+1}) + \eta_{y} E_{t} \delta_{t+1} \tag{39}\]

where \(\eta_{y} = 1 - \frac{(1-q)\beta}{1+q} > 0\), \(\eta_{z} = \beta \frac{(1-q)}{1+q} > 0\), \(\eta_{q} = \frac{(1-q)\beta}{1+q} > 0\)

Market clearing:

\[\frac{c}{y} \dot{c}_{t} + \frac{R}{y} \eta_{g} \delta_{t+1} = \dot{\psi}_{t} \tag{40}\]

\[\dot{\psi}_{t} = \dot{M}_{t} + \dot{L}_{t} \tag{41}\]

Wage setting:

\[\hat{n}_{t}^{w} = \hat{\beta} E_{t} \hat{n}_{t+1} + \kappa_{w} [\hat{\omega}_{t} + v \hat{L}_{t} - \hat{\omega}_{t}] + \kappa_{w} \hat{\lambda}_{w} \tag{42}\]

\[\hat{\omega}_{t} = \hat{M}_{t} \tag{43}\]

\[\hat{n}_{t}^{w} = \hat{\omega}_{t} - \hat{\omega}_{t-1} + \hat{n}_{t} + \hat{\delta}_{t} \tag{44}\]

where \(\kappa_{w} \equiv \frac{(1-\theta_{w})(1-\beta_{w})}{\theta_{w}(1+v(1+\theta_{w}))} > 0\)

Monetary policy rule:

\[\hat{i}_{t} = \max \left( \frac{-1}{1+\gamma} \right) \psi_{r} \hat{n}_{t}^{w} + \psi_{g} \hat{L}_{t} + \hat{\epsilon}_{t} \tag{45}\]

B Impulse Responses under Taylor rule eq 27

We show the detailed derivation for impulse response under the Taylor rule eq 27 and liquidity demand shock. For monetary policy shock, productivity shock and markup shock, the proof is similar. Assume that the liquidity demand shock follows the AR(1) process:

\[\delta_{t} = \rho_{\delta} \delta_{t-1} + \hat{\delta}_{t} \]

Guess the solution takes the form:

\[
\hat{\delta}_{t} = \psi_{c} \hat{\delta}_{t-1} + \hat{\delta}_{t}
\]

From Euler equation, we get:

\[(1-\rho_{\delta}) \psi_{c} = - (\psi_{r} - \rho_{\delta}) \psi_{r} + \psi_{g} \psi_{y} - 1 \tag{46}\]

From the Endogeneous Growth equation:

\[(1-\rho_{\delta}) \psi_{c} + \rho_{\delta} \psi_{y} = [(q-1)\eta_{g} + 1] \psi_{g} \tag{47}\]

From the Resource constraint:

\[\frac{c}{y} \psi_{c} + \frac{R}{y} \psi_{g} = \psi_{y} \tag{48}\]

From the Wage Phillips curve

\[(1-\beta \rho_{\delta}) \psi_{p} = \kappa_{w} (\psi_{c} + v \psi_{y}) \tag{49}\]

From equations 47, 48, and 50, we can find a relation between \(\psi_{c}\) and \(\psi_{y}\). Rest of the system is pretty standard NK system where we can solve for \(\psi_{p}\) and \(\psi_{y}\) from equations 46 and 49 using:

\[
\psi_{c} = \frac{1-\eta_{V} \rho_{l} (e-1) \eta_{g} + 1}{\rho_{l} \frac{1}{\psi_{y}}} + \eta_{g} + \eta_{y} \frac{c}{y} \psi_{c} + \eta_{q} (1-\rho_{l}) \psi_{y} = A_{1} \psi_{y}; \quad 0 < A_{1} < 1
\]

\[
\psi_{y} = \frac{1-\eta_{V} \rho_{l} (e-1) \eta_{g} + 1}{\rho_{l} \frac{1}{\psi_{y}}} + \eta_{g} + \eta_{y} \frac{c}{y} \psi_{c} + \eta_{q} (1-\rho_{l}) \psi_{y} = A_{1} \psi_{y}; \quad 0 < A_{1} < 1
\]
We get:
\[
\psi_p = \frac{\kappa (A_1 + \nu)}{1 - \beta \rho_i} \psi y = A_2 \psi y
\]
And thus:
\[
\psi y = \frac{-1}{(1 - \rho_i)A_1 + (\phi_\pi - \rho_i)A_2 + \phi y} < 0
\]
Further, from the resource constraint we find:
\[
\psi_y = \frac{\psi y - \frac{\phi}{\gamma} \psi c}{\frac{\phi}{\epsilon} \psi y} = \frac{1 - \frac{\phi}{\epsilon} A_1}{\frac{\phi}{\epsilon} \psi y}
\]
Since \(A_1 < 1\), it follows that \(A_1 < \frac{\phi}{\epsilon}\). Hence there is a positive co-movement of output and growth rate under liquidity demand shock. Further it must be that the following holds
\[
\psi_y = \frac{(\phi - 1) \psi y + 1) \psi y - (1 - \rho_i) \psi c}{\rho_i} = \frac{\psi y - (\psi y - \frac{\phi}{\epsilon} A_1 \psi c_1 - (1 - \rho_i) \psi c}{1 - \psi y - \rho_i}
\]

C Solution to Social Planner’s Problem

C.1 Social Planner problem I

The Social Planner chooses \{C_t, L_t, A_{t+1}, z_t\} to maximize the welfare function:
\[
\max \log C_t - \frac{\omega}{1 + \nu} t^{1 + \nu}
\]
subject to the constraints:
\[
C_t + RD_t = a \frac{A_t}{\gamma - 1} (1 - a) A_t L_t
\]
\[
\frac{A_{t+1} - A_t}{A_t} = (\gamma - 1) z_t
\]
\[
R_t = \delta \frac{A_t}{A_t} z_t
\]
\[
z_t \geq 0
\]
Combining the constraints and using the functional form for R&D Investment, we get:
\[
C_t + \delta \left( \frac{A_{t+1} - A_t}{A_t} \right)^{\epsilon} A_t = a \frac{\psi c_1}{\psi c_1} (1 - a) A_t L_t
\]
Let \(\lambda_t\) be the Lagrange multiplier on the constraint. Solution to this problem is thus:
\[
\lambda_t = \frac{1}{C_t}
\]
\[
L_t^\psi C_t = \frac{\psi c_1}{\psi c_1} (1 - a)
\]
\[
\frac{-\lambda_t \delta \eta}{\gamma - 1} \left( \frac{A_{t+1} - A_t}{A_t} \right)^{\epsilon - 1} A_t + \frac{\lambda_{t+1} \beta \delta \eta}{\gamma - 1} \left( \frac{A_{t+2} - A_{t+1}}{A_{t+1}} \right)^{\epsilon - 1} A_{t+1} \frac{A_{t+2} - A_{t+1}}{A_{t+1}} + \frac{a \frac{\psi c_1}{\psi c_1} (1 - a) L_{t+1} \lambda_{t+1} \beta}{\psi c_1} = 0
\]
Since growth rate is defined as \(g_{t+1} = \frac{A_{t+1} - A_t}{A_t}\), we can rewrite the above condition as:
\[
\frac{C_{t+1} \delta \eta}{\gamma - 1} \left( \frac{g_{t+2} - g_{t+1}}{g_{t+1}} \right)^{\epsilon - 1} = \frac{\frac{\beta \delta \eta}{\gamma - 1} \left( \frac{g_{t+2} - g_{t+1}}{g_{t+1}} \right)^{\epsilon - 1} \left( \frac{g_{t+2} - g_{t+1}}{g_{t+1}} \right) \delta \eta}{\gamma - 1} + \frac{a \frac{\psi c_1}{\psi c_1} (1 - a) L_{t+1} \gamma - 1}{\psi c_1}
\]
This can be rewritten as:
\[
\frac{C_{t+1} \delta \eta}{\gamma - 1} \left( \frac{g_{t+2} - g_{t+1}}{g_{t+1}} \right)^{\epsilon - 1} = \frac{\beta \delta \eta}{\gamma - 1} \left( \frac{g_{t+2} - g_{t+1}}{g_{t+1}} \right)^{\epsilon - 1} + \frac{a \frac{\psi c_1}{\psi c_1} (1 - a) L_{t+1} \gamma - 1}{\psi c_1}
\]
This the Euler equation for R&D investment in the Social Planner’s allocation. The right hand side gives the return
on R&D investment. Writing the LHS in normalized terms i.e. \( C_t = c_t A_t \), we get

\[
c_{t+1} \left( 1 + \frac{\delta}{\delta t+1} \right) = \beta \left[ \left( \frac{\delta}{\delta t+1} \right)^{-1} - \frac{1 - \theta}{\theta} \frac{\delta}{\delta t+2} \left( \frac{\delta}{\delta t+1} \right)^{\alpha - 1} + \frac{\alpha}{\theta} (1 - \alpha) L_t \left[ \gamma - (\gamma - 1) \right] \right] (51)
\]

The (interior) equilibrium (with positive growth) is thus given by the sequence of three variables \( \{c_t, L_t, \delta t+1\} \) such that equation 51 and following two conditions (intra-temporal labor supply and budget constraint) are satisfied:

\[
\omega L_t c_t = a \frac{\delta}{\delta t+1} (1 - \alpha) (52)
\]

\[
c_t + \delta \left( \frac{\delta}{\delta t+1} \right)^{\alpha} = a \frac{\delta}{\delta t+1} (1 - \alpha) L_t (53)
\]

C.2 Policy Relevant Welfare Function

The representative agent’s lifetime welfare function at time \( t \) can be rewritten as

\[
V_t = \sum_{s=t}^{\infty} \beta^{s-t} [\log C_s - v(L_s)] = \sum_{s=t}^{\infty} \beta^{s-t} \left[ \log c_s - v(L_s) + \frac{\beta}{1 - \beta} \log(1 + \delta t+1) \right] + \frac{1}{1 - \beta} \log A_t
\]

We redefine the terms in the square brackets as the policy relevant per period welfare function:

\[
W_t = \log c_t - v(L_t) + \frac{\beta}{1 - \beta} \log(1 + \delta t+1)
\]

Thus the policy relevant lifetime welfare function is given by

\[
W_t = \sum_{s=t}^{\infty} \beta^{s-t} \left[ \log c_s - v(L_s) + \frac{\beta}{1 - \beta} \log(1 + \delta t+1) \right]
\]

C.3 Social Planner problem II

The Social Planner chooses \( \{c_t, L_t, \delta t+1, z_t\} \) to maximize lifetime-policy relevant welfare function:

\[
\max \sum_{s=t}^{\infty} \beta^{s-t} \left[ \log c_s - \frac{\omega}{1 + \nu} L_{s+1}^{1+\nu} + \frac{\beta}{1 - \beta} \log(1 + \delta t+1) \right]
\]

subject to

\[
c_t + \delta L_t = a \frac{\delta}{\delta t+1} (1 - \alpha) L_t = y_t
\]

\[
\delta t+1 = z_t (\gamma - 1)
\]

\[
\delta t+1 = \delta z_t^0
\]

\[
z_t \geq 0
\]

Solution (for \( z > 0 \)) is given by:

\[
\frac{\delta t' (z_t)}{c_t} = (\gamma - 1) \frac{\beta}{1 - \beta} \frac{1}{1 + \delta t+1}
\]

\[
\omega L_t^t = a \frac{\delta}{\delta t+1} (1 - \alpha)
\]

\[
c_t + \delta L_t = a \frac{\delta}{\delta t+1} (1 - \alpha) L_t = y_t
\]

\[
\delta t+1 = \delta z_t^0
\]

\[
\delta t+1 = z_t (\gamma - 1)
\]

Substituting out for research intensity \( z_t \) in terms of growth rate and using the functional form for R&D Investment, Solution is given by Intra-temporal labor supply condition eq 52, Budget constraint eq 53 and the following R&D investment condition:

\[
\omega \delta \left( \frac{\delta}{\delta t+1} \right)^{\alpha - 1} = (\gamma - 1) \frac{\beta}{1 - \beta} \frac{1}{1 + \delta t+1}
\]
C.4 Equivalence of two solutions

It is clear that Euler condition derived in eq 51 is not as amenable to analytical manipulations as is the corresponding R&D investment condition eq 54 derived under the modified Social Planner problem II. Remains to be shown that the resulting equilibrium is identical in both scenarios.

In Steady State eq 51 simplifies to:

\[ (1 + g) = \beta \left( 1 - \frac{1-q}{q} \right) + a \frac{\bar{\tau}_t (1 - \alpha)}{\omega c} \frac{L(\gamma - 1)^{\bar{\tau}}}{\bar{\epsilon}^{\bar{\tau}} - 1} \]

It is straightforward to show that eq 54 combined with eq 53 also yields the above condition. Thus, the solutions are identical at the steady state.\(^4\) As regards the dynamics away from the steady state, eq 51 can be rewritten as:

\[ \frac{c_{t+1}(1 + g_{t+1})}{c_t} = \beta \left[ \frac{(g_{t+2})^{e_{t+1}}}{(1 + g_{t+2})} + \frac{c_{t+1} (\gamma - 1)^{\bar{\tau}}}{\bar{\epsilon}^{\bar{\tau}} - 1} \right] \]

From eq 54, we can write out the RHS of the above equation 55 as

\[ \frac{c_{t+1}(1 + g_{t+1})}{c_t} = \left( \frac{g_{t+2}}{g_{t+1}} \right)^{e_{t+1}} (1 + g_{t+2}) \]

Thus, it remains to show that the LHS of two equations 55 and 56 are equal. We prove by reduction. Substitute LHS of equation 56 into RHS of eq 55 to get:

\[ \left( \frac{g_{t+2}}{g_{t+1}} \right)^{e_{t+1}} (1 + g_{t+2}) = \beta \left[ \frac{(g_{t+2})^{e_{t+1}}}{(1 + g_{t+2})} + \frac{c_{t+1} (\gamma - 1)^{\bar{\tau}}}{\bar{\epsilon}^{\bar{\tau}} - 1} \right] \]

Simple algebraic manipulation yields:

\[ (1 - \beta) \left( \frac{g_{t+2}}{g_{t+1}} \right)^{e_{t+1}} (1 + g_{t+2}) = \beta \frac{c_{t+1} (\gamma - 1)^{\bar{\tau}}}{\bar{\epsilon}^{\bar{\tau}} - 1} \]

which can be simplified to yield:

\[ \frac{1 - \beta}{\beta} \frac{\phi \delta}{\gamma - 1} \left( \frac{g_{t+2}}{g_{t+1}} \right)^{e_{t+1}} (1 + g_{t+2}) = c_{t+1} \]

which is true since it is eq 54 forwarded by one period. Since we do not use the labor-supply intra-temporal condition to show the equivalence between the two solutions under flexible wages, the two approaches are also equivalent under nominal wage rigidities which introduces a wedge in the labor-supply intra-temporal condition.

C.5 Efficient Steady State

Efficient Steady State is given by following system of equations in three variables \( c, L, g \):

\[ L = \left[ \frac{a \frac{\bar{\tau}_t (1 - \alpha)}{\omega c}}{\omega c} \right]^{\frac{1}{\gamma - 1}} \quad c = \frac{\phi \delta}{\gamma - 1} \frac{1 - \beta}{\beta} (1 + g) \left( \frac{g}{\gamma - 1} \right)^{e_{t+1}} \]

\[ c + \delta \left( \frac{g}{\gamma - 1} \right)^{e_{t+1}} = a \frac{\bar{\tau}_t (1 - \alpha)}{\omega \chi_1 (1 + g)} \]

When \( q = 1 \), the solution is given by a fixed point of the following equation:

\[ \chi_1 (1 + g) + \frac{\delta}{\gamma - 1} = \left[ \frac{a \frac{\bar{\tau}_t (1 - \alpha)}{\omega c}}{\omega \chi_1 (1 + g)} \right]^{\frac{1}{\gamma - 1}} \quad \text{where} \quad \chi_1 = \frac{\delta}{\gamma - 1} \frac{1 - \beta}{\beta} \]

\[ \frac{\delta c (\gamma - 1)^{\bar{\tau}}}{\phi \delta - \gamma - 1} = (1 - \beta)(1 + g); \quad c = a \frac{\bar{\tau}_t (1 - \alpha)}{\omega \chi_1 (1 + g)} \]

These yield the above Euler equation.

A.7
The LHS is a linear monotonically increasing function of \( g \). RHS is a monotonically decreasing function of \( g \). By single crossing, one can show that there is a unique locally determinate solution for a given condition on \( \chi_1 \). For higher values of \( \varrho \), numerically we verify local determinacy.

C.6 Unconventional Policy away from the ZLB: Implementable Allocation

Now we show that the first-best equilibrium allocation can be implemented as the competitive equilibrium using the time-varying fiscal and monetary instruments - nominal interest rate \( i_t \), Tax on interest income \( \tau^b_t \), Tax on intermediate goods \( \tau^p_t \), Research subsidy for entrepreneurs \( \tau^r_t \) and Labor tax for household \( \tau^w_t \) as follows:

\[
\tau^b_t = \xi^t \frac{1}{\beta} - \frac{1}{\alpha} (1 + g)^{-1} \eta^t \frac{1}{1 + \eta^t} \\
\tau^p_t = 1 - \frac{1}{\alpha} \\
\tau^w_t = \lambda^t \\
1 - \tau^r_t = 1 - \frac{1}{\beta} \frac{1}{\gamma - 1} c_t \tilde{V}_{t+1} \\
\tau^m_t = 0
\]

and the nominal interest rate is set such that \( W_t = \bar{\pi}_t W - 1 \) - consistent with perfect nominal wage inflation targeting.

Proof. Follows from comparing the system of equations derived under first-best allocation in Appendix C.3 and the (normalized) competitive equilibrium defined in Definition A.2.

C.7 Unconventional Fiscal Policy at the ZLB: Implementable Allocation

At the zero lower bound, the nominal interest rate is stuck at 0. However, the first best can still be implemented using the tax subsidy on interest income \( \tau^b_t \), Tax on intermediate goods \( \tau^p_t \), Research subsidy for entrepreneurs \( \tau^r_t \) and Labor tax for household \( \tau^w_t \) as follows:

\[
\tau^b_t = \xi^t \frac{1}{\beta} c_t^{-1} (1 + g_t + 1)^{-1} \eta^t \\
\tau^p_t = 1 - \frac{1}{\alpha} \\
\tau^w_t = \lambda^t \\
1 - \tau^r_t = 1 - \frac{1}{\beta} \frac{1}{\gamma - 1} c_t \tilde{V}_{t+1} \\
\tau^m_t = 0
\]

and the nominal interest rate is set such that \( W_t = \pi^t W \) - consistent with perfect nominal wage inflation targeting.

Proof. Follows from comparing the system of equations derived under first-best allocation in Appendix C.3 and the (normalized) competitive equilibrium defined in Definition A.2.

As in Correia et al. (2013), it can be shown that the resulting equilibrium is revenue-neutral and time-consistent.

We can re-define the first-best allocation as the equilibrium allocation defined in Definition 1 such that the government provides the time-varying fiscal and monetary instruments listed in eq 57-60.

C.8 Approximate First-Best Equilibrium

We log-linearize the non-linear equilibrium conditions around the non-stochastic efficient steady state. Approximate first-best equilibrium is given by a sequence of 4 quantities: \( \{\tilde{L}_t, \tilde{c}_t, \tilde{g}_t, \tilde{\delta}_t\} \) that solve the following equations for a given exogenous process of shocks \( M_t \):

\[
v \tilde{L}_t + \tilde{c}_t = M_t \\
(\varrho - 1) \eta^t \tilde{S}_{t+1} = \tilde{c}_t - \tilde{g}_t \\
\frac{c}{y} \tilde{c}_t + \frac{rd}{y} \theta^t \eta^t \tilde{S}_{t+1} = \tilde{g}_t \\
\tilde{M}_t + \tilde{L}_t = \tilde{g}_t
\]
Efficient solution
The above system can be solved to derive the following closed form solution:

\[ S_{t+1}^f = \psi^*_g \hat{M}_{t}; \quad \ell_t^f = \psi^*_y \hat{M}_{t}; \quad \hat{y}_f^t = \psi^*_g \hat{M}_{t}; \quad \hat{l}_t^f = \psi^*_t \hat{M}_{t} \]

where \( \psi^*_g = \frac{1+\rho}{(\varphi g + 1) + \psi_g \varphi g} > 0 \),

\( 0 < \psi^*_y = \frac{(\nu g + 1) \psi^*_g}{\psi_g \varphi g} > 1 \),

\( \psi^*_y = \frac{\psi^*_g + rd \psi g \psi^*_g \varphi g}{\psi g \varphi g} > 0 \), and

\( \psi^*_t = \frac{1-\varphi g}{\varphi g} > 0 \)

C.9 Time-t vs. time-0 flexibility
There are two concepts of flexibility in the presence of a pre-determined state variable. One is the Neiss and Nelson (2003) definition of flexible wages, under which wages have been been flexible since time 0 and remain flexible indefinitely. Wages set under this concept are called time-0 flexible wages. Based on two concepts of flexible wages, there are time-0 first best, time-0 natural rate, time-t first best and time-t natural rate allocations.

Since the normalized equilibrium can be written without any reference to the level of productivity \( A_t \), the normalized allocations based on the two flexibility concepts coincide.

Definition C.1 (normalized natural rate allocation). The normalized natural rate allocation is given by a sequence of variables \( \{\ell_t^f, g_t^f, S_{t+1}^f, V_{t+1}^f\} \) such that these satisfy the following equations for a given sequence of shocks \( \{\hat{S}_t^f, \hat{c}_t^f, \hat{M}_t, \hat{\lambda}_{wt}\} \):

\[ \ell_t^f + v g_t^f - (1 + v) \hat{M}_t + \hat{\lambda}_{wt} = 0 \tag{65} \]

\[ \frac{c}{y} \ell_t^f + \frac{rd}{y} \psi g \hat{S}_{t+1}^f = \hat{y}_f^t \tag{66} \]

\[ (\nu - 1) \psi g \hat{S}_{t+1}^f = -(\psi g \hat{c}_{t+1}^f - \psi g \hat{c}_t^f + \hat{S}_{t+1}^f) + \hat{\omega}_g \hat{g}_{t+1}^f \tag{67} \]

\[ \hat{V}_{t+1}^f = \hat{\eta}_g \hat{g}_{t+1}^f - \hat{\eta}_g \hat{S}_{t+1}^f + \hat{\eta}_g (\psi g \hat{c}_{t+1}^f - \psi g \hat{c}_t^f + \hat{S}_{t+1}^f) + \hat{\eta}_v \hat{V}_{t+1}^f \tag{68} \]

In other words, if \( x_t^f = [\ell_t^f, g_t^f, S_{t+1}^f, V_{t+1}^f, \hat{V}_{t+1}^f] \) is vector of endogenous variables and \( e_t = [\hat{S}_t^f, \hat{c}_t^f, \hat{M}_t, \hat{\lambda}_{wt}] \) is a vector of shocks, then there is a unique flexible allocation independent of history of nominal distortions and is given by the solution of following rational expectations system:

\[ F x_{t+1}^f + G x_t^f + H e_t = 0 \tag{69} \]

where \( F, G, \) and \( H \) are matrices of coefficients corresponding to definition A.2. Using a standard rational expectations solution method, the system can be solved as:

\[ x_t^f = M e_t \tag{70} \]

Therefore the major difference that these two flexibility concepts generate in the context of our framework is that under time-t flexibility setting, productivity \( A_{t,-\infty}^f \) is a hypothetical construct that would have occurred had prices and wages been flexible since the beginning of time. Under time-t flexibility, the level of productivity \( A_{t,t}^f \) is the pre-determined level of productivity corresponding to the data \( A_{data}^t \). Following are the law of motions of the two productivity concepts:

\[ A_{t+1}^f = A_{t,-\infty}^f (1 + \hat{S}_{t+1}^f) \]

\[ A_{t+1}^f = A_{data}^f (1 + \hat{S}_{t+1}^f) \]

where \( A_{t,-\infty}^f \) is the level of productivity under flexible wages at time \( t \) when wages have been flexible since the infinite past. \( A_{data}^t \) is the level of productivity given by the Definition A.1 of the competitive equilibrium and \( S_{t+1}^f \) is the flexible-wage productivity growth rate solved in the system 70.

C.9.1 time-0 allocations
We can therefore define the time-0 allocations as follows:
Definition C.2 (time-0 first-best allocation). The \textit{time-0 first best allocation} is defined as sequence of variables \(\{Y_{t}^{*}, \quad A_{t+1}^{*}, \quad C_{t+1}^{*}, \quad \hat{t}_{t+1}, \quad \hat{g}_{t+1}, \quad \hat{L}_{t+1}\}\) which satisfy the equations 61-64 and the following equations, given a sequence of shocks \(\{\hat{e}_{t}, \hat{e}_{t}, \hat{M}_{t}, \hat{\lambda}_{w,t}\}\) and initial level of productivity \(A_{0}\):

\[
\begin{align*}
A_{t+1}^{*} &= A_{t}^{*} + \log(1 + g_{ss}) \\
Y_{t}^{*} &= Y_{t}^{*} + \log(y_{ss}) \\
C_{t}^{*} &= C_{t}^{*} + \log(c_{ss})
\end{align*}
\]

Definition C.3 (time-0 natural rate allocation). The \textit{time-0 natural rate allocation} is defined as sequence of variables \(\{Y_{t}^{f}, \quad A_{t+1}^{f}, \quad C_{t+1}^{f}, \quad \hat{c}_{t}, \quad \hat{g}_{t}, \quad \hat{L}_{t}\}\) which satisfy the equations 65-68 and the following equations, given a sequence of shocks \(\{\hat{s}_{t}, \hat{e}_{t}, \hat{M}_{t}, \hat{\lambda}_{w,t}\}\) and initial level of productivity \(A_{0}\):

\[
\begin{align*}
A_{t+1}^{f} &= A_{t}^{f} + \log(1 + g_{ss}) \\
Y_{t}^{f} &= Y_{t}^{f} + \log(y_{ss}) \\
C_{t}^{f} &= C_{t}^{f} + \log(c_{ss})
\end{align*}
\]

C.9.2 time-\(t\) allocations

Similarly, we define the time-\(t\) allocations as follows:

Definition C.4 (time-\(t\) first-best allocation). The \textit{time-\(t\) first best allocation} is defined as sequence of variables \(\{Y_{t}^{*}, \quad A_{t+1}^{*}, \quad C_{t+1}^{*}, \quad \hat{c}_{t}, \quad \hat{g}_{t}, \quad \hat{L}_{t}\}\) which satisfy the equations 61-64 and the following equations, given a sequence of shocks \(\{\hat{e}_{t}, \hat{e}_{t}, \hat{M}_{t}, \hat{\lambda}_{w,t}\}\) and the actual level of productivity at date \(t\), \(A_{t}^{data}\):

\[
\begin{align*}
A_{t+1}^{*} &= A_{t}^{data} + \log(1 + g_{ss}) \\
Y_{t}^{*} &= Y_{t}^{data} + \log(y_{ss}) \\
C_{t}^{*} &= C_{t}^{data} + \log(c_{ss})
\end{align*}
\]

Definition C.5 (time-\(t\) natural rate allocation). The \textit{time-\(t\) natural rate allocation} is defined as sequence of variables \(\{Y_{t}^{f}, \quad A_{t+1}^{f}, \quad C_{t+1}^{f}, \quad \hat{c}_{t}, \quad \hat{g}_{t}, \quad \hat{L}_{t}\}\) which satisfy the equations 65-68 and the following equations, given a sequence of shocks \(\{\hat{e}_{t}, \hat{e}_{t}, \hat{M}_{t}, \hat{\lambda}_{w,t}\}\) and the actual level of productivity at date \(t\), \(A_{t}^{data}\):

\[
\begin{align*}
A_{t+1}^{f} &= A_{t}^{data} + \log(1 + g_{ss}) \\
Y_{t}^{f} &= Y_{t}^{data} + \log(y_{ss}) \\
C_{t}^{f} &= C_{t}^{data} + \log(c_{ss})
\end{align*}
\]

C.9.3 sticky-wage allocation

Definition C.6 (sticky-wage allocation). The \textit{sticky-wage allocation} is defined as sequence of variables \(\{Y_{t}, \quad A_{t+1}, \quad C_{t}, \quad \hat{\pi}_{t}, \quad \hat{c}_{t}, \quad \hat{g}_{t}, \quad \hat{L}_{t}, \quad \hat{\nu}_{t}, \quad \hat{\lambda}_{t}, \quad \hat{V}_{t}\}\) which satisfy the equations 37-45 and the following equations, given a sequence of shocks \(\{\hat{e}_{t}, \hat{e}_{t}, \hat{M}_{t}, \hat{\lambda}_{w,t}\}\) and initial level of productivity \(A_{0}\):

\[
\begin{align*}
A_{t+1} &= A_{t} + \log(1 + g_{ss}) \\
Y_{t} &= Y_{t} + \log(y_{ss}) \\
C_{t} &= C_{t} + \log(c_{ss})
\end{align*}
\]

\(A_{t}^{data}\) corresponds to \(A_{t}\) defined under the sticky-wage allocation.
D Proposition Proofs

Proposition (Proposition 1: Output hysteresis). Given the monetary policy rule (eq 27) and in the absence of a zero lower bound constraint on the nominal interest rate, transitory (modeled as AR(1) process) liquidity demand shocks or monetary policy shocks induce a permanent deviation in the time series of output from the counterfactual (flexible wage-) level of output if and only if monetary policy is not a strict targeting rule i.e.

\[ Y_T \neq Y_T^e \iff \{ \varphi_T, \varphi_y > 0 : \varphi_T \not\to \infty \cup \varphi_y \not\to \infty \} \]

where \( 1 < T < \infty \) such that \( Y_T \equiv \frac{Y_T}{Y_T^e} = y \) (steady state value).

Proof. We give the proof for liquidity demand shocks. The proof is identical for monetary policy shocks. Note that

\[ Y_T^e = (1 + g_{ss})^T A_0 y; \quad Y_T = \prod_{k=0}^{T-1} (1 + g_k) A_0 y \]

Taking a log difference in the two series

\[
\log Y_T - \log Y_T^e = \sum_{k=0}^{T-1} \delta_{k+1} = \psi_S^k \sum_{k=0}^{T-1} \epsilon_k
\]

where \( \psi_S^k \) is the coefficient derived in Appendix B above. For a given sequence of shocks that does not add to zero (which is the case with AR(1) process), the difference in the two series depends on \( \psi_S^k \). This parameter is 0 if and only if monetary policy rule is either a strict inflation targeting (\( \varphi_T \to \infty \)) or a strict employment targeting rule \( \varphi_y \to \infty \).

\[ \square \]

Proposition (Proposition 2: Output Hysteresis at the ZLB). Given the monetary policy rule (eq 27), a positive shock to liquidity demand such that the zero lower bound is binding for finite time \( T^e \) results in a permanent gap in output from the flexible wage counterfactual.

Proof. A positive shock to the liquidity demand that induces the ZLB under the Taylor rule results in wage deflation and drop in output for the duration of ZLB. Under Eggertsson and Woodford (2003) two-state Markov Chain assumption, the system at time \( t < T^e \) is in state \( S \) (short run) and can be expressed as:

\[
(1 - \mu) \hat{c}_S = \mu \hat{e}_S + \hat{f}_S \\
(1 - \beta \mu) \hat{r}_S^y = \kappa_w (\hat{c}_S + \nu \hat{g}_S) \\
[(e - 1) \eta_S + 1] \hat{g}_S = \mu \hat{V}_S + (1 - \mu) \hat{c}_S \\
\frac{r_d}{\nu} \hat{g}_S \hat{g}_S = \hat{y}_S - \frac{c}{\nu} \hat{c}_S
\]

\[ \hat{V}_S = \frac{1}{1 - \eta \nu \mu} \left[ \eta V \hat{y}_S + \eta_d (1 - \mu) \hat{c}_S - (\eta + \eta_d) \hat{g}_S \right] \]

We can solve the last three equations to find a relationship between \( c \) and \( y \):

\[ \hat{c}_S = \eta c \hat{g}_S; \quad \eta c \equiv \frac{1 - \eta \nu \mu (e - 1) \eta_d + 1}{\eta \nu \mu} + \frac{\eta + \eta_d}{\eta \nu \mu} - \frac{\eta}{\eta \nu \mu} < 1 \]

We can solve the system for \( t < T^e \):

\[ \hat{g}_t = \psi_y \hat{r}_S^y < 0; \hat{r}_S^y = \psi_y \hat{r}_S^y < 0; \hat{g}_t = \psi_y \hat{r}_S^y < 0 \]

where \( \psi_y = \frac{(1 - \beta \mu) \eta c}{1 - (1 - \beta \mu)(1 - \nu) - \kappa_w (1 - \beta \mu) \psi_y} \) > 0, \( \psi_y = \frac{\kappa_w (1 - \beta \mu) \psi_y}{1 - \beta \mu} > 0 \), and \( \psi_y = \frac{1 - \mu}{\eta \nu \mu} \psi_y > 0 \). We assume (by A2 in the main text) the system is locally determinate around the state \( S \) equilibrium defined above. Therefore using the accounting identity eq 71 derived in the proof of Proposition 1, we can derive:

\[ \log Y_t - \log Y_t^e = \sum_{k=0}^{T-1} \delta_{k+1} = (T^e - 1) \psi_S^k \hat{r}_S^y < 0; \quad \forall t \geq T^e \]

A.11
This is the permanent output hysteresis in our framework following a ZLB episode.

**Proposition (Proposition 3: Steady State Efficiency).** Assuming the policy maker has access to non-distortionary lump-sum taxes, the steady state of the competitive equilibrium can be made efficient using the following three fiscal tools:

- a) sales subsidy \( \tau' = 1 - \frac{1}{a} \)
- b) wage tax cut \( \lambda_w \)
- c) research tax/subsidy \( \tau' = 1 - \left[ \left( \frac{(1-a)\lambda(1-\beta)}{1-\beta(1-\xi)} \right) \left( \frac{1-\beta}{(1-\xi)} \right) \right], \) where terms with * denote the efficient steady state values.

**Proof.** Follows from Appendix C.6 above.

**Proposition (Proposition 4).** The (time-0) natural rate allocation coincides with the (time-0) first-best allocation under liquidity demand and monetary policy shocks.

**Proof.** From Appendix C.9, the time-0 natural rate allocation under liquidity demand shocks and monetary policy shocks is characterized by:

\[
\begin{align*}
\gamma'_t &= 0, \quad \ell'_t = 0, \quad \delta^*_{t+1} = 0; \quad \forall t \geq 0
\end{align*}
\]

Because of the presence of time-varying taxes, the time-0 first-best allocation has the same solution for the corresponding variables \( \{\gamma'_t, \ell'_t, \delta^*_{t+1}\} \). Hence, output at any time under (time-0) natural rate and (time-0) first-best allocations coincide (follows from the accounting identity eq 71). Moreover, time-t natural rate and time-t first best allocations also coincide with each other.

**Proposition (Proposition 5).** Assume that the economy is at the efficient steady state at time \( t = 0 \), with given productivity level \( A_0 \). Under sticky wage allocation, quadratic approximation of representative agent’s lifetime utility function \( W_0 \) around the non-stochastic efficient steady state is given by

\[
\frac{W_0 - W_0^*}{U_{c_a} y_{ss}} = -\frac{1}{2} \sum_{t=0}^{\infty} \beta^t \left[ \lambda_y \left( \tilde{g}_t - \frac{1}{1 - \beta} \frac{1}{\psi + \frac{\beta}{1 + \psi}} \delta^*_{t+1} \right) + \lambda_x \delta^2_{t+1} \right] + \lambda_{\pi} \left( \frac{\pi_w^m}{\pi_w^*} \right)^2 + O(\|\tilde{g}_t, \ell_t^*\|^3) + \text{i.t.p.}
\]

where \( \lambda_y = (v + \frac{\beta}{\psi}) > 0, \lambda_x = \frac{\beta}{\psi} \left[ \frac{v}{(v + 1) + \frac{\beta}{1 + \psi}} + [(\psi - 1)\eta_g + 1] \right] > 0, \lambda_{\pi} = \frac{1 + \lambda_w}{\lambda_w} \frac{1}{\psi} > 0, \kappa_w = \frac{(1 - \beta)(1 - \beta\psi)}{(1 + \psi)(1 + 1 + \frac{\beta}{1 + \psi})} > 0. \eta_g = \frac{1 + \frac{\beta}{\psi}}{\beta} > 1 \) and t.i.p. stands for “terms independent of policy”. \( W^* \) denotes welfare under the (time-0) first-best allocation. The approximation is scaled by the constant \( U_{c_a} y_{ss} = \frac{\kappa_w}{\psi} \) (evaluated at the efficient steady state).

**Proof.** The proof for this is detailed and builds on results shown above. First, note from Appendix C.4 that the solution of welfare function of the representative household is equivalent to the solution of the policy-relevant welfare function derived in Appendix C.2.

We then derive a quadratic approximation of the policy-relevant lifetime welfare function. Since the problem is relatively complicated, we break the approximation into first solving for a setting with flexible wages. We show the derivation in the case of flexible wages, i.e. no pricing distortions, in Lemma 1 below. This simplifies the exposition. It is relatively standard to extend this proof to include nominal wage setting frictions. The extended proof is similar to the textbook proof of Gali (2015, Ch. 4) and is available on request.

**Lemma 1.** Quadratic approximation of \( W_t \) under flexible wages is given by

\[
-\frac{1}{2} \left[ \lambda_y \left( \tilde{g}_t - \tilde{g}^*_t \right) + \lambda_x \left( \tilde{\delta}_{t+1} - \delta^*_{t+1} \right) \right] + h.o.t. + \text{i.t.p.}
\]

**Proof.** We will make use of following two approximation results as in Erceg Henderson Levin 2000:

\[
\frac{dx}{x} \approx \ddot{x} + \frac{1}{2} \dot{x}^2, \quad \ddot{x} = \ln x - \ln \ddot{x}
\]

A.12
If \( x = \left[ \int_0^1 x(j) \phi \, dj \right]^{\frac{1}{2}} \), the logarithmic approximation of \( x \) is

\[
\hat{x} \approx \int_0^1 \hat{x}(j) \, dj + \frac{1}{2} \varphi \hat{x}(j) = \int_0^1 \hat{x}(j) \, dj + \frac{1}{2} \phi \left[ \int_0^1 \hat{x}(j)^2 \, dj - \left( \int_0^1 \hat{x}(j) \, dj \right)^2 \right]
\]

Writing the per period utility as sum of three components:

\[
W_t = u(c_t) - \int_0^1 v(L_t(h)) \, dh + \frac{\beta}{1 - \beta} w(g_{t+1})
\]

At the Efficient Steady state,

\[
y = \alpha \hat{x} = (1 - \alpha) L_t, \quad \omega = \frac{y}{c} L^{1+v}
\]

\[
c + \delta \left( \frac{\phi}{\gamma - 1} \right)^e = y
\]

\[
u_t = \omega L^v = \frac{1}{c}; v_g = \frac{\omega v L^{1+v}}{y^2} = \frac{\nu}{y c}
\]

\[
w_g = \frac{\beta}{1 - \beta} \frac{1}{1 + g}; w_{gg} = -\frac{\beta}{1 - \beta} \frac{1}{(1 + g)^2}
\]

Second Order approximation of individual components of the welfare function is given by:

\[
\hat{u}_t = \hat{u}_t + y u_y \frac{d\hat{u}_t}{y} + (1 + g) u_{\hat{g}_{t+1}} \frac{d\hat{g}_{t+1}}{1 + g} + y(1 + g) \frac{d\hat{u}_t}{y} \frac{d\hat{g}_{t+1}}{1 + g} + \frac{y^2}{2} u_{gg} \left( \frac{d\hat{u}_t}{y} \right)^2 + \frac{(1 + g)^2}{2} u_{gg} \left( \frac{d\hat{g}_{t+1}}{1 + g} \right)^2 + \text{h.o.t.}
\]

\[
v_t = \sigma + y v_y \frac{d\hat{u}_t}{y} + \frac{y^2}{2} v_{yy} \left( \frac{d\hat{u}_t}{y} \right)^2 + \text{h.o.t.}
\]

\[
w_t = \omega + (1 + g) w_{\hat{g}_{t+1}} \frac{d\hat{g}_{t+1}}{1 + g} + \frac{(1 + g)^2}{2} w_{gg} \left( \frac{d\hat{g}_{t+1}}{1 + g} \right)^2 + \text{h.o.t.}
\]

Using the Taylor approximation result that

\[
\frac{dx}{x} = \hat{x} + \frac{1}{2} \hat{x}^2
\]

where \( \hat{x} = \log(x) - \log(x_{ss}) \), we can write down the quadratic approximation as:

\[
\hat{u}_t = y u_y \left[ \hat{g}_{t+1} + \frac{1}{2} \hat{g}_{t+1}^2 \right] + (1 + g) u_g \left[ \hat{g}_{t+1} + \frac{1}{2} \hat{g}_{t+1}^2 \right] + y(1 + g) \hat{g}_{t+1} \left[ \hat{g}_{t+1} + \frac{1}{2} \hat{g}_{t+1}^2 \right] + \frac{y^2}{2} u_{yy} \hat{g}_{t+1}^2 + \frac{(1 + g)^2}{2} u_{gg} \hat{g}_{t+1}^2 + \text{h.o.t.} + \text{t.i.p.}
\]

\[
v_t = y v_y \left[ \hat{g}_{t+1} + \frac{1}{2} \hat{g}_{t+1}^2 \right] + \frac{y^2}{2} v_{yy} \hat{g}_{t+1}^2 + \text{h.o.t.} + \text{t.i.p.}
\]

\[
w_t = (1 + g) w_g \left[ \hat{g}_{t+1} + \frac{1}{2} \hat{g}_{t+1}^2 \right] + \frac{(1 + g)^2}{2} w_{gg} \hat{g}_{t+1}^2 + \text{h.o.t.} + \text{t.i.p.}
\]

where \( \hat{y}_t = \log y_t - \log y \), and \( \hat{g}_{t+1} = \log(1 + g_{t+1}) - \log(1 + g) \).
Combining the three components, per period welfare function can be expressed as:

\[ W_t = [y u_y - y^2 v_y] g_t + [(1 + g) u_x + (1 + g) w_x] \hat{g}_{t+1} + y (1 + g) u_y \hat{g}_{t+1} \]

\[ + \frac{1}{2} (y u_y + y^2 u_y - y v_y - y^2 v_y) \hat{g}_t^2 \]

\[ + \frac{1}{2} ((1 + g) u_g + (1 + g)^2 u_g + (1 + g) w_g + (1 + g)^2 w_g) \hat{g}_{t+1} \]

\[ + h.o.t. + t.i.p. \]

(note that following relations hold true at the efficient steady state)

\[ y u_y = y v_y, \quad (1 + g) u_x + (1 + g) w_x = 0; \quad y (1 + g) u_y = \frac{y}{\beta} - \frac{1}{1 - \beta} \]

\[ y u_y + y^2 u_y - y v_y - y^2 v_y = -\frac{y}{\beta} \left[ \frac{y}{c} + v \right] \]

\[ (1 + g) u_g + (1 + g)^2 u_g + (1 + g) w_g + (1 + g)^2 w_g = -\left[ \frac{\beta}{1 - \beta} + \left( \frac{\beta}{1 - \beta} \right)^2 + \frac{e (c - 1) \delta (1 + g)^2 g v^2}{c (\gamma - 1)^2} \right] \]

Using these into the quadratic approximation of \( W_t \) and completing the squares we get

\[ W_t = -\frac{1}{2} \left[ \frac{y}{\beta} \right] \left( \hat{g}_t - \hat{g}_t^* \right)^2 \]

\[ \left( \frac{\beta}{1 - \beta} \right) + \frac{1}{2} \frac{\beta}{\frac{\beta}{1 - \beta} + \frac{1}{v} [(\gamma - 1) \eta_g + 1] \left( \hat{g}_{t+1} - \hat{g}_{t+1}^* \right)^2 + h.o.t. + t.i.p. \]

the term in the first bracket is the labor wedge. 

\[ \square \]

**Lemma 2. Labor Wedge is given by**

\[ v (\hat{L}_t - \hat{L}_t^*) + (\hat{c}_t - \hat{c}_t^*) - (\hat{\omega}_t - \hat{\omega}_t^*) = (\hat{y}_t - \hat{y}_t^*) = \frac{\beta}{1 - \beta} \frac{1}{v} (\hat{g}_{t+1} - \hat{g}_{t+1}^*) \]

**Proof.** Use equations 40, 41 and 43 from definition A.3 to substitute for \( \hat{L}_t, \hat{c}_t \) and \( \hat{\omega}_t \). Finally note that under efficient allocation, the labor wedge is zero, that is, \( v \hat{L}_t^* + \hat{c}_t^* - \hat{\omega}_t^* = 0 \). 

\[ \square \]

**Corollary (Corollary 1: Importance of Growth Stabilization).** The relative weight on growth rate gap is higher than the relative weight on labor efficiency wedge if

\[ \frac{\beta}{1 - \beta} > \frac{\beta}{\frac{\beta}{1 - \beta} + \frac{1}{v} [(\gamma - 1) \eta_g + 1]} \]

(73)

**Proof.** If \( \frac{\beta}{1 - \beta} > \frac{\beta}{\frac{\beta}{1 - \beta} + \frac{1}{v} [(\gamma - 1) \eta_g + 1]} \), then it follows directly that:

\[ \frac{\beta}{1 - \beta} \left[ \frac{v}{\frac{\beta}{1 - \beta} + \frac{1}{v} [(\gamma - 1) \eta_g + 1]} > \frac{\beta}{\frac{\beta}{1 - \beta} + \frac{1}{v} [(\gamma - 1) \eta_g + 1]} \right] \]

since all the terms in the square bracket on the LHS are positive and add to more than 1. 

\[ \square \]

**Proposition (Proposition 6: Optimal Policy away from ZLB).** Given a process for liquidity demand and monetary policy shocks, optimal policy under sticky wage allocation tracks the natural rate of interest when the Zero Lower Bound constraint is slack.

**Proof.** When the nominal interest rate is set equal to the natural interest rate (and is non-negative), the unique solution to the competitive equilibrium is

\[ \hat{y}_t = 0; \quad \hat{c}_t = 0; \quad \hat{\omega}_t = 0; \quad \hat{g}_{t+1} = 0 \]

which corresponds to the first-best allocation as shown in proof of Proposition 4. 

\[ \square \]
Corollary (Corollary 2). When the ZLB is slack, the time series of output under optimal policy is a trend stationary process (integrated of order zero), that is,

$$\log Y_t = a + b * t$$

where $a = \log Y_0$ is the initial level of output, and $b = \log(1 + g_{ss})$ is the steady state productivity growth rate.

Proof. Under optimal policy, the productivity growth rate does not deviate from the steady state growth rate. Hence the series of output can be expressed as:

$$\log Y_T = \log Y_0 + \sum_{k=0}^{T-1} (1 + g_{ss}) = \log Y_0 + (T - 1)(1 + g_{ss}); \quad \forall t \geq 1$$

\[\Box\]

D.1 Optimal Policy at the Zero Lower Bound

D.1.1 Optimal Commitment Solution at the ZLB

$$L_0 = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left\{ \frac{1}{2} \left[ \lambda_1 (\gamma_t - \hat{\gamma}_{t+1})^2 + \lambda_2 \hat{\gamma}_{t+1}^2 + (\hat{\lambda}_t)^2 \right] + \phi_{1t} \left[ \hat{c}_t - \hat{c}_{t+1} - \hat{\lambda}_{t+1}^w - \hat{\lambda}_t^w \right] + \phi_{2t} \left[ \hat{\lambda}_{t+1}^w - \beta \hat{\lambda}_t^w - \kappa_w (\hat{c}_t + v \hat{y}_t) \right] + \phi_{3t} \left[ - (E_t \hat{c}_{t+1} - \hat{c}_t + \hat{g}_{t+1} + E_t \hat{V}_{t+1} - (q - 1) \eta \hat{S}_{t+1} \right] + \phi_{4t} \left[ \frac{1}{y} \eta \hat{S}_t + \sigma \eta \hat{g}_{t+1} - \hat{y}_t \right] + \phi_{5t} \left[ - \hat{V}_t + \eta \hat{g}_t - q \hat{S}_{t+1} - \eta_q (E_t \hat{c}_{t+1} - \hat{c}_t + \hat{g}_{t+1} + \eta \hat{V}_{t+1} \right] \right\}$$

First Order conditions:

$$\phi_{1t} - \kappa_w \phi_{2t} + \phi_{3t} - \frac{c}{y} \phi_{4t} + \eta_q \phi_{5t} - \beta^{-1} \left[ \phi_{3t-1} + \phi_{5t-1} + \eta_q \phi_{5t-1} \right] = 0$$

$$\lambda_1 (\gamma_t - \hat{\gamma}_{t+1}) - \phi_{2t} \kappa_w v - \phi_{4t} + \phi_{5t} \eta_q = 0$$

$$-\lambda_1 \hat{\gamma} (\gamma_t - \hat{\gamma}_{t+1}) + \lambda_2 \hat{\gamma}_{t+1} - [(q - 1) \eta \hat{S}_t + 1] \phi_{5t} + \frac{rd}{y} \sigma \eta \phi_{4t} - (q \eta + \eta_q) \phi_{5t} = 0$$

$$\hat{\lambda}_t^w + \phi_{2t} - \phi_{2t-1} - \beta^{-1} \phi_{5t-1} = 0$$

$$-\phi_{5t} + \beta^{-1} \left[ \phi_{5t-1} + \eta \phi_{5t-1} \right] = 0$$

$$\phi_{1t} \geq 0, \quad \phi_{5t} \geq 0, \quad \phi_{1t} \phi_{5t} = 0$$

D.1.2 Optimal Discretionary Solution at the ZLB

Following is the Lagrangian for the Discretion policy

$$L_0 = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left\{ \frac{1}{2} \left[ \lambda_1 (\gamma_t - \hat{\gamma}_{t+1})^2 + \lambda_2 \hat{\gamma}_{t+1}^2 + (\hat{\lambda}_t)^2 \right] + \phi_{1t} \left[ \hat{c}_t - \hat{c}_{t+1} - \hat{\lambda}_{t+1}^w - \hat{\lambda}_t^w \right] + \phi_{2t} \left[ \hat{\lambda}_{t+1}^w - \beta \hat{\lambda}_t^w - \kappa_w (\hat{c}_t + v \hat{y}_t) \right] + \phi_{3t} \left[ - (\hat{c}_{t+1} - \hat{c}_t + \hat{g}_{t+1} + \hat{V}_{t+1} - (q - 1) \eta \hat{S}_{t+1} \right] + \phi_{4t} \left[ \frac{1}{y} \eta \hat{S}_t + \sigma \eta \hat{g}_{t+1} - \hat{y}_t \right] + \phi_{5t} \left[ - \hat{V}_t + \eta \hat{g}_t - q \hat{S}_{t+1} - \eta_q (\hat{c}_{t+1} - \hat{c}_t + \hat{g}_{t+1} + \eta \hat{V}_{t+1} \right] \right\}$$

$$\lambda_1 = \kappa_w \left( v + \frac{v}{\bar{r}} \right), \quad \hat{\lambda}_t = \frac{\beta}{1 - \beta} \frac{1}{q r} \left[ \frac{v}{r + \sigma} \beta + [(q - 1) \eta \hat{S}_t + 1] \right]$$

First Order conditions:

$$\phi_{1t} - \kappa_w \phi_{2t} + \phi_{3t} - \frac{c}{y} \phi_{4t} + \eta_q \phi_{5t} = 0$$

$$\lambda_1 (\gamma_t - \hat{\gamma}_{t+1}) - \phi_{2t} \kappa_w v - \phi_{4t} + \phi_{5t} \eta_q = 0$$

$$-\lambda_1 \hat{\gamma} (\gamma_t - \hat{\gamma}_{t+1}) + \lambda_2 \hat{\gamma}_{t+1} - [(q - 1) \eta \hat{S}_t + 1] \phi_{5t} + \frac{rd}{y} \sigma \eta \phi_{4t} - (q \eta + \eta_q) \phi_{5t} = 0$$

A.15
\[ \tilde{\mu}_t + \phi_{2t} = 0 \]
\[ \phi_{5t} = 0 \]
\[ \phi_{1t} \geq 0, \quad i_t \geq 0, \quad \psi_t i_t = 0 \]

**Proposition** (Proposition 7: Optimal Discretionary Policy at the ZLB). If Assumptions A1 and A2 hold and for a given level of productivity at time 0, \( A_0 \), the Markov equilibrium is characterized by:

\[ \log A_1 = \log A_0 + \log(1 + \gamma_{ss}) \]

for \( 0 < t < T^c \)

\[ \tilde{g}_t = \psi_p S^t < 0; \tilde{\mu}_t = \psi_p S^t < 0; \hat{g}_t = \psi_y S^t < 0 \]

\[ \log A_{t+1} = \log A_t + \psi_y S^t \]

and when \( t \geq T^c \)

\[ \tilde{g}_t = \tilde{\mu}_t = \hat{g}_t = 0 \]

\[ \log A_{t+1} = \log A_{t+1} + (T^c - 1)\psi_y S^t \leq \log A_{t+1}^c \]

where \( \psi_y = \frac{1 - \beta \mu}{(1 - \beta \mu)(1 - \rho)\eta_y - \kappa_{c y} (\nu + \gamma_{ss})} > 0, \psi_p = \frac{\kappa_{c y} (\nu + \gamma_{ss})}{1 - \beta \mu}, \psi_y > 0, \) and \( \psi_y = \frac{1 - \beta \mu}{\gamma_{ss}} \psi_y > 0 \). \( A_{t+1}^c \) is the (time-0) first-best output at time \( t + 1 \).

Proof. First note that the policymaker sets the policy rate to the unconstrained optimal policy rate as soon as the zero lower bound stops binding that is for \( t \geq T^c \). The discretionary policy (MPE) taking into account the ZLB constraint is defined by the first order conditions derived above and the structural relations. The optimal policy when the ZLB stops binding involves setting \( \hat{g}_t \), the Lagrange multiplier on the zero lower bound constraint, to 0. This reduces the system of equations to the familiar unconstrained policy of setting interest rate equal to the natural interest rate such that output and inflation are back to the (unconstrained) steady state. This constitutes a unique bounded solution and proves that there is no inertia in the discretionary policy. Remains to show that under the zlb, it is optimal to set interest rate to 0. Suppose it is not then, as discussed above, the Lagrange multiplier on ZLB constraint must be 0 and output and inflation must be at the steady state. But this leads to a violation of the AD equation, which is not satisfied. Next we solve for the values of endogenous variables. Under the assumed Eggertsson and Woodford two-state Markov Chain, the system at time \( t < T^c \) is in state \( S \) (short run) and can be expressed as:

\[
(1 - \mu)\tilde{c}_S = \mu \tilde{\mu}_S + \tilde{r}_S \\
(1 - \beta \mu)\tilde{\mu}_S = \kappa_{c y} (\tilde{c}_S + \nu \tilde{g}_S) \\
\left[ (1 - \gamma_s)\eta_y + 1 \right] \tilde{g}_S = \mu \tilde{V}_S + (1 - \mu)\tilde{c}_S \\
\frac{r_d}{\gamma} \psi_y \tilde{g}_S = \tilde{g}_S - \frac{c}{\gamma} \tilde{c}_S \\
\tilde{V}_S = \frac{1}{1 - \gamma_{ss} \gamma} \left[ \gamma_{y} \tilde{g}_S + \eta \gamma (1 - \mu)\tilde{c}_S - (\eta_{c y} + \eta) \tilde{g}_S \right]
\]

We can solve the last three equations to find a relationship between \( c \) and \( y \):

\[ \tilde{c}_S = \eta c \tilde{g}_S; \quad \eta_c \equiv \frac{1 - \gamma_{ss} \gamma (\nu + \gamma_{ss})}{\gamma_{ss} \eta_{c y} \gamma + (1 - \mu)} < 1 \]

We can solve the system for \( t < T^c \):

\[ \tilde{g}_t = \psi_y S^t < 0; \tilde{\mu}_t = \psi_p S^t < 0; \hat{g}_t = \psi_y S^t < 0 \]

where \( \psi_y = \frac{1 - \beta \mu}{(1 - \beta \mu)(1 - \rho)\eta_y - \kappa_{c y} (\nu + \gamma_{ss})} > 0, \psi_p = \frac{\kappa_{c y} (\nu + \gamma_{ss})}{1 - \beta \mu}, \psi_y > 0, \) and \( \psi_y = \frac{1 - \beta \mu}{\gamma_{ss}} \psi_y > 0 \). We assume (by A2 in the main text) the system is locally determinate around the state \( S \) equilibrium defined above. Therefore by the law of motion of productivity, we can derive that:

\[ \log A_{t+1} = \log A_t + \psi_y S^t; \quad \forall 0 < t < T^c \]

Second part of the proposition (when \( t \geq T^c \)) follows from Proposition 6.
D.2 Supply Shocks

Proposition (Proposition 8). Given i.i.d. aggregate productivity shocks, output follows a unit root under the (time-0) first best allocation.

Proof. As shown in Appendix C.6, the first-best allocation has a unique solution given by following conditions:

\[ \hat{g}_t + \hat{M}_t = \psi^*_y \hat{M}_t; \quad \bar{c}_t = \psi^*_c \hat{M}_t; \quad \bar{y}_t = \psi^*_y \hat{M}_t; \quad \bar{l}_t = \psi^*_l \hat{M}_t \]

where \( \psi^*_y = \frac{1+v}{(v+1)+\varphi V_g} > 0 \),

\[ 0 < \psi^*_y = (\eta_\lambda (\eta + \gamma_2) + \eta_\lambda (\eta + 1)) \psi^*_y < 1, \]

\[ \psi^*_y = \frac{\psi^*_y}{\varphi \psi^*_y + \varphi_2} \eta_\lambda \psi^*_y > 0, \]

\[ \psi^*_l = \frac{1-\varphi_2}{\varphi \psi^*_y} > 0 \]

Since output can be decomposed as \( Y_t = y_t A_t \), the following accounting identity follows:

\[ \log Y_t = \log \bar{y}_t + \hat{g}_t - \hat{g}_t + \hat{M}_t - \hat{g}_t \]

which can be expressed using the solution of system of equations as

\[ \log Y_t = \log \bar{y}_t + \psi^*_y \hat{M}_t + (\psi^*_y - \psi^*_y) \hat{M}_t - \psi^*_y \hat{M}_t \]

For i.i.d. shocks \( \tilde{M}_t \), output is thus a random walk process with drift.

D.2.1 Optimal Policy with Supply shocks

\[ \max \sum_{i=0}^{\infty} \beta^i \sum_{t=0}^{\infty} \beta^j \left[ \lambda_\phi (\hat{y}_t - \hat{y}_t) - \lambda_\psi (\hat{g}_t + \hat{M}_t - \hat{g}_t) \right] \]

subject to following constraints:

Wage Phillips Curve: \( \hat{n}_t = \beta \hat{n}_t + \kappa_2 (\hat{c}_t + \varphi_2 \hat{M}_t + \hat{\lambda}_t) \)

Resource Constraint: \( \frac{c}{\hat{y}} + \frac{\varphi_2 \hat{y}}{\varphi_2} \eta_\lambda \hat{g}_t = \frac{\hat{c}_t}{\hat{y}} + \varphi_2 \hat{M}_t \)

Endogenous Growth eq 1: \( \varphi_2 \hat{g}_t + \hat{M}_t + \lambda_2 \hat{g}_t = \hat{c}_t + \hat{M}_t + \hat{g}_t \)

Endogenous Growth eq 2: \( \hat{V}_t = \eta_\lambda \hat{y}_t - \eta_\lambda \hat{g}_t + \eta_\lambda \hat{g}_t + \eta_\lambda \hat{M}_t \)

where the terms with asterisk denote corresponding values under the first-best allocation.

First Order conditions:

\[ -\kappa_2 \hat{g}_t + \hat{M}_t + \frac{\varphi_2}{\hat{y}} \hat{M}_t + \eta_\lambda \hat{M}_t - \varphi_2 \left[ \hat{M}_t + \eta_\lambda \hat{M}_t \right] = 0 \]

\[ \lambda_2 (\hat{g}_t - \hat{g}_t) - \lambda_2 \hat{g}_t - \lambda_2 \hat{M}_t + \eta_\lambda \hat{M}_t - \varphi_2 \left[ \hat{M}_t + \eta_\lambda \hat{M}_t \right] = 0 \]

Hence, the optimal allocation under supply shocks (away from ZLB) is given by set of 5 variables \( \{\hat{y}_t, \hat{c}_t, \hat{g}_t, \hat{V}_t, \hat{n}_t\} \), 4 Lagrange multipliers \( \{\lambda_2, \psi^*_y, \psi^*_y, \psi^*_y\} \) which satisfy the above 5 first order conditions and 4 structural relations for a given process of shocks \( \{\hat{M}_t, \hat{\lambda}_t, \hat{\lambda}_t\} \) and the corresponding first-best allocation.

E Computational Appendix

E.1 consumption-equivalent welfare loss

We derive the consumption equivalent welfare loss relative to the (time-0) first best allocation as follows:

\[ \Psi_0 = \sum_{s=0}^{\infty} \beta^s \left[ \log L_s - \varphi (L_s) + \frac{\beta}{1-\beta} \log (1 + s + 1) \right] \]
Assuming a permanent gain in consumption \( b \geq 0 \) percent, the welfare at the efficient allocation is given by:

\[
W^*_0(b) = \sum_{s=0}^{\infty} \beta^s \left[ \log(c_s(1+b)) - v(L_s) + \frac{\beta}{1-\beta} \log(1+g_{s+1}) \right] = W^*_0(b = 0) + \frac{1}{1-\beta} \log(1+b)
\]

Equating this to the welfare under the sticky wage allocation:

\[
W_0 = W^*_0(b) \\
\iff (1-\beta)(W_0 - W^*_0(b = 0)) = \log(1+b) \approx b
\]

Further we derived the quadratic approximation of the welfare relative to the first-best allocation above in Appendix D:

\[
\frac{W_0 - W^*_0(b = 0)}{U_{c_0}y_{ss}}
\]

Thus, the consumption equivalent welfare loss:

\[
b = U_{c_0}y_{ss}(1-\beta) \left[ \frac{W_0 - W^*_0(b = 0)}{U_{c_0}y_{ss}} \right] < 0
\]

where the term in square brackets is the quadratic approximation we derived above in Proposition 4 and the welfare terms are the conditional welfare gains or losses, starting at the efficient steady state at \( t = 0 \).

Under monetary policy shocks, liquidity demand shocks and wage markup shocks, the first-best allocation corresponds with the no-fluctuations allocation. Hence the consumption-equivalent welfare loss is relative to the Balanced Growth path. However under productivity shocks, the first-best allocation departs from the Balanced Growth path and the consumption equivalent welfare loss derived above is non-standard.

### F Fiscal Policy Multipliers at the ZLB

The results echo the findings of Eggertsson (2011). The key insight here is that the policy has long-run implications. We here present the analytical solution for investment subsidy for research spending. Results on paradox of toil, paradox of thrift, and expansionary government spending can be derived similarly.

To analyze fiscal policy, we allow government to run budget deficits, by issuing the nominal risk-free bond \( B_{t+1} \), which is not in net zero supply anymore.\(^4\)

\[
P_i T_i + B_{t+1} = -\tau^f \int_0^{T_f} p_i x_i d_i + \tau^w R_i - \tau^w \int_0^{T_f} W_i(h)L_i(h) + (1 - \tau^w)(1 + i_{t-1})B_t
\]

And the following transversality condition on government debt holds

\[
\lim_{T \to \infty} E_t \frac{B_T}{P_T(1 + \tau^w)} u_c(C_T) = 0
\]

### F.1 R&D Investment Subsidy

Assume a temporary research subsidy is implemented \( \hat{\tau}^f > 0 \) for \( S \in [1,T_f) \). Under Eggertsson and Woodford (2003)'s two-state Markov Chain assumption, the system at time \( t < T_f \) is in state \( S \) (short run) and can be expressed as:

\[
(1 - \mu)\hat{c}_S = \mu \hat{c}_S + \hat{r}_S \\
(1 - \beta \mu) \hat{c}_S = \kappa_w(\hat{c}_S + \nu \hat{y}_S) \\
[(q - 1)\hat{y}_S + 1] \hat{g}_S - \hat{r}_S = \mu \hat{V}_S + (1 - \mu)\hat{c}_S \\
\frac{rd}{y} \nu \hat{y}_S \hat{g}_S = \hat{g}_S - \frac{c}{y} \hat{c}_S \\
\hat{V}_S = \frac{1}{1 - \nu \mu} [\eta \hat{V} \hat{g}_S + \eta \eta (1 - \mu)\hat{c}_S - (\eta z + \eta \eta) \hat{g}_S]
\]

\(^4\)Since Ricardian Equivalence holds, it does not matter if the government finances expenditure by running a balanced budget or via deficit financing.
We can solve the last three equations to find a relationship between $c$ and $y$:

$$
\dot{S} = \eta_c \dot{g} - \eta_x \dot{c}^2; \quad \eta_c = \frac{1 - \eta \mu (e-1)y_0 + 1 + \eta \frac{c^0}{y_0}}{y^0} - \eta Y < 1
$$

$$
\eta_x = \frac{1 - \eta \mu (e-1)y_0 + 1 + \eta \frac{c^0}{y_0}}{y^0} \eta_x \eta_y + \eta \mu (1 - \mu)
$$

Using this, the resulting AD-AS system can be expressed as:

$$
(1 - \mu) \eta_c \dot{g} = \mu \dot{t}^2 + (1 - \mu) \eta_x \dot{c}^2 + \dot{r}_S
$$

$$
(1 - \beta \mu) \dot{t}^w = \kappa_w (\eta_c + \nu) \dot{g} - \kappa_w \eta_x \dot{c}^2
$$

We can solve the system for $t < T^*$:

$$
\dot{g}_t = \psi_y r^b + \psi_x c^2
$$

$$
\dot{t}^w = \psi_p r^b + \psi^p c^2
$$

$$
\dot{g}_t + 1 = \psi_y r^b + \psi^x c^2
$$

where

$$
\psi_y = \frac{1 - \beta \mu \eta c^0}{(1 - \beta \mu)(1 - \mu) - \kappa_w (\nu + \eta c^0) \eta c^0} > 0,
$$

$$
\psi_x = \frac{1 - \beta \mu \eta c^0}{1 - \beta \mu (1 - \mu) - \kappa_w \eta c^0} > 0
$$

$$
\psi_p = \frac{\kappa_w (\nu + \eta c^0)}{1 - \beta \mu \eta c^0} \psi_y > 0, \text{ and } \psi_S = \frac{1 - \beta \mu \eta c^0}{\eta c^0} \psi_y > 0
$$

$$
\psi^p = \frac{\kappa_w (\nu + \eta c^0)}{1 - \beta \mu \eta c^0} \left( (1 + \gamma c^0)(1 - \beta \mu)(1 - \mu) - \kappa_w (\nu + \eta c^0) \eta c^0 \right) - 1 < 0.
$$

$$
\psi_S = \frac{1 - \beta \mu \eta c^0}{\eta c^0} \psi_y > 0
$$

$$
\psi^S = \frac{1 - \beta \mu \eta c^0}{\eta c^0} \psi_y - \frac{\kappa}{\eta c^0} \eta y = \frac{1}{\eta c^0} \left[ \psi_y - \frac{\kappa}{\eta c^0} \eta x \left[ \frac{\eta c^0}{\kappa} \eta c^0 - 1 \right] \right] > 0
$$

Hence research tax subsidy is expansionary at the ZLB. This is equivalent to the investment tax credit studied by Eggertsson (2011). Note that a supply side expansionary policy is contractionary at the ZLB if it reduces expectations of inflation. Here, this supply side policy increases the potential output of the economy without inducing the corresponding deflationary pressures. Instead the expectations of increased demand for research spending boosts inflation. Hence, a tax subsidy for non-tangible investment can be expansionary at the ZLB.

The Long-run Output is given by:

$$
\log Y_{t+1} = \log Y_{t+1} + (T^* - 1) \psi_y r^b + (T^* - 1) \psi^p c^2; \quad \forall t \geq T^*
$$

The Long-run output is higher by the increase in productivity growth rate achieved by higher research subsidies during the binding ZLB. Thus the long-run output multiplier for research subsidy is given by:

$$
\frac{\partial Y_t}{\partial t^S} = (T^* - 1) \psi^S > 0
$$

G Optimal Policy under discount rate shocks

Eggertsson (2008), following Auerbach and Obstfeld (2005), and Eggertsson and Woodford (2003) modeled shocks to the discount rate in the utility function of the representative agent as the "demand shocks" (purely inter-temporal shocks) that induce a drop in output, risk-free nominal interest rate and price level in events such as the Great Depression. In this paper, we primarily analyzed the liquidity demand shock and monetary policy shock. This is because using Eggertsson (2008)'s discount rate shock $\xi_{Eggs}$ need not always imply the Great Depression outcome when the natural interest rate drops.

$$
E_t \sum_{s=0}^{t} \frac{E_{t+s}^\beta}{(1 + \psi)^{t+s}} \left[ \log (C_{t+s}) - \frac{\omega}{1 + \psi} \int_0^1 L_{t+s} (j + 1 + v) \left[ 1 + B_{t+s} + \xi_{t+s} - \psi_{t+s} \right] \right]
$$

Remember from the entrepreneur's decision to invest equation that her decision to invest in R&D depends on two
forces - a) discounting effect and b) market size effect. A higher stochastic discount factor increases the entrepreneur’s incentive to innovate (discounting effect) because of lower discounting of future profits. Second, higher expected future output increases her incentive to invest in innovation because of the market size effect. A negative shock to the discount factor such that the household wants to postpone his consumption increases the stochastic discount factor of the household. While the market size contracts for the entrepreneur in the presence of nominal rigidities, she discounts future profits less. Thus these two effects are opposing in nature. Depending on the degree of nominal rigidities, the entrepreneur may want to increase investment into R&D and therefore the output may increase in response to a negative discount rate shock. If wages are sufficiently rigid, the market size effects dominates the discounting effect and output contracts. However, in the absence of nominal rigidities a negative discount rate shock exerts only the discounting effect. The market size effect disappears. Thus the entrepreneur increases investment in R&D under such preference shock when there are no nominal rigidities. The dashed (brown) line in Figure 27 plots output under flexible wages. There is over-investment in R&D because it is inexpensive to do so. This captures the lower “opportunity cost” of doing R&D during recessions hypothesis. However, with sufficient nominal rigidities firms may expect their market to contract sizably. As a result, they reduce their R&D investment and output contracts following such a shock. Circled (red) line plots the equilibrium path of output under sticky wages when the central bank follows a Taylor rule.

Since entrepreneurs do not internalize the benefits of R&D to the entire economy over the long-run, there are “dynamic externalities” (as also discussed in response to TFP shocks). As a result, the impulse response under first-best allocation (solid blue line) differs from the flexible wage allocation. In this calibration, first-best output is higher than the flexible-wage competitive equilibrium allocation because the entrepreneur chooses investment to maximize her own profits taking into account the probability that she will be replaced in the future by a new entrant. Since we wanted to focus our analysis to a setting where the first-best or the flexible wage allocation would not respond to shocks, liquidity demand shocks presented themselves as a useful modeling alternative.

**Figure 27: Output: discount rate shock**

What is the optimal policy in response to such demand shocks? The solid green line plots the optimal commitment solution. Optimal solution is to induce higher R&D investment in response to such demand shocks. The level of output is considerably higher under optimal policy solution than under Taylor rule.
H Medium scale DSGE model

We follow Howitt and Aghion (1998) and Aghion and Howitt (2008) in introducing capital in the endogenous growth framework. We however extend our model to allow for investment adjustment costs in sync with the New Keynesian literature following (Christiano, Eichenbaum and Evans, 2005), Smets and Wouters (2007) and Justiniano, Primiceri and Tambalotti (2013). The new ingredients are (i) a monopolistically competitive retail sector that sets prices in a staggered fashion, (ii) endogenous capital accumulation by households subject to investment adjustment costs, (iii) habit formation in consumption, (iv) variable capital utilization rate, and (v) partial indexation of prices and wages to the respective lagged inflation rates. We discuss these in turn:

H.1 Monopolistically Competitive Retailers

There is a continuum of monopolistically competitive retailers that sell the final good $Y_t(k)$. These goods can be aggregated into a Dixit-Stiglitz composite $Y_t$ as follows:

$$Y_t = \left[ \int_0^1 Y_t(k) \frac{1}{1 - \rho_p} dk \right]^{1 + \lambda_{pt}}$$

where $\lambda_{pt} > 0$ is the (time-varying) price markup. We assume that $\lambda_{pt}$ follows the exogenous ARMA process:

$$\log \lambda_{pt} = (1 - \rho_p) \log \lambda_p + \rho_p \log \lambda_{pt-1} + \epsilon_{pt} p^\theta - \mu \epsilon_{pt-1}; \quad \epsilon_{pt} \sim N(0, \sigma_p)$$

Each retailer $k$ purchases one unit of intermediate good composite $Y_t(k, m)$ at a given price of $P_t^M$ to package it into one unit of final good and is assumed to set prices on a staggered basis following Calvo (1983). With probability $(1 - \theta_p)$, a retailer gets to reset its price. It solves the following problem:

$$\max_{P_t(k)} \sum_{s=0}^{\infty} (\beta \theta_p)^s Q_{t,s} \left[ P_t(k) \Pi_{t,s} - P_t^M \right] Y_{t+s}(k)$$

subject to demand for its product

$$Y_{t+s}(k) = \left( \frac{P_t(k) \Pi_{t,s}}{P_{t+s}} \right)^{1 + \lambda_{pt+s}}$$

where the stochastic discount factor period $t + s$ is given by:

$$Q_{t,s} = \frac{\Lambda_{t+s}}{\Lambda_t} \frac{P_t}{P_{t+s}}$$

where $\Lambda_t$ is the marginal utility of consumption defined later and

$$\Pi_{t,s} \equiv \prod_{b=1}^{s} \left( \pi_{bs} \frac{\pi_{t+s}}{\pi_{t+b-1}} \right)$$

is the automatic adjustment that firms make to their price when they do not get to reset them, $\theta_p \in (0, 1)$ is the indexation coefficient and $\pi_{bs}$ is the steady state price inflation rate. Let $\bar{P}_t$ be the reset price at time $t$. The first order condition is:

$$\mathbb{E}_t \sum_{s=0}^{\infty} (\beta \theta_p)^s Q_{t,s} \left[ P_t \Pi_{t,s} - (1 + \lambda_{pt+s}) P_t^M \right] Y_{t+s}(k) = 0$$

The law of motion of the aggregate price index $P_t$ is given by:

$$P_t \frac{1}{1 - \rho_p} = (1 - \theta_p)(\bar{P}_t) \frac{1}{1 - \rho_p} + \theta_p \left( \pi_{t-1}^{1 - \rho_p} \pi_{t-1}^{1 - \rho_p} P_{t-1} \right) \frac{1}{1 - \rho_p}$$

H.2 Perfectly Competitive Composite Good Production

Each of the intermediate good composites is produced by a perfectly competitive firm that uses a CES composite of labor and secondary intermediate goods.\(^5\) As a result, all intermediate good firms are identical and we omit the subscripts $(k, m)$ and simply denote the intermediate output at $Y_t^M$.

\(^5\)Such a convoluted market structure is assumed to introduce price-rigidity in a staggered fashion. Basically, there is a single consumption good that is produced by a perfectly competitive firm, but is retailed by monopolistically competitive retailers.
where each \( x_{it} \) is the flow of intermediate product \( i \) used at time \( t \), and the productivity parameter \( A_{it} \) reflects the quality of that product and \( M_t \) is the aggregate (stationary) productivity shock which follows the process:

\[
\log M_t = (1 - \rho_m) \log M_t + \rho_m \log M_{t-1} + e_t^m; \quad e_t^m \sim N(0, \sigma_m)
\]

The composite good producer’s maximization problem is as follows

\[
\max_{L_t, \{x_{it}\}_{i \in [1,n]}} \left\{ p_t^m M_t L_t^{1-\alpha} \int_0^1 A_{it} x_{it}^\alpha di - W_t L_t - \int_0^1 p_t x_{it} di \right\}
\]

Solving this gives the (inverse) factor demands:

\[
p_{it} = \alpha P_t^m M_t L_t^{1-\alpha} A_{it} x_{it}^{\alpha - 1}
\]

\[
W_t = (1 - \alpha) P_t^m \frac{Y_t^m}{L_t}
\]

**H.3 Monopolist Intermediate Good Producer**

Intermediate good producers are monopolists and use capital to produce one unit of intermediate good. Following Howitt and Aghion (1998), we assume the following production function for the intermediate good:

\[
x_{it} = \frac{K_{it}}{A_{it}}
\]

The intermediate monopolistic firm sets prices flexibly every period in order to maximize profits:

\[
\max_{p_t} (1 - \tau_t^p) p_t x_{it} - R_t^K K_{it}
\]

subject to the demand for the intermediate good (eq H.1). \( \tau_t^p \) is the sales tax/subsidy imposed on the monopolist’s price. Further, we assume that there is a competitive fringe in every sector that faces a marginal cost of \( \chi A_{it} R_t^K \), where \( \chi \in (1, \frac{1}{2}) \). As a result, the intermediate monopolist cannot charge a price higher than \( p_{it} = \chi A_{it} R_t^K \). In equilibrium, the monopolist charges a price given by:

\[
p_{it} = \zeta A_{it} R_t^K \equiv \min \left( \chi, \frac{1}{(1 - \tau_t^p)\zeta} \right) A_{it} R_t^K
\]

This yields

\[
x_{it} = \frac{K_t}{A_t} = \left( \frac{\frac{\alpha P_t^m M_t L_t^{1-\alpha}}{R_t^K}}{\frac{1}{\zeta}} \right)^{\frac{\alpha}{1-\alpha}}, \quad R_t^K = \frac{\alpha P_t^m Y_t^m}{\zeta - K_t}
\]

and profits are given by \( \Gamma_t(A_{it}) = (\zeta - 1) \frac{P_t^m Y_t^m A_{it}}{K_t} \). Define aggregate productivity \( A_t \equiv \int_0^1 A_{it} di \). Substituting for the intermediate goods’ production levels, we can rewrite the production function purely in the form of aggregates:

\[
Y_t^m = M_t (A_t L_t)^{1-\alpha} K_t^\alpha
\]

Define \( k_t = \frac{K_t}{\alpha} \) and \( y_t^m = \frac{Y_t^m}{A_t} = M_t k_t L_t^{1-\alpha} \).

**H.4 Innovation and research arbitrage**

There is a single entrepreneur in each sector who spends final output in research. The entrepreneur at time \( t \) decides her research inputs and if successful, she gets to be the intermediate monopolist in the following period producing goods with productivity \( A_{it+1} = \gamma A_{it} \). She is successful with probability \( \Omega_{z_{it}} \), where \( \Omega_t \) is the exogenous shock to innovation success and is assumed to follow the following process:

\[
\log \Omega_t = \rho_t \log \Omega_{t-1} + \epsilon_t^\Omega; \quad \epsilon_t^\Omega \sim N(0, \sigma_t^\Omega)
\]

competitive retailers in different packaging.
\( z_t \) is the innovation intensity chosen by the entrepreneur. In order to achieve this, she needs to spend the amount of

\[
R_t = c(z_t)A_{it}
\]

in research, where \( c(z_t) \equiv \delta z_t^\varphi \varphi > 1 \) is the research cost. Entrepreneur maximizes the net expected profits from investing in research:

\[
\max_{z_t \in [0,1]} \{ \Omega_t z_t Q_{t+1} V_{t+1}(\gamma A_{it}) - (1 - \tau_t) P_t c(z_t) A_{it} \}
\]

where the lifetime discounted profits are given by the value function:

\[
V_t(A_{it}) = \Gamma_t(A_{it}) + (1 - \Omega_t z_t) \sum_{j=0}^{\infty} \nu_{t+j} M.3
\]

Because of the linearity of production function, as we showed above in the Appendix A, the Value function is also linear in productivity. Writing the normalized Value function as \( \tilde{V}_t \equiv \frac{V_t}{\tau_t A_{it}} \) and focusing on the symmetric equilibrium, we solve for interior solution (where \( z_t > 0 \)):

\[
\varphi z_t^{\varphi-1} = \beta \frac{\Lambda_{t+1}}{\Lambda_t} \frac{\gamma \tilde{V}_{t+1}}{(1 - \tau_t) \delta}
\]  

(H.4)

Total amount of the final good used in research and innovation:

\[
R_t = \int_0^1 R_t d i = c(z_t) A_t
\]

**H.5 Households & Wage Setting**

**H.5.1 Households**

Each household supplies differentiated labor indexed by \( j \). Household \( j \) chooses consumption \( C_t \), risk-free nominal bonds \( B_t \), investment \( I_t \) and capital utilization \( u_t \) to maximize the utility function:

\[
E_t \sum_{s=0}^{\infty} \beta^s \left[ \log(C_{t+s} - h C_{t+s-1}) - \frac{\omega}{1 + \nu} L_{t+s}(j)^{1+\nu} + \zeta_t^u B_{t+1}^1 \right]
\]

where \( h \) is the degree of habit formation, \( \nu > 0 \) is the inverse Frisch elasticity of labor supply, \( \omega > 0 \) is a parameter that pins down the steady-state level of hours, the discount factor \( \beta \) satisfies \( 0 < \beta < 1 \) and \( \zeta_t^u \) is the liquidity demand shock. We assume that in the steady state \( \zeta = 0 \). We assume perfect consumption risk sharing across the households. As a result, household’s budget constraint in period \( t \) is given by

\[
P_t C_t + P_t I_t + B_{t+1} = B_t(1 + i_t) + B_t^S(j) + (1 + \tau^{\infty}) W_t L_t(j) + \Gamma_t + T_t + R_t^K u_t K_t^u - P_t a(u_t) K_t^a
\]

(H.5)

where \( I_t \) is investment, \( B_t^S(j) \) is the net cash-flow from household \( j \)’s portfolio of state-contingent securities. Labor income \( W_t L_t(j) \) is subsidized at a fixed rate \( \tau_w \). Households own an equal share of all firms, and thus receive \( \Gamma_t \) dividends from profits. Finally, each household receives a lump-sum government transfer \( T_t \). Since households own the capital and choose the utilization rate, the amount of effective capital that the households rent to the firms at nominal rate \( R_t^K \) is:

\[
K_t = u_t K_t^u
\]

The (nominal) cost of capital utilization is \( P_t a(u_t) \) per unit of physical capital. As in the literature (SW 2007, JPT 2010) we assume \( a(1) = 0 \) in the steady state and \( a'' > 0 \). Following CEE 2005, we assume investment adjustment costs in the production of capital. Law of motion for capital is as follows:

\[
K_{t+1}^u = v_t \left[ 1 - S \left( \frac{I_t}{(1 + g_{ss}) I_{t-1}} \right) \right] I_t + (1 - \delta_t) K_t^u
\]

where \( g_{ss} \) is the steady state growth rate of productivity, \( \epsilon_t^u \) is a shock to the relative price of investment and In the steady state \( S(1) = S'(1) = 0, S'' > 0 \). JPT consider this as shock to marginal efficiency of investment (MEI) and is

---

† This could further be generalized to allow for adoption probability for this entrepreneur’s technology in the next period, which would better match the data. Secondly, we can also add a financial frictions constraint to get more action.

M.3
assumed to follow the following process:

$$\log v_t = \rho_v \log v_{t-1} + \epsilon^v_t; \quad \epsilon^v_t \sim N(0, \sigma_v)$$

Utility maximization delivers the first order condition linking the inter-temporal consumption smoothing to the marginal utility of holding the risk-free bond

$$1 = \beta \mathbb{E}_t \left[ \frac{\Lambda_{t+1}}{\Lambda_t} \left(1 + \left[ \frac{P_{t+1}}{P_t} \right] \right) \right] + \Lambda_t^{-1} \epsilon_t \quad \text{(H.6)}$$

The stochastic discount factor in period \( t + 1 \) is given by:

$$Q_{t,t+1} = \beta \frac{\Lambda_{t+1}}{\Lambda_t} \frac{P_t}{P_{t+1}}$$

where \( \Lambda_t \) is the marginal utility of consumption given by:

$$\Lambda_t = \frac{1}{C_t - h C_{t-1}} - \frac{h \beta}{C_{t+1} - h C_t}$$

The household does not choose hours directly. Rather each type of worker is represented by a wage union who sets wages on a staggered basis. Consequently the household supplies labor at the posted wages as demanded by firms.

We introduce capital accumulation through households. Solving household problem for investment and capital yields the Euler condition for capital:

$$q_t = \beta \mathbb{E}_t \left[ \frac{\Lambda_{t+1}}{\Lambda_t} \left( \frac{R_{t+1}}{P_{t+1}} \right) - a(u_{t+1}) + \rho_t (1 - \delta_t) \right]$$

where the (relative) price of installed capital \( q_t \) is given by

$$q_t v_t \left[ 1 - S \left( \frac{I_t}{(1 + g_w)I_{t-1}} \right) - S' \left( \frac{I_t}{(1 + g_w)I_{t-1}} \right) \right] = \frac{\beta \Lambda_{t+1}}{\Lambda_t} q_{t+1} v_{t+1} \frac{1}{(1 + g_w)} \left( \frac{I_{t+1}}{I_t} \right)^2 S' \left( \frac{I_{t+1}}{I_t} \right) = 1$$

Choice of capital utilization rate yields:

$$\frac{R^K}{P_t} = \frac{a'(u_t)}{a(u_t)}$$

### H.5.2 Wage Setting

Wage Setting follows Erceg, Henderson and Levin (2000) and is relatively standard. Perfectly competitive labor agencies combine \( j \) type labor services into a homogeneous labor composite \( L_t \) according to a Dixit-Stiglitz aggregation:

$$L_t = \left[ \int_0^1 L_t(j) \frac{1}{\Gamma_{\lambda w,j}} \right]^{-1 + \lambda w, t}$$

where \( \lambda w, t > 0 \) is the (time-varying) nominal wage markup. We assume that \( \lambda w, t \) follows the exogenous ARMA process:

$$\log \lambda w, t = (1 - \rho \mu) \log \lambda w + \rho \mu \log \lambda w, t-1 + \epsilon^w \mu - \mu \epsilon^w \mu; \quad \epsilon^w \mu \sim N(0, \sigma_w)$$

Labor unions representing workers of type \( j \) set wages (with indexation) on a staggered basis following Calvo (1983), taking given the demand for their specific labor input:

$$L_t(j) = \left[ \int_0^1 W_t(j) \frac{1}{\Gamma_{\lambda w,j}} \right]^{-1 - \lambda w, j} L_t, \quad \text{where} \quad W_t = \left[ \int_0^1 W_t(j) \frac{1}{\Gamma_{\lambda w,j}} \right]^{-\lambda w, j}$$

In particular, with probability \( 1 - \theta \), the type-\( j \) union is allowed to re-optimize its wage contract and it chooses \( W_t \) to minimize the disutility of working for laboror of type \( j \), taking into account the probability that it will not get to reset wage in the future. If a union is not allowed to optimize its wage rate, it indexes wage to steady state wage inflation \( \Pi^w \). Workers supply whatever labor is demanded at the posted wage. The first order condition for this problem is
given by:
\[ E_t \sum_{s=0}^{\infty} (\beta \theta_w)^s \Lambda_t^{s+} \left[ (1 + \eta^W_t) \Pi^p_{t+s} - (1 + \lambda_{w,t}) \omega \frac{L_t^{\gamma}(j)}{\Lambda_t^{s+}} \right] L_{t+s}(j) = 0 \]  

(H.7)

where
\[ \Pi^p_{t+s} \equiv \prod_{b=1}^{s} \left( (\pi^w_t)^{-\omega}(\pi^w_{t+b-1})^{-\omega} \right) \]

where \( \eta_w \in (0,1) \) is the indexation coefficient as discussed under price-setting above. By the law of large numbers, the probability of changing the wage corresponds to the fraction of types who actually change their wage. Consequently, the nominal wage evolves according to:
\[ W_{t+1}^{\omega} = (1 - \theta_w)(W_t^{\omega})^{1-\omega} + \theta_w \left( (\pi^p_t)^{2a} (\pi^w_t)^{1-\omega} W_{t-1}^{\omega} \right)^{1-\omega} \]

where the nominal wage inflation and price inflation are related to each other:
\[ \pi_t^{\omega} = \frac{W_t}{W_{t-1}} = \frac{w_t}{w_{t-1}} \frac{1}{\pi_t(1 + g_t)} \]

where \( \pi_t \equiv \frac{p_t}{p_{t-1}} \) is the inflation rate, \( w_t \equiv \frac{\pi_t}{\pi_t} \) is the productivity adjusted real wage and \( g_t \) is the (endogenous) productivity growth rate.

**H.6 TFP and growth rate**

Aggregate (endogenous) productivity follows:
\[ A_t = \int_0^1 A t \, d t = \int_0^1 [\Omega_{t-1} z_{t-1} \gamma A_{t-1} + (1 - \Omega_{t-1} z_{t-1}) A_{t-1}] d t = A_{t-1} + \Omega_{t-1} z_{t-1}(\gamma - 1) A_{t-1} \]

The growth rate of the productivity:
\[ g_t = \frac{A_t - A_{t-1}}{A_{t-1}} = \Omega_{t-1} z_{t-1}(\gamma - 1) \]

Measured TFP (total factor productivity) is given by product of stationary exogenous component and the non-stationary endogenous component:
\[ TFP_t = M_t \times A_t \]

**H.7 Government**

The central bank follows a Taylor rule in setting the nominal interest rate. It responds to deviations in inflation, output and output growth rate from time-t natural allocations.

\[ \frac{1 + i_t}{1 + i_{ss}} = \left( \frac{1 + i_{t-1}}{1 + i_{ss}} \right)^{\rho_k} \left[ \left( \frac{\pi_t}{\pi_{ss}} \right)^{\phi_\pi} \left( \frac{Y_t}{Y_{t-1}} \right)^{\phi_Y} \left( \frac{Y_t / Y_{t-1}}{Y_{t-1}^{1+\gamma}} \right)^{\phi_{\mu}} \right] \epsilon_t^{mp} \]

where \( i_{ss} \) is the steady state nominal interest rate, \( Y_t^{f,t} \) is the time-t natural output, \( \rho_k \) determines interest-rate smoothing and \( \epsilon_t^{mp} \sim N(0, \sigma) \) is the monetary policy shock.

We assume government balances budget every period:
\[ P_t T_t = \tau_p \int_0^1 p_t X_t d t + \tau_p P_t R_t + \tau_t W_t L_t + P_t G_t \]

where \( G_t \) is the government spending, which is determined exogenously as as a fraction of GDP
\[ G_t = \left( 1 - \frac{1}{\lambda_g} \right) Y_t \]

where the government spending shock follows the process:
\[ \log \lambda_g = (1 - \rho_g) \lambda_g^{\mu} + \rho_g \log \lambda_{g-1}^{\mu} + \epsilon_t^{\lambda} ; \epsilon_t^{\lambda} \sim N(0, \sigma) \]
H.8 Market Clearing

\[ Y_t = C_t + I_t + R_t + a(u_t)K_t^\beta + G_t \]

H.9 Stationary Allocation

We normalize the following variables:

- \( y_t = Y_t/A_t \)
- \( y_t^m = Y_t^m/A_t \)
- \( c_t = C_t/A_t \)
- \( k_t = K_t/A_t \)
- \( k_t^m = K_t^m/A_{t-1} \)
- \( I_t = I_t/A_t \) capital investment
- \( R_t = R_t/A_t \) R&D investment
- \( G_t = G_t/A_t \) Govt Spending

Further note that because of the linearity assumption in the production of final goods, the Value function is a linear function in productivity with which an entrepreneur enters the sector:

\[ \hat{V}_t \equiv \frac{V_t(A_t)}{P_t A_t} = \hat{\Gamma}_t + (1 - z_t) \frac{\lambda_{t+1}}{\lambda_t} \hat{V}_{t+1} \]

where \( \hat{V} \) is normalized by the productivity with which the entrepreneur enters the sector.

**Definition H.1 (normalized equilibrium).** 25 endogenous variables \( \{ \lambda_t, \xi_t, \eta_t, \pi_t, \rho_t, X_{1t}, X_{2t}, Y_{1t}, Y_{2t}, m_t, a_t, b_t, \lambda_t^\beta, \lambda_t^\gamma, \lambda_t^\delta, \lambda_t^\zeta, \lambda_t^\xi, \lambda_t^\lambda, \lambda_t^\rho, \lambda_t^\sigma, \lambda_t^\tau, \lambda_t^\upsilon, \lambda_t^\phi, \lambda_t^\chi, \lambda_t^\psi, \lambda_t^\omega, \lambda_t^\nu, \lambda_t^\xi, \lambda_t^\pi, \lambda_t^\sigma, \lambda_t^\tau, \lambda_t^\upsilon, \lambda_t^\phi, \lambda_t^\chi, \lambda_t^\psi, \lambda_t^\omega \} \) given the natural rate allocation variables defined in Definition 2 \( \{ y_t^s, q_t^s \} \).

**Consumption Euler Equation**

\[ 1 = \beta E_t \left[ \frac{\lambda_{t+1}}{\lambda_t} \left( 1 + \frac{1 + \eta_t}{\xi_{t+1}} \right)^{-1} \xi_{t+1} \right] \quad \text{(H.8)} \]

\[ \lambda_t = \frac{1}{c_t} = \frac{c_{t+1}(1 + \zeta)}{c_t(1 + \xi_{t+1})} \quad \text{(H.9)} \]

**Price-Setting**

\[ \frac{X_{1t}^p}{X_{2t}^p} = \left( 1 - \theta_p \left( \left( \pi_t \right)^{1-\eta_t} \left( \pi_{t-1} \right)^{\eta_t} \right) \frac{1}{\tau_{t+1}} \left( \pi_t \right)^{\frac{1}{\tau_{t+1}}} \right)^{\frac{-\lambda_t}{\lambda_t^\beta}} \quad \text{(H.10)} \]

\[ X_{1t}^p = (1 + \lambda_{t} \lambda_t^\beta) y_t p_t A_t \left( \left( \pi_t \right)^{1-\eta_t} \left( \pi_{t-1} \right)^{\eta_t} \right)^{-\frac{1}{\tau_{t+1}}} \pi_{t+1}^{-\frac{1}{\tau_{t+1}}} X_{1t+1}^p \quad \text{(H.11)} \]

\[ X_{2t}^p = \lambda_t \left( y_t \right) + \theta_p \beta \left( \left( \pi_t \right)^{1-\eta_t} \left( \pi_t \right)^{\eta_t} \right)^{-\frac{1}{\tau_{t+1}}} \pi_{t+1}^{-\frac{1}{\tau_{t+1}}} X_{2t+1}^p \quad \text{(H.12)} \]

**Wage-Setting**
\[
X_{it}^w = (1 + \lambda_t w_t) X_{it-1} + \theta_t \beta \left( (\pi_{it-1}^w)^{1-v} \right) \frac{\omega_t}{\omega_{it-1}} - \lambda_t u_t + (1 + \lambda_t w_t) v
\]

(13)

\[
X_{it}^w = (1 + \lambda_{it} w_t) X_{it-1} + \theta_t \beta \left( (\pi_{it-1}^w)^{1-v} \right) \frac{\omega_t}{\omega_{it-1}} - \lambda_t u_t + (1 + \lambda_{it} w_t) v
\]

(14)

\[
X_{2it}^w = (1 + \lambda_{it} w_t) X_{2it-1} + \theta_t \beta \left( (\pi_{2it-1}^w)^{1-v} \right) \frac{\omega_t}{\omega_{2it-1}} - \lambda_t u_t + (1 + \lambda_{it} w_t) v
\]

(15)

Endogenous Growth Block

\[
\frac{\delta t}{\delta t} = \beta \frac{\lambda_{t+1}}{\lambda_t} \frac{\gamma V_{t+1}}{(1 - \delta_t)}
\]

(16)

\[
\dot{V}_t = \hat{V}_t + (1 - z_t) \beta \frac{\lambda_{t+1}}{\lambda_t} \frac{1}{\lambda_t} \dot{V}_{t+1}
\]

(17)

\[
\dot{V}_t = (\zeta - 1) \frac{\beta}{\delta_t} P^m_t y^m_t
\]

(18)

\[
\text{where } \zeta = \min \left( \frac{1}{1 - \gamma} \right)
\]

(19)

\[
R_t = \delta_t e^t
\]

(20)

Capital Investment

\[
k^m_{t+1} = u_t \left[ 1 - \left( \frac{I_t}{I_{t-1}} \right) \frac{1 + g_{tt}}{1 + g_{st}} \right] I_t + (1 - \delta_t) \frac{k^m_t}{1 + g_{tt}}
\]

(21)

\[
q_t = \beta \text{E} \left[ \frac{\lambda_{t+1}}{\lambda_t} \frac{1}{1 + g_{st}} \left( \frac{S^*_{t+1} u_{t+1} + a(u_{t+1}) + q_{t+1}(1 - \delta_k)}{1 + g_{st}} \right) \right]
\]

(22)

\[
q_t^m u_t \left[ 1 - S \left( \frac{I_t}{I_{t-1}} \frac{1 + g_{tt}}{1 + g_{st}} \right) - S' \left( \frac{I_t}{I_{t-1}} \frac{1 + g_{tt}}{1 + g_{st}} \right) \frac{1 + g_{tt}}{1 + g_{st}} + \beta \frac{\lambda_{t+1}}{\lambda_t} q_{t+1} q_{t+1} \frac{1 + g_{tt} + g_{st}}{1 + g_{st}} \frac{(I_{t+1}/I_t)^2}{(I_{t+1}+1)/I_t} = 1
\]

(23)

Capital utilization rate

\[
k_t = u_t \frac{k^m_t}{1 + g_{tt}}
\]

(24)

\[
\dot{r}_t = a' (u_t)
\]

(25)

Production Technologies

\[
y_t = y^m_t \quad \text{(to a first order)}
\]

(26)

\[
y^m_t = M_t k^m_t l^t_t - a
\]

(27)

\[
r^t_t = \frac{a p^m_t y^m_t}{k_t}
\]

(28)

\[
\omega_t = (1 - a) \frac{p^m_t y^m_t}{k_t}
\]

(29)

\[
g_t = \Omega_{t-1} z_{t-1} (\gamma - 1)
\]

(30)

Government

\[
1 + \frac{u_t}{1 + g_{st}} = \left( \frac{1 + u_{t-1}}{1 + g_{st}} \right) \rho_x \left[ \left( \frac{\pi_t}{\pi_{t-1}} \right) \phi_e \left( \frac{y_t}{y_{t-1}} \right) \right]^{1-\rho_x} \left( \frac{y_t / y_{t-1} + 1 + g_{tt}}{y_t / y_{t-1} + 1 + g_{tt}} \right) \phi_{ly} e^m_t
\]

(31)

Market Clearing

\[
y_t = c_t + I_t + R_t + a(u_t) \frac{k^m_t}{1 + g_{tt}} + \left( 1 - \frac{1}{\lambda_t} \right) y_t
\]

(32)

Shocks

\[
\log \lambda_{it} = (1 - \rho_p) \log \lambda_p + \rho_p \log \lambda_{it-1} + \epsilon_t^p - \mu_p e^m_{t-1} \quad \epsilon_t^p \sim N(0, \sigma_p)
\]

(33)
\[
\log M_t = (1 - \rho_m) \log M_t + \rho_m \log M_{t-1} + \epsilon_m^m; \quad \epsilon_m^m \sim N(0, \sigma_m) \quad \text{(H.34)}
\]

\[
\log \Omega_t = \rho_\Omega \log \Omega_{t-1} + \epsilon_\Omega^\Omega; \quad \epsilon_\Omega^\Omega \sim N(0, \sigma_\Omega) \quad \text{(H.35)}
\]

\[
\log v_t = \rho_v \log v_{t-1} + \epsilon_v^v; \quad \epsilon_v^v \sim N(0, \sigma_v) \quad \text{(H.36)}
\]

\[
\log \lambda_{w,t} = (1 - \rho_w) \log \lambda_w + \rho_w \log \lambda_{w,t-1} + \epsilon_w^w - \mu_w \epsilon_{t-1}^w; \quad \epsilon_w^w \sim N(0, \sigma_w) \quad \text{(H.37)}
\]

\[
\log \lambda_i^g = (1 - \rho_g) \lambda^g + \rho_g \log \lambda_{i,t-1}^g + \epsilon_i^g; \quad \epsilon_i^g \sim N(0, \sigma_g) \quad \text{(H.38)}
\]

\[
\log \xi_t^g = \rho_v \log \xi_{t-1}^g + \epsilon_v^g; \quad \epsilon_v^g \sim N(0, \sigma_v^g) \quad \text{(H.39)}
\]

\[
\epsilon_t^{mp} \sim N(0, \sigma_t^{mp}) \quad \text{(H.40)}
\]

### H.10 Natural Rate Allocation

#### Definition H.2 (normalized natural rate allocation).
19 endogenous variables \( \{ \lambda_{i,t}^f, r_{i,t}^f, S_{i+1}^f, \hat{c}_{t}^f, z_{i,t}^f, \hat{V}_{i+1}^f, \hat{V}_{i}^f, \hat{P}^m_{i,t}, y_{i,t}^m, k_{i,t}^{u,f}, k_{i,t}^{K,f}, \hat{I}_{i}^f, q_{i,t}, \epsilon_i^v, \epsilon_i^p, \text{ and shocks} \{ \lambda_{p,t}, \lambda_{w,t}, M_t, \lambda_t^g, \Omega_t, \xi_t^g \} \).

\[
1 = \beta \mathbb{E}_t \left[ \frac{\lambda_{i+1}^f}{\lambda_t^f (1 + S_{t+1}^f)} (1 + r_t^f) \right] + (\lambda_t^f)^{-1} \xi_t^f \quad \text{(H.41)}
\]

\[
\lambda_t^f = \frac{1}{c_t^f - \frac{\rho^c_t}{1 + S_t^f} - c_{t+1}^f (1 + S_{t+1}^f) - \rho^c_t} \quad \text{(H.42)}
\]

\[
(1 + \lambda_{p,t}) \hat{P}_t^m = 1 \quad \text{(H.43)}
\]

\[
(1 + \lambda_{w,t}) \omega (L_t^f)^{1+v} = (1 + \tau_t^v) \lambda_t^f w_t^f L_t^f \quad \text{(H.44)}
\]

\[
\hat{V}_t = \Gamma_t^f + (1 - z_t^f) \beta \mathbb{E}_t \left[ \frac{\lambda_{i+1}^f}{\lambda_t^f (1 + S_{t+1}^f)} \hat{V}_{i+1}^f \right] \quad \text{(H.45)}
\]

\[
\Gamma_t^f = (\xi - 1) \frac{1}{\xi} \hat{P}_t^m y_t^m, \quad \text{where} \quad \xi = \min \left( \lambda_t^f, \frac{1}{(1 - \tau_t^v) \alpha} \right) \quad \text{(H.46)}
\]

\[
\mathbb{R}_t^f = \delta (z_t^f)^e \quad \text{(H.47)}
\]

\[
k_{i+1}^{u,f} = u_t \left[ 1 - S \left( \frac{\hat{I}_{i}^f}{\hat{I}_{i+1}^f} \right) \right] \hat{I}_t^f + (1 - \delta_k) \frac{k_{i+1}^{u,f}}{1 + S_t^f} \quad \text{(H.48)}
\]

\[
q_t^f = \beta \mathbb{E}_t \left[ \frac{\lambda_{i+1}^f}{\lambda_t^f (1 + S_{t+1}^f)} \left( z_{i,t+1}^f u_{i+1}^f - a u_{i+1}^f + q_{i+1}^f (1 - \delta_k) \right) \right] \quad \text{(H.49)}
\]

\[
q_t^f u_t \left[ 1 - S \left( \frac{\hat{I}_{i+1}^f}{\hat{I}_{i+1}^f} \right) \right] = S_t^f \left( \frac{\hat{I}_{i}^f}{\hat{I}_{i+1}^f} \right) + \beta \frac{\lambda_{i+1}^f q_{i+1}^f u_{i+1}^f}{\lambda_t^f (1 + S_{t+1}^f) S_t^f} \left( \frac{\hat{I}_{i+1}^f}{\hat{I}_t^f} \right)^2 + \left( \frac{\hat{I}_{i+1}^f}{\hat{I}_t^f} \right) ^2 = 1 \quad \text{(H.50)}
\]

\[
k_t^f = \frac{u_t k_{i+1}^{u,f}}{1 + S_t^f} \quad \text{(H.51)}
\]

\[
r_t^{K,f} = a' (u_t^f) \quad \text{(H.52)}
\]

M.8
Shocks

\[ y_t^f = y_t^{m,f} \] (to a first order) (H.54)

\[ y_t^{m,f} = M_t(k_t^f a(L_t^f)^{1-a} \] (H.55)

\[ r_t^f = \frac{\alpha_p v_t^m y_t^{m,f}}{k_t^f} (H.56) \]

\[ w_t^f = (1 - a) \frac{v_t^m y_t^{m,f}}{L_t^f} \] (H.57)

\[ \delta_t^f = \Omega_{t-1} k_{t-1}^f (\gamma - 1) \] (H.58)

\[ y_t^f = c_t^f + I_t^f + R_t^f + a(u_t^f) \frac{k_t^{w,f} y_t^{w,f}}{1 + \delta_t^{w,f}} + \left( 1 - \frac{1}{\lambda_t^w} \right) y_t^f \] (H.59)

\[
\begin{align*}
\log \lambda_{P,t} &= (1 - \rho_p) \log \lambda_p + \rho_p \log \lambda_{P,t-1} + \epsilon_t^p \sim N(0, \sigma_p) \\
\log M_t &= (1 - \rho_m) \log M_t + \rho_m \log M_{t-1} + \epsilon_t^m \sim N(0, \sigma_m) \\
\log \Omega_t &= \rho_\Omega \log \Omega_{t-1} + \epsilon_t^\Omega \sim N(0, \sigma_\Omega) \\
\log v_t &= \rho_v \log v_{t-1} + \epsilon_t^v \sim N(0, \sigma_v) \\
\log \lambda_{w,t} &= (1 - \rho_w) \log \lambda_w + \rho_w \log \lambda_{w,t-1} + \epsilon_t^w \sim N(0, \sigma_w) \\
\log \lambda_t^f &= (1 - \rho_\delta) \lambda_t^f + \rho_\delta \log \lambda_{t-1}^f + \epsilon_t^f \sim N(0, \sigma_\delta) \\
\log \xi_t^f &= \rho_\xi \log \xi_{t-1}^f + \epsilon_t^\xi \sim N(0, \sigma_\xi) \\
\end{align*}
\]

\[ \Gamma = (\xi - 1) \frac{\alpha}{\beta} y_t^p \; y_t^{m,p}; \quad \text{where} \quad \xi \equiv \min \left( \lambda_t \frac{1}{(1 - \tau^f)\alpha} \right) \]

\[ y = c + I + R + \left( 1 - \frac{1}{\lambda_t^\delta} \right) y \]

\[ gdp = y \]

\[ R = \delta z^e \]
where growth is defined as a sequence of 21 sticky-economy variables

\[ (1 - \frac{1 - \delta_k}{1 + g}) k^u = \Pi \]

\[ 1 = \beta \left[ \frac{1}{1 + g} \left( r^K + (1 - \delta_k) \right) \right] \]

\[ k = \frac{k^u}{1 + g} \]

\[ r^K = a'(1) \]

\[ y = y^m \]

\[ y^m = k^u L^{1 - \alpha} \]

\[ r^k = \frac{\alpha}{\bar{e}} \frac{p^m y^m}{k} \]

\[ w = (1 - \alpha) \frac{p^m y^m}{L} \]

\[ g = z(\gamma - 1) \]

H.12 Approximate Equilibrium

We log-linearize the variables around the steady state as follows: for any variable \( x \),

\[ \hat{x}_t = \log \left( \frac{x_t}{x} \right) \]

where \( x \) is the steady state, except for the following variables

\[ \hat{g}_{t+1} = \log \left( \frac{1 + g_{t+1}}{1 + g} \right) \]

\[ \hat{h}_t = \log \left( \frac{1 + h_t}{1 + h} \right) \]

\[ \hat{e}_t = \lambda - 1 \hat{A}_t \hat{e}_t \]

\[ \hat{\lambda}_{p,J} = \log(1 + \lambda_{p,J}) - \log(1 + \lambda_p) \]

\[ \hat{\lambda}_{w,J} = \log(1 + \lambda_{w,J}) - \log(1 + \lambda_w) \]

Definition H.3 (Approximate Equilibrium). An approximate competitive equilibrium in this economy with endogenous growth is defined as a sequence of 21 sticky-economy variables \( \{ \hat{A}_t, \hat{g}_{t+1}, \hat{h}_t, \hat{e}_t, \hat{\lambda}_{t+1}, \hat{\lambda}_{p,J}, \hat{\lambda}_{w,J}, \hat{\lambda}_t, \hat{\lambda}_h, \hat{\lambda}_p, \hat{\lambda}_{w,J}, \hat{\lambda}_{p,J}, \hat{\lambda}_{w,J} \} \), 19 natural-rate-economy-variables \( \{ \hat{\lambda}_t, \hat{g}_{t+1}, \hat{h}_t, \hat{e}_t, \hat{\lambda}_{t+1}, \hat{\lambda}_{p,J}, \hat{\lambda}_{w,J}, \hat{\lambda}_t, \hat{\lambda}_h, \hat{\lambda}_p, \hat{\lambda}_{w,J} \} \) which satisfy the following equations, for a given sequence of 8 exogenous shocks \( \{ \hat{\xi}_t, \hat{\theta}_t, \hat{\lambda}_p', \hat{\lambda}_w', \hat{\lambda}_h, \hat{\lambda}_p, \hat{\lambda}_w, \hat{\lambda}_h \} \).

Consumption Euler Equation

\[ (E_t \hat{\lambda}_{t+1} - \hat{\lambda}_t - \hat{g}_{t+1}) + \hat{h}_t - E_t \hat{g}_{t+1} + \hat{e}_t = 0 \]  \hspace{1cm} (H.67)

\[ \hat{\lambda}_t = \frac{1 + g}{1 + g - h \beta} \left[ \hat{g}_t - \left( \frac{1 + g}{1 + g - h} (\hat{e}_t + \hat{g}_t) - \frac{h}{1 + g - h} \hat{\lambda}_{t+1} - \hat{g}_{t+1} \right) \right] \]  \hspace{1cm} (H.68)

Price-Setting

\[ \hat{\lambda}_t = \frac{\beta}{1 + \tau_p \beta} E_t \hat{\lambda}_{t+1} + \frac{\tau_p}{1 + \tau_p \beta} \hat{\lambda}_{t+1} + \kappa_p \hat{\lambda}_{t+1} + \kappa_p \hat{\lambda}_{t+1} \]  \hspace{1cm} (H.69)

where \( \kappa_p = \frac{(1 - \theta_p)(1 - \theta_p)}{\theta_p (1 + \tau_p \beta)} \).
Market Clearing

\[ \hat{A}_t^w = \frac{\beta}{1 + \iota w \beta} E_t \hat{A}_t^w + \frac{\iota_w}{1 + \iota w \beta} \hat{A}_{t-1}^w + \kappa_w [-\lambda_t + \nu \hat{L}_t - \hat{w}_t] + \kappa_w \lambda_{w,t} \]  

(H.70)

where \( \kappa_w \equiv \frac{(1-\theta_v)(1-\beta_w)}{\theta_w(1+\iota w \beta)(1+(1+\frac{1}{\beta_w}))} > 0 \)

\[ \hat{A}_t^m = \hat{A}_t - \hat{A}_{t-1} + \hat{A}_t + \hat{\xi}_t \]  

(H.71)

Endogenous Growth Block

\[ (\hat{\xi} - 1) \hat{z}_t = (E_t \hat{\lambda}_{t+1} - \hat{\lambda}_t - \hat{g}_{t+1}) + E_t \hat{V}_{t+1} \]  

(H.72)

\[ \hat{V}_t = \eta_f (\hat{p}_t^m + \hat{g}_t) - \eta_z \hat{z}_t + \eta_{\theta} (E_t \hat{\lambda}_{t+1} - \hat{\lambda}_t - \hat{g}_{t+1}) + \eta_q E_t \hat{V}_{t+1} \]  

(H.73)

where \( \eta_f = 1 - \frac{(1-z)}{1+g} > 0, \eta_z = \frac{\beta z}{1+g} > 0, \eta_{\theta} = \frac{(1-z)}{1+g} > 0 \)

\[ \hat{V}_t = \eta_f \hat{V}_t \]  

(H.74)

Capital Investment

\[ \hat{k}_t = \hat{a}_t + \hat{k}_t^m - \hat{g}_t \]  

(H.78)

\[ \hat{p}_t^m = \frac{a''(1)}{a'(1)} \hat{a}_t \]  

(H.79)

Production Technologies

\[ \hat{g}_t = \hat{g}_t^m \]  

(H.80)

\[ \hat{g}_t^m = \hat{M}_t + \alpha \hat{k}_t + (1-\alpha) \hat{L}_t \]  

(H.81)

\[ \hat{p}_t^m = \hat{p}_t^m + \hat{g}_t^m - \hat{t}_t \]  

(H.82)

\[ \hat{w}_t = \hat{p}_t^m + \hat{g}_t^m - \hat{L}_t \]  

(H.83)

\[ \eta_{\theta} \hat{g}_{t+1} = \Omega_t + \hat{z}_t \]  

(H.84)

Government

\[ \hat{u} = \max \left( -\log(1+i_{st}), p_R \hat{u}_{t-1} + (1-p_R) \right) \phi_{pt} \hat{r}_t + \phi_{g} (\hat{g}_t - \hat{g}_t') + \phi_{\hat{w}} \left( (\hat{w}_t - \hat{w}_{t-1}) - (\hat{g}_t' - \hat{g}_{t-1}) + (\hat{g}_t - \hat{g}_t') + \xi_{mt} \right) \]  

(H.85)

Market Clearing

\[ \frac{1}{\lambda^t} \hat{g}_t = \frac{c}{y} \hat{c}_t + \frac{1}{y} \hat{L}_t + \frac{R}{y} \hat{R}_t + \frac{a'(1)k}{y} \hat{a}_t + \frac{1}{\lambda^t} \hat{\lambda}_t \]  

(H.86)

\[ \hat{g} \hat{d}_p_t = \hat{g}_t - \frac{a'(1)k}{y} \hat{a}_t \]  

(H.87)

Natural Rate Allocation Variables

Consumption Euler Equation

\[ (E_t \lambda_{t+1}^f - \lambda_t^f - \hat{g}_{t+1}^f) + \hat{p}_t^f + \hat{\xi}_t = 0 \]  

(H.88)

\[ \hat{\lambda}_t^f = \frac{1}{1+g-h\beta} \left( \hat{g}_t^f - \left( \frac{1+g}{1+g-h} (\hat{c}_t' + \hat{g}_t') - \frac{h}{1+g-h} \hat{c}_t' \right) \right) \]  

(H.89)

\[ + E_t \frac{h\beta}{(1+g-h\beta)} \left( \hat{g}_t' + \hat{g}_{t+1} + \hat{g}_{t+1} - \frac{h}{1+g-h} \hat{c}_t' \right) \]  

M.11
where \( \eta > 1 - \frac{(1-z)R}{\gamma} > 0 \), \( \eta_z = \frac{\beta_z}{\gamma} > 0 \), \( \eta_q = \frac{(1-z)\eta R}{\gamma} > 0 \)

\[
\tilde{R}^f_i = \tilde{q}^f_i
\]

Capital Investment

\[
k_{i+1}^f = \frac{1}{k^f} (\tilde{a}_i + \tilde{q}^f_i) + \frac{1 - \delta_k}{1 + \frac{1}{k^f}} [k_{i+1}^f - \tilde{g}_{i+1}^f]
\]

\[
\tilde{q}^f_i = \left[ \mathbb{E}_i \hat{a}_{i+1} + \hat{\lambda}_{i+1}^f - \hat{g}_{i+1}^f \right] + \frac{1}{r^k + (1 - \delta_k)} \hat{q}^k_{i+1} + \frac{1 - \delta_k}{r^k + (1 - \delta_k)} \mathbb{E}_i \tilde{q}^f_{i+1}
\]

\[
\tilde{q}^f_i + \hat{a}_i - S'' \left( \tilde{w}_i^f - \tilde{w}_{i+1}^f + \tilde{g}_{i+1}^f \right) + \beta S'' \left( \tilde{w}_{i+1}^f - \tilde{w}_i^f + \tilde{g}_{i+1}^f \right) = 0
\]

Capital utilization rate

\[
\hat{k}_i^f = \tilde{a}_i + k_{i+1}^f - \tilde{g}_{i+1}^f
\]

\[
\hat{p}_i^f = \frac{\alpha''(1)}{\alpha'} \tilde{q}_i^f
\]

Production Technologies

\[
\tilde{q}_i^f = \tilde{q}_i^\text{m,f}
\]

\[
\hat{q}_i^\text{m,f} = \tilde{M}_i + \alpha k_i^f + (1 - \alpha) \hat{L}_i^f
\]

\[
\hat{p}_i^\text{k,f} = \hat{p}_i^\text{m,f} \hat{y}_i^\text{m,f} - k_i^f
\]

\[
\hat{a}_i^f = \hat{p}_i^\text{m,f} \hat{y}_i^\text{m,f} - \hat{L}_i^f
\]

\[
\eta \hat{a}_{i+1} = \tilde{\Omega}_i + \tilde{z}_i^f
\]

Market Clearing

\[
\frac{1}{\lambda y} \tilde{g}_i^f = \frac{c}{y} e_i^f + \frac{1}{y} \tilde{h}_i^f + \frac{R}{y} \tilde{R}_i^f + \frac{\alpha''(1)k}{y} \tilde{a}_i^f + \frac{1}{\lambda y^2} \tilde{\lambda}_i
\]

\[
\tilde{g} \tilde{p}_i^f = \tilde{g}_i^f - \frac{\alpha''(1)k}{y} \tilde{a}_i^f
\]