Borrowing Costs and the Equity Premium in Standard OLG Models*

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Abstract

Simulating a realistic-sized equity premium in macroeconomic models has proved a daunting challenge, hence the "equity premium puzzle". "Resolving" the puzzle requires heavy lifting. Precise choices of particular preferences, shocks, technologies, and hard borrowing constraints can do the trick, but haven't stopped the search for a simpler and more robust solution.

This paper suggests that soft, but rapidly rising borrowing costs, imbedded in an otherwise standard overlapping generations model can work. Its model features ten periods, isoelastic preferences with modest risk aversion, Cobb-Douglas production, realistic shocks, and reasonable fiscal policy. Absent borrowing costs, the model's equity premium is extremely small. Adding the costs readily produces large equity premiums.

These results differ from those of Constantinides, Donaldson, and Mehra (2002). In their model, hard borrowing constraints on the young can produce large equity premiums. Here soft, but rising borrowing costs on all generations are needed.

Keywords: Equity Premium; Borrowing Constraints; Aggregate Shocks; Incomplete Markets; Stochastic Simulation.

JEL Classification: D91, D58, E1, E2, C63, C68

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1 Introduction

The U.S. equity premium—the difference between the mean return on U.S. stocks (the market portfolio) and short-term Treasuries (3-month T-Bills)—has averaged 6 percent on an annual basis, since 1925. The average risk-free rate during this period has averaged 1 percent. In their seminal paper Mehra and Prescott (1985) showed that a reasonably calibrated representative-agent model produces an equity premium of at most 0.35 percent and a risk-free rate of 4 percent—hence the "equity premium puzzle".

A large literature has tried to resolve the puzzle, invoking a lot of machinery. For example, hard borrowing constraints were used by Constantinides, Donaldson, and Mehra (2002). Alternative preferences were used by Constantinides (1990), Abel (1990), Epstein and Zin (1991), Jermann (1997), Campbell and Cochrane (1999), Campbell (2001), Ju and Miao (2012), and Liu and Miao (2014). Capital frictions were used by Jermann (1997), Tallarini (2000), Boldrin, Christiano, and Fisher (2001), and Liu and Miao (2014). Rare disasters in technology processes were used by Rietz (1988), Barro (2006), Gabaix (2012), and Wachter (2013). Significant stochastic depreciation shocks were used by Krueger and Kubler (2006). Non-fundamental shocks (sunspots) were used by Farmer (2014). Finally, behavioral models, such as those based on prospect theory, were used by Barberis, Huang, and Santos (2001). See also the survey by Mehra (2006) and the references therein.

This paper shows that borrowing costs, which are rapidly rising in the amount borrowed, imbedded in an otherwise standard general equilibrium overlapping generation (OLG) model, can easily generate a sizable equity premium. The model features ten periods (referencing ages 20 to 80), isoelastic preferences with risk aversion of 2, Cobb-Douglas production, and realistically calibrated total factor productivity (TFP) shocks. It also includes government consumption at 20 percent of GDP and pay-as-you-go social security. The model features no government debt, but can be relabeled (see Green and Kotlikoff, 2008) to have as much or as little debt as desired.

In the model, agents work prior to retirement and rent capital to firms. They are uncertain about the compensation accruing to their supplies of labor and capital due to the uncertainty about the economy's future productivity. However, they are affected differently by this uncertainty, since they are in different stages in their life cycle and hence possess human and physical capital in different proportions. To hedge their labor and capital income risks, they save and invest in risky capital and borrow and lend from each other in the private market for one-period safe bonds. Borrowing costs imply a wedge between the borrowing and lending rates. These costs are specified exogenously and are increasing, in a convex fashion, in the size of the loan.

Absent borrowing costs, the model's equity premium is extremely small. This is due to the fact that the risk-free rate is just as high as the rate of return on capital, which is set at a reasonable value by the saving behavior of the young (i.e., their time preference rate). By contrast, borrowing costs reduce the private supply of bonds dramatically, increasing the bond price and lowering the safe rate of return. The equity premium rises from essentially zero without borrowing costs to over 5 percent with them, and the risk-free rate declines from roughly 6 – 8 percent to roughly 1 – 2.5 percent.

The definition of equity premium employed in this paper is the equilibrium gap between the mean return to capital and the return to risk-free bonds, rather than this gap per unit of risk. This is in the space of Mehra and Prescott (1985) and Constantinides, Donaldson, and Mehra (2002). For example, Mehra and Prescott (1985) stress that while their exercise is well suited for addressing this equilibrium gap, it is poorly suited for other issues, in particular issues such as the volatility of asset prices. That is because in their paper, like in this one, the risky asset traded is the claim to the entire capital in the economy, rather than just to corporate dividends. The return to the economy's total capital stock is not highly volatile because a significant fraction of it is government infrastructure.[1]

[1] Government infrastructure includes bridges, highways, water systems, schools, etc. According to the COB's estimates, in 2004 the US invested 400 billion dollars in this type of capital.

Although it is not needed to generate equity premium, including modest capital depreciation shocks or capital adjustment costs allows the model to emit a pattern of increasing bond holdings by age. Without either of these features, the old are the borrowers (the suppliers), with them they are the lenders (the demanders) of bonds. This is because capital depreciation shocks and capital adjustment costs expose investment principal to risk. Since, for the elderly, the only source of income is their investments, they demand bonds when the principal is risky, otherwise they supply them.

The model builds on Hasanhodzic and Kotlikoff (2013) by adding borrowing costs, stochastic depreciation, and capital adjustment costs. Like that paper's, its solution method relies on the general framework of Judd, Maliar, and Maliar's (2009, 2011) algorithm—a numerically stable and accurate extension of Marcet (1988)—to overcome the curse of dimensionality.

Comparison to Constantinides, Donaldson, and Mehra (2002)

Constantinides, Donaldson, and Mehra (2002) posit a three-period OLG model with pure exchange and a hard borrowing constraint on the young to resolve the equity premium puzzle. They justify this constraint based on the inability of the young to borrow against their future earnings. The authors suggest that their results would also hold in more realistic extensions, as long as the young face hard borrowing constraints. The extensions they mention include a larger number of generations, standard technology, and the inclusion of government policy.

Recall that this paper features ten, not three generations, Cobb-Douglas technology, and government spending and generational policy. But it assumes soft, but rapidly rising borrowing costs, which seem more realistic (see the next subsection). Interestingly, limiting the borrowing costs to a subset of generations does not have the effect of lowering the risk-free rate and increasing the equity premium. Only when *all* generations face the costs does the supply of bonds become limited enough for the risk-free rate to decline substantially and the equity premium to emerge. Indeed, even if there is a single generation that is unconstrained, *it* becomes the low-cost supplier of bonds to the economy and hence the marginal investor relevant for their pricing. This makes perfect sense in light of the finding by Hasanhodzic and Kotlikoff (2013) of minimal differences across generations in standard, realistically-calibrated models.

Borrowing Costs

Representing agents' borrowing opportunities via soft, but rising borrowing costs seems a reasonable alternative to hard borrowing constraints, which prevent borrowing altogether. Most households appear able to borrow very small amounts at low rates. But taking out more sizable loans on an unsecured basis entails credit card-level interest rates.

This is consistent with the empirical evidence of Scott (1996), who documents significant variation in interest rates charged on unsecured loans in the U.K. Specifically, he finds that the deposit rates are less than half the interest rates on bank personal loans, that the latter are significantly lower than credit card rates, which in turn are lower than retail loan rates. He then shows that individuals should rationally first use bank personal loans, only when that is no longer available turn to credit cards, and only when those are exhausted resort to retail cards. He concludes that "the most appropriate way of modeling agents' borrowing opportunities is by an upward-sloping interest rate schedule" (page 2). In other words, as an individual's debt increases, their marginal borrowing rate rises.

Scott (1996) further suggests that such representation is consistent with the theoretical model of Milde and Riley (1988), where banks screen borrowers by offering larger loans at higher interest rates and borrowers signal their quality by accepting larger loans at higher interest rates. Another theoretical model which predicts that the cost of borrowing increases with the size of the loan is Chatterjee, Corbae, Nakajima, and Rios-Rull (2007). In that paper, a larger loan commands a higher interest rate since it induces a higher probability of default, and the intermediaries who make unsecured loans take this probability into account.[2]

[2]Other studies permit the borrowing rate to increase in the amount borrowed in various contexts, see, e.g., Altonji and Siow (1987), Wirjanto (1995), and Fernandez-Corugedo (2002).

Although not modeled explicitly in order to explore the effect of borrowing costs on the risk-free rate in a parsimonious paradigm, the default risk is at the heart of increasing marginal borrowing costs in this paper as well. Because of it, as well as due to informational time-consistency issues, the intermediary exerts effort to verify the borrower's creditworthiness. Moreover, the larger is the amount borrowed, the higher is the risk of default, and the more resources are committed to ensuring more precise verification of the agent's creditworthiness. For example, FICO scores, tax returns, conversations with employers, interviews with the borrower, and the appraisals of the borrower's property may be added successively to the checklist as the size of the loan increases. In addition, the larger is the amount borrowed, the greater is the labor supplied by the borrower who processes the paperwork and proves the validity of their identity to the intermediary. As a result, in intermediation there exist two interest rates: one that the intermediary receives, and another one that the borrower pays in interest. For example, a borrower may be paying a credit card rate of 25 percent, while the ultimate sources of funds (the owners of the corporation) may be getting 5 percent—the difference is interpreted as real resources consumed in the process of intermediation.

As one would expect, the steeper is the interest rate schedule, the smaller is the gross bond supply, the lower is the risk-free rate, and the larger is the equity premium. For example, in the equilibrium with the risk-free rate of roughly 1 – 2.5 percent and the equity premium of roughly 5 percent, the ratio of the marginal borrowing cost to the risk-free rate is typically 15 to 1. This seems reasonable. It is not uncommon for credit card companies to charge 20 percent interest rates, a ratio, again, of roughly 15 to 1 with the risk-free rate of 1 – 2 percent. Even for someone who is very rich, has excellent credit score, owns a million dollar house (on which the mortgage company owns the insurance policy), a rate on an effectively very small home equity loan is at least 10 times higher than the 1-year T-Bill rate. The reason is the inability of the lender to observe what the borrower's actions may be at some point in the future. But in the equilibrium where this ratio is typically 2 to 1, the risk-free rate and the equity premium are roughly 3 – 4.5 percent and 3 percent, respectively.

Another measure of the plausibility of these borrowing costs is how much the welfare changes when they are introduced. It changes very little, suggesting that they are not implausible. The reason is that realistically calibrated aggregate or macroeconomic risk is too small to have significant microeconomic consequences, as documented by Lucas (1987), Krusell and Smith (1999), and Hasanhodzic and Kotlikoff (2013). As a result, the agents are not eager to hold bonds even if they are unconstrained, and large changes in the price of bonds translate into small changes in welfare.

This paper is not the first to introduce exogenous costs or benefits to asset holding or loan creation. For example, Krishnamurthy and Vissing-Jorgensen (2012) argue that safe assets such as treasury debt provide a convenience yield which drives the yield on treasuries down relative to other assets such as corporate bonds. Also, Goodfriend (2005) and Goodfriend and McCallum (2007) expand a standard DSGE model with a production function for loan creation that takes as inputs monitoring effort and collateral, and use it to study the role of money and banking in monetary policy analysis.

The paper is organized as follows. Sections 2 and 3 describe the model and its calibration. Section 4 presents the results, Section 5 conducts a sensitivity analysis, and Section 6 evaluates the accuracy of solutions. Section 7 concludes. The solution algorithm is described in the Appendix.

2 The Model

The model features G overlapping generations with shocks to total factor productivity and either capital depreciation shocks or capital adjustment costs. Each agent works through retirement age R, dies at age G, and maximizes expected lifetime utility. There is an increasing cost of supplying bonds, i.e. of borrowing. If there are adjustment costs, firms maximize their financial value, i.e. the present value of their revenue flow, otherwise they maximize static profits.

2.1 Endowments and Preferences

The economy is populated by G overlapping generations that live from age 1 to age G. All agents within a generation are identical and are referenced by their age g and time t. Each cohort of workers supplies 1 unit of labor each period. Using a hump-shaped age-earnings profile as in Hansen (1983) and Krueger and Kubler (2006), where labor supplies by age are given by $[0.624, 0.794, 0.909, 0.985, 1.032, 1.041, 1.039, 0.576, 0, 0]$, does not change the results. Hence, total labor supply equals the retirement age R. Utility is time-separable and isoelastic, with risk aversion coefficient γ:

$$u(c) = \frac{c^{1-\gamma} - 1}{1 - \gamma}. \tag{1}$$

2.2 Financial Markets

Households save and invest in either risky capital or one-period safe bonds. Investing 1 unit of consumption in bonds at time t yields $1 + \bar{r}_t$ units in period $t + 1$. The safe rate of return, $\bar{r}_t$, is indexed by t since it is known at time t although it is received at time $t + 1$. The total demand for assets of household age g at time t is denoted by $\theta_{g,t}$, and its share of assets invested in bonds is denoted by $\alpha_{g,t}$. Households enter period t with $\theta_{g-1,t-1}$ in assets, which corresponds to the total assets they demanded the prior period. Since investment decisions are made at the end of the period, the aggregate supply of capital in period t, K_t, is the sum of assets brought by the households into period t, i.e.

$$K_t = \sum_{g=1}^{G} \theta_{g,t-1}, \tag{2}$$

normalized by q_{t-1} in the case of adjustment costs. Bonds are in zero net supply, hence by being short (long) bonds, households are borrowing (lending) to each other.

2.3 Technology

Production is Cobb-Douglas with output Y_t given by

$$Y_t = z_t K_t^{\alpha} L_t^{1-\alpha}, \tag{3}$$

where z is total factor productivity, α is capital's share of output, and L_t is labor demand, which equals R, labor supply. Equilibrium factor prices are given by

$$w_t = z_t(1-\alpha)\left(\frac{\sum_{g=1}^{G}\theta_{g,t-1}}{R}\right)^{\alpha}, \tag{4}$$

$$r_t = z_t\alpha\left(\frac{\sum_{g=1}^{G}\theta_{g,t-1}}{R}\right)^{\alpha-1} - \delta_t, \tag{5}$$

where depreciation $\delta_t \sim \mathcal{N}(\mu_\delta, \sigma_\delta^2)$, as in Ambler and Paquet (1994).

With capital adjustment costs, r is given by

$$r_t = \frac{z_t\alpha(\frac{K_t}{R})^{\alpha-1} + \frac{m}{2}(\frac{I_t}{K_t})^2 + q_t - q_{t-1}}{q_{t-1}}, \tag{6}$$

where $q_t = 1+m\frac{I_t}{K_t}$ is the price of capital, $K_t = \frac{\sum_{g=1}^{G}\theta_{g,t-1}}{q_{t-1}}$ is the capital stock, $I_t = K_{t+1}-K_t$ is the investment at time t, and m is the adjustment cost parameter (see, e.g., Auerbach and Kotlikoff, 1987 and the references therein).

Total factor productivity, z, obeys

$$\ln(z_{t+1}) = \rho\ln(z_t) + \epsilon_{t+1}, \tag{7}$$

where $\epsilon_{t+1} \sim \mathcal{N}(0, \sigma^2)$.

2.4 Borrowing Costs

The model's borrowing costs are implemented via a smoothing function proposed by Chen and Mangasarian (1996). The function is smooth and rising for negative bond holdings, and is essentially zero when bond holdings are close to zero or positive (see Figure 1).

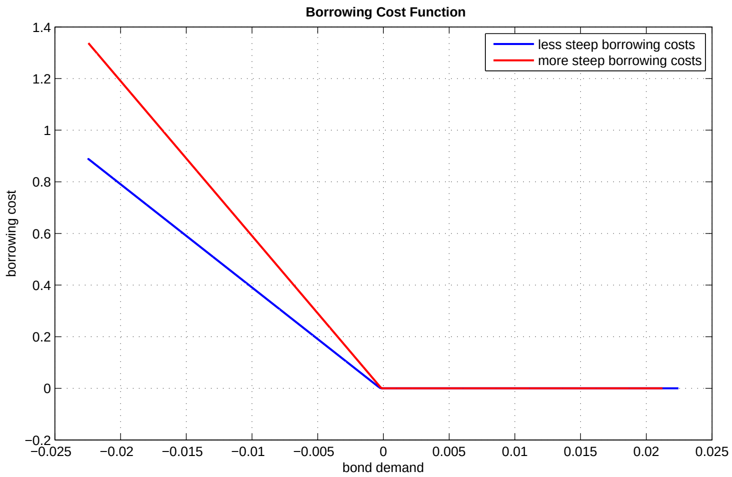

Figure 1: Borrowing cost functions with different slope parameters for a fixed level of assets. The x-axis displays bond demands. Since the asset level is fixed, the functions are increasing in bond shares.

Specifically, to borrow the amount of $\alpha\theta$ households have to pay the borrowing cost of $f(\alpha)\theta$, where

$$f(\alpha) = 0.2\left(-b\alpha - 1 + \frac{1}{5}\ln(1 + e^{5b\alpha+5})\right) \tag{8}$$

and b is the parameter described in Section 3 governing slope of f. Since f is increasing in bond shares (α), for a given level of assets (θ) the marginal borrowing cost is increasing in total amount borrowed ($\alpha\theta$).

Specifying the borrowing costs via $f(\alpha)\theta$ rather than $f(\alpha\theta)$ insures that the model remains scalable: if all of the "level" variables were to double, the borrowing cost would double as well. This specification also makes economic sense. With 2θ in assets and some α, the marginal costs would be the same as with θ in assets and that same alpha, since the extra assets could be used as collateral. This is in line with Goodfriend (2005) and Goodfriend and McCallum (2007), where collateral is as a valuable input in loan production because it enables a bank to enforce the repayment of loans with less monitoring (i.e., the greater is the borrower's collateral, the more productive is the intermediary's monitoring effort).

2.5 Government

Government spending equals a fixed share ξ of GDP each period. It is financed by the payroll tax τ. Two types of intergenerational redistribution policy are considered: a fixed benefit policy and a proportional tax policy. Under the former, the government takes a variable share of the wage from the workers and gives a fixed benefit to the elderly. Under the latter, it takes a fixed share of the wage from the workers and gives a variable amount to the elderly. The transfers to the elderly are denoted by H_t.

2.6 Household Problem

Households of age g in state (s, z, δ) maximize expected remaining lifetime utility given by

$$V_g(s, z, \delta) = \max_{c,\theta,\alpha} \left\{u(c) + \beta \mathrm{E}\left[V_{g+1}(s', z', \delta')\right]\right\} \tag{9}$$

subject to

$$c_{1,t} = \ell_1(1 - \tau_t)w_t - \theta_{1,t} + (1 - \ell_1)H_t, \tag{10}$$

$$\begin{aligned} c_{g,t} = {} & \ell_g(1 - \tau_t)w_t + \left[\alpha_{g-1,t-1}(1 + \bar{r}_{t-1}) + (1 - \alpha_{g-1,t-1})(1 + r_t)\right]\theta_{g-1,t-1} - \theta_{g,t} \\ & - f(\alpha)\theta_{g-1,t-1} + (1 - \ell_g)H_t, \\ & \text{for } \ 1 < g < G, \ \text{ and} \end{aligned} \tag{11}$$

$$\begin{aligned} c_{G,t} = {} & \ell_G(1 - \tau_t)w_t + \left[\alpha_{G-1,t-1}(1 + \bar{r}_{t-1}) + (1 - \alpha_{G-1,t-1})(1 + r_t)\right]\theta_{G-1,t-1} \\ & - f(\alpha)\theta_{G-1,t-1} + (1 - \ell_G)H_t, \end{aligned} \tag{12}$$

where $c_{g,t}$ is the consumption of a g-year old at time t, τ_t is the payroll tax financing government spending and transfers to the elderly, H_t is the benefit given to the elderly, and (10)–(12) are budget constraints for age group 1, those between 1 and G, and that for age group G.

2.7 Equilibrium

At time t, the economy's state is (s_t, z_t, δ_t), with $s_t = (\theta_{1,t-1}, \ldots, \theta_{G-1,t-1})$ denoting the set of age-specific asset holdings. Given the initial state of the economy s_0, z_0, δ_0, where $s_0 = (\theta_{1,-1}, \ldots, \theta_{G-1,-1})$, the recursive competitive equilibrium is defined as follows:

Definition. The recursive competitive equilibrium is governed by the collection of the value functions and the household policy functions for total savings $\theta_g(s, z, \delta)$, the share of savings invested in bonds $\alpha_g(s, z, \delta)$, and consumption $c_g(s, z, \delta)$ for each age group g, the choices for the representative firm $K(s, z, \delta)$ and $L(s, z, \delta)$, as well as the pricing functions $r(s, z, \delta)$, $w(s, z, \delta)$, and $\bar{r}(s, z, \delta)$ such that:

1. Given the pricing functions, the value functions (9) solve the recursive problem of the households subject to the budget constraints (10)–(12), and θ_g, α_g, and c_g are the associated policy functions for all g and for all dates and states.

2. Wages and rates of return on capital satisfy (4) and either (5) or (6), i.e. at each point, for given w and r the firm maximizes profits if there are no adjustment costs and maximizes the firm value otherwise.

3. All markets clear: Labor and capital market clearing conditions are implied by $L_t = R$ and (2). Since bonds are in zero net supply, bond market clearing requires

$$\sum_{g=1}^{G} \alpha_g(s, z, \delta)\theta_g(s, z, \delta) = 0. \tag{13}$$

Market clearing conditions in labor, capital, and bond markets and satisfaction of household budgets imply market clearing in consumption.

4. The government balances its budget, i.e.,

$$\tau_t = \frac{\xi Y_t + H_t(G - R)}{w_t R}. \tag{14}$$

Finally, for all age groups $g = 1, \ldots, G-1$, optimal intertemporal consumption and investment choice satisfies

$$\begin{aligned} 1 = \beta \mathrm{E}_z \big[\big(1 + r(s', z', \delta') + \alpha_g(s, z, \delta)(\bar{r}(s, z, \delta) - r(s', z', \delta')) \\ - f(\alpha_g(s, z, \delta))\big) \frac{u'(c_{g+1}(s', z', \delta'))}{u'(c_g(s, z, \delta))} \big] \end{aligned} \tag{15}$$

and

$$0 = \mathrm{E}_z \left[u'(c_{g+1}(s', z', \delta'))(\bar{r}(s, z, \delta) - r(s', z', \delta') - f'(\alpha_g(s, z, \delta))) \right], \tag{16}$$

where E_z is the conditional expectation of z' given z, and $f'(\alpha) = 0.2b \left(-1 + \frac{\mathrm{e}^{5b\alpha+5}}{1+\mathrm{e}^{5b\alpha+5}}\right)$ is the derivative of f given by (8). Note that the endogenous part of the state next period, s', is determined by the asset demands chosen the period before.

3 Calibration

The parameters are calibrated as follows.

3.1 Endowments and Preferences

The risk aversion parameter γ is set to 2. Agents work for 7 periods and live for 10. Hence, each period represents 6 years. The quarterly subjective discount factor, β, is set at 0.99, as is standard in the macroeconomics literature.

3.2 Technology

Quarterly values for ρ and σ are 0.95 and 0.01, respectively, as estimated in the empirical literature (see, e.g., Hansen (1985) or Prescott (1986)). Capital share of output, α, equals 0.33. The quarterly value for σ_δ is 0.0026. The adjustment cost parameter m is set to 10.

3.3 Borrowing Costs

The borrowing cost parameter b equals 200 in the flatter case and 300 in the steeper case. The steeper calibration produces a reasonable ratio of the marginal borrowing cost to the risk-free rate at the equilibrium (see the Borrowing Costs subsection of Section 1).

3.4 Government

The government spending share, ξ, equals 20 percent. Under the fixed benefit policy, the benefit equals 20 percent of the average wage. Under the proportional tax policy, the tax equals 20 percent of the wage.

4 Results

4.1 Fluctuations

Before discussing the demand for bonds and the associated equity premium, it is instructive to describe the fluctuations against which the agents might want to insure. Figure 2 plots the evolution over 640 years of the capital stock, output, the wage, and the rate of return on capital for three cases—the model with stochastic depreciation, the model with adjustment costs, and the base model without either. The fixed benefit policy is in place in each case.

Total factor productivity z across the 640 years has a mean of 0.999 (1.000) and a standard deviation of 0.033 (0.032) in models without (with) stochastic depreciation. This produces sizable fluctuations in capital, output, the wage, and the annualized rate of return,

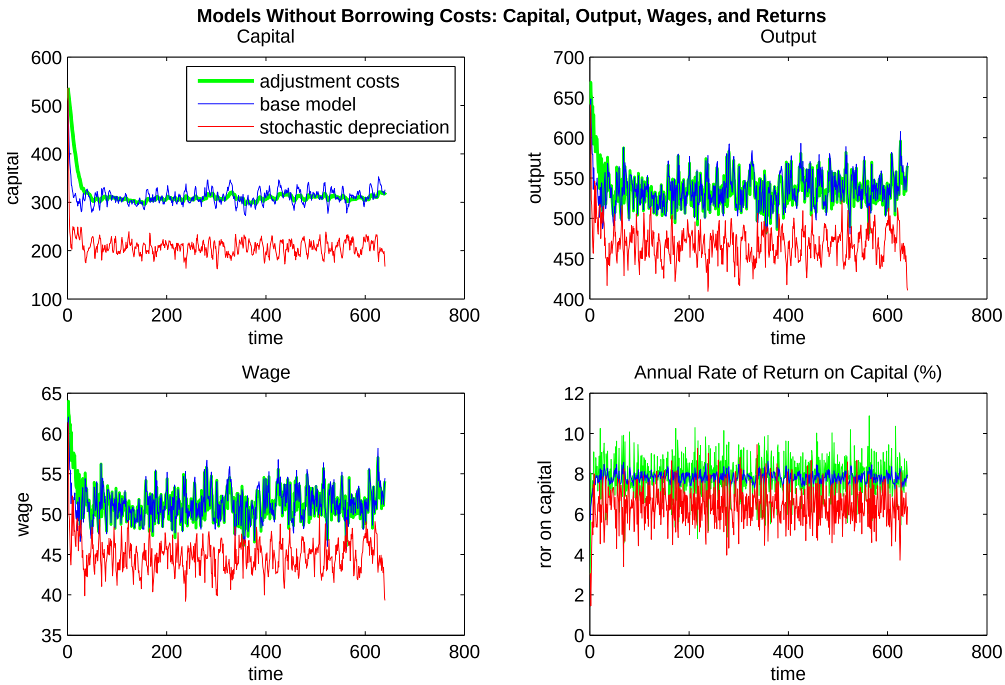

Figure 2: Capital, output, wage, and rate of return on capital in the base model, and in models with stochastic depreciation or adjustment costs, with the fixed benefit policy and without borrowing costs.

with standard deviations of 0.136, 0.203, 0.019, and 0.003 around the means of 3.108, 5.356, 0.513, and 0.078 in the base model.[3]

Over the same time period, stochastic depreciation has a mean of 0.293 and a standard deviation of 0.064. It has the effect of lowering the capital stock to 2.063 on average and increasing its standard deviation somewhat to 0.138. Output and the wage decrease to 4.673 and 0.447 on average. The rate of return on capital is directly hit by depreciation shocks, hence its standard deviation triples to 0.009 around the mean of 0.064.

With adjustment costs, the firms "smooth" or "partially adjust" their investment behavior over time. Hence, the standard deviation of capital, 0.048, is less than half that in the base model around roughly the same mean (3.083). Consequently, output and the wage fluctuate less than in the base model: the standard deviation is 0.181 around the mean of 5.342 for the output, and 0.017 around the mean of 0.511 for the wage. The rate of return on capital is even more volatile than in the base model—it has a standard deviation of 0.010 around the mean of 0.078—since the marginal cost of investment, q, fluctuates substantially, exhibiting a standard deviation of 0.048 around the mean of 1.001.

4.2 Bond Demands and the Equity Premium Without Borrowing Costs

Figure 3 plots age-specific average bond shares, assets, and bond demands for the three models without borrowing costs. The bond share is the proportion of assets invested in bonds, and the bond demand is the absolute amount demanded in bonds. This figure shows that bonds are supplied by the young in the model with stochastic depreciation, by the middle-aged in the model with adjustment costs, and by the old in the base model.

This pattern is intuitive since the old live off their equity income, i.e., the assets they have accumulated (the principal) and the return earned on them. With stochastic depreciation or

[3] The first 50 observations are excluded from computations of statistics in this section so that results are insensitive to initial conditions.

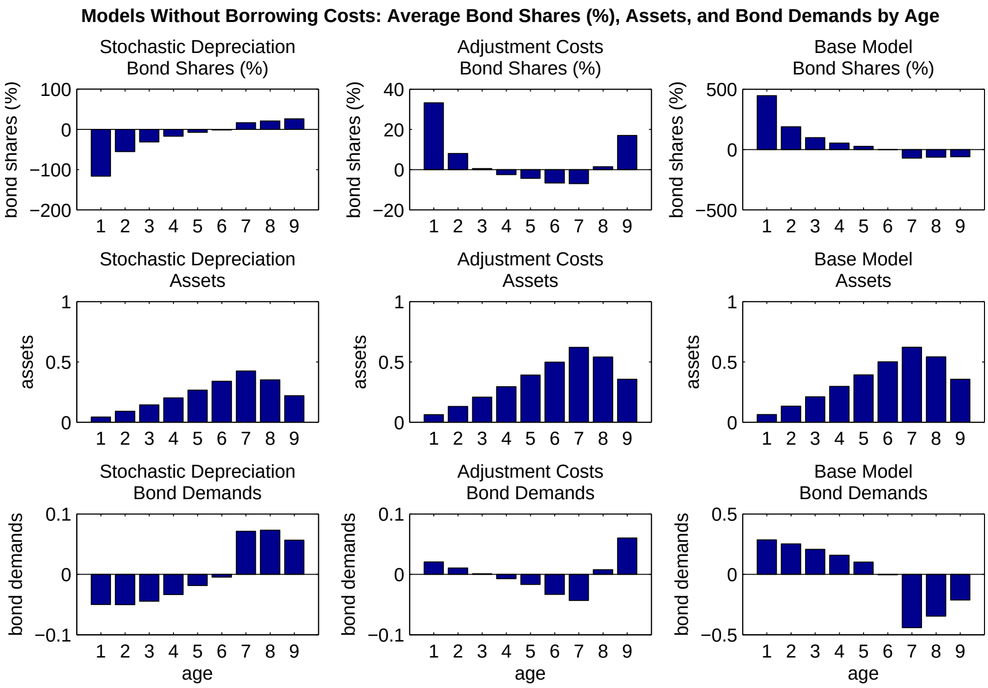

Figure 3: The average bond shares (%), assets, and bond demands by age in the models with stochastic depreciation or adjustment costs, and in the base model, with the fixed benefit policy and without borrowing costs.

adjustment costs, both the principal and the return on assets are uncertain and, thus, can be lost. Thus, the old demand bonds.

On the other hand, the young live off both the wage income and, except for the first generation, the equity income. With stochastic depreciation, the wages and the equity returns are uncorrelated, exhibiting a correlation coefficient of -2.77 percent. Hence, the income sources of the young are more diversified than those of the old. Consequently, the young are in a position to insure the old by selling bonds to them and going long capital. Indeed, this model exhibits a pattern of increasing bond shares with age.

With adjustment costs, the income sources of the young are less diversified: the wages and the returns exhibit a correlation coefficient of 45.07 percent. This makes the young less willing to supply bonds. In fact, the first three generations demand bonds, while the relatively wealthier middle-aged supply them.

In the base model, the age pattern of bond holdings is reversed, with the old supplying the bonds that the young demand. As above, the main asset of the young—their wages—is positively correlated with the rate of return on capital, exhibiting a correlation coefficient of 38.97. But without stochastic depreciation or adjustment costs, the old face less risks: while the return on their assets is uncertain, the principal cannot be lost and, thus, is safe. Consequently, the old are in a position to insure the young against productivity shocks by selling bonds to them and going long capital.

The bond market provides effective insurance. For example, in the base model, it covers about one-third of the young's potential loss in wages that might arise due to an adverse shock.[4]

Figure 4 plots the demands for bonds of different age groups in specific states of the world, as characterized by good or bad z's and δ's. It shows that the bond demands are responsive to different economic conditions. For example, the one-year-olds supply more than twice as many bonds and the seven-year-olds demand three times as many bonds in the state associated with favorable realizations of the shocks.

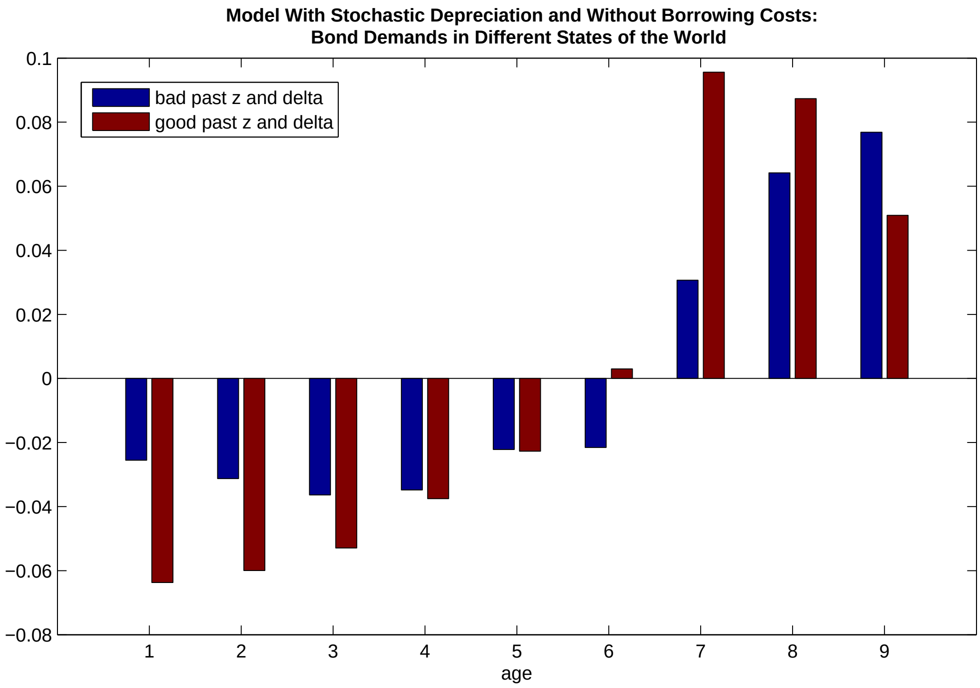

Figure 4: The average bond demands by age starting from good and bad states of the world as characterized by the z's and the δ's in the model with stochastic depreciation, the fixed benefit policy, and without borrowing costs.

The top panel of Table 1 shows that without borrowing costs, the annual equity premium

[4]To see this, note that the young short capital to insure against an adverse shock in z and the resulting decline in wage. Consider two scenarios. In one, the beginning-of-period capital is equal to the average capital stock over 640 periods, 5.3600, and z is equal to the average z, 0.9999, implying a wage of 0.6134 and a rate of return on capital of 0.3946 per period. In the other, capital is again equal to its average value and z is one standard deviation below average, at 0.9668, implying a wage of 0.5931 and a rate of return on capital of 0.3815. One measure of the young's potential loss in wages is the difference in wage between the two scenarios, 0.0203. Since the difference in the rates of return on capital between the two scenarios is 0.0131 and the average bond demand of the youngest age group is 0.4801, the youngest gain 0.0063 in consumption units when the adverse shock hits. Hence the potential capital gain covers about one-third of the loss.

is very small. For example, in the model with stochastic depreciation, it is a mere 0.021 percent. The return on capital averages 6.439 percent annually, which is in line with the empirical estimate of 6.243. However, at 6.418 percent, the bond returns are almost as high. The other two models exhibit similar patterns.

Statistic (Annual, in Percent)	Stochastic Depreciation	Adjustment Costs	Base Model
	No Borrowing Costs		
Equity Premium	0.021	0.024	0.005
Mean Stock Return	6.439	7.835	7.797
Mean Bond Return	6.418	7.811	7.793
	Flatter Borrowing Costs		
Equity Premium	3.579	3.332	3.309
Mean Stock Return	6.440	7.838	7.826
Mean Bond Return	2.860	4.506	4.517
	Steeper Borrowing Costs		
Equity Premium	5.615	5.205	5.179
Mean Stock Return	6.441	7.837	7.827
Mean Bond Return	0.827	2.632	2.647

Table 1: The equity premium and the average risky and safe returns in the models with stochastic depreciation or adjustment costs, and in the base model, with the fixed benefit policy, and with different specification of the borrowing costs.

4.3 Bond Demands and the Equity Premium With Borrowing Costs

Just as the demand for bonds is affected by different market conditions, it is also affected by the introduction of borrowing costs. Figure 5 shows that borrowing costs limit the supply of bonds, more so when they are steeper. The two bottom panels of Table 1 show that this increases the bond price and reduces the safe rate of return, regardless of the model. Figure 6 plots the returns and the equity premium through time. It shows that the realized equity premium fluctuates quite a bit regardless of the presence of borrowing costs.

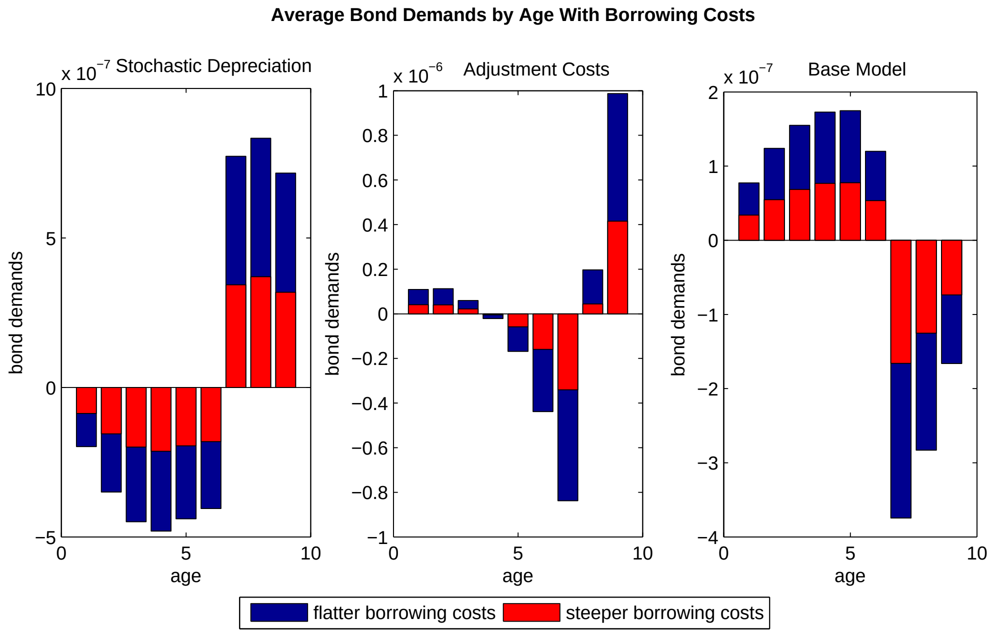

Figure 5: The average bond demands by age in the model with stochastic depreciation, the fixed benefit policy, and the borrowing costs.

4.4 Leverage Considerations

Recall that the risky asset priced in this paper corresponds to the economy's entire capital stock, rather than the stocks traded in the U.S. stock market. This choice is motivated by the following leverage considerations. First, note that by changing the leverage ratio, the equilibrium difference between the average return on equity and the average risk-free return, as well as the volatility of the risky asset, can be made arbitrarily large. By the Modigliani-Miller theorem, this would not change anything real in the economy (without borrowing costs). Now, the leverage ratio itself suffers from a labeling problem, as illustrated by the following example.

Suppose there is one company, General Motors (GM), two people (person A and person B), and two states of the world (a good state and a bad state). Consider the following two descriptions of the economy. Under the first description, GM describes itself as 50 percent equity financed and 50 percent debt-financed. I.e., it has some capital it is using to produce,

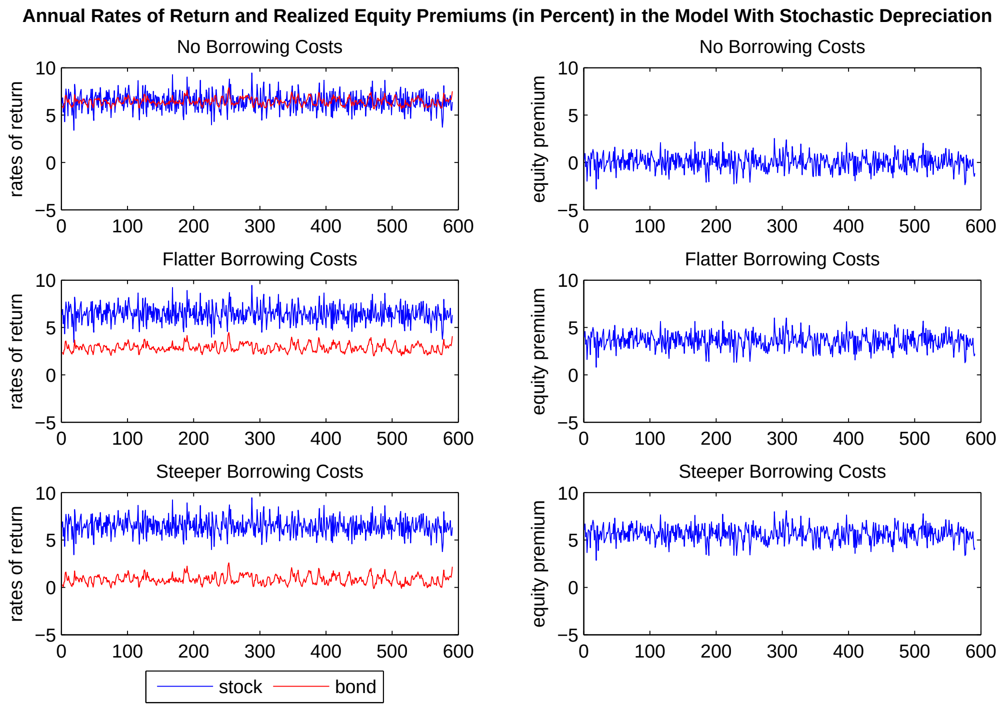

Figure 6: The annual risky and safe returns and the realized equity premiums (in percentage points) for the model with stochastic depreciation, the fixed benefit policy, with or without borrowing costs.

50 percent of which it borrowed from person B, and 50 percent of which it raised from person A by issuing pieces of paper called equity. GM earns some $\tilde{r} = (\bar{r}D + \tilde{r}^e E)/K$, where capital (K) equals debt (D) plus equity (E). It then pays out $\bar{r}$ to B and $\tilde{r}^e$ to A. Suppose that total payouts (including principal) are such that person A gets 500 dollars in the bad state and 4000 dollars in the good state, while person B gets 1000 dollars in either state. Under the second description, GM describes itself as 100 percent equity financed. Here person A holds all the shares, and is entitled to 1500 in the bad state and 5000 in the good state. However, A directs the corporation to mail a check to B in the amount of 1000 dollars on his behalf. Person A then takes the residual.

Note that the outcome for all agents (GM, A, B) is the same regardless of the set of words used to describe the economy. However, in the second description where there is no debt whatsoever, GM is reporting a return to equity that is much safer and with lower mean than under the first description.[5]

4.5 Volatility

In the model, the volatility of returns on the risky asset is 0.5 percent without stochastic depreciation or adjustment costs, and 1 percent with either of these features. This is in line with the data because this paper models all capital in the entire economy, rather than the stock market. The latter is just one sector of the former. To see this, note that in the data, the rate of return on capital has an annual mean and volatility of 6 percent and 0.5 percent, respectively. This volatility is very different from the stock market volatility of 15 percent annually (the mean is about the same). Hence, the Sharpe ratios for the stock return and the return on capital are different. Since the Sharpe ratio is a leverage-insensitive measure, this confirms that the stock market is just one sector of the economy.

[5]On the other hand, the risk-free rate is the relative price of swapping consumption today for sure consumption tomorrow, and hence does not suffer from this labeling issue.

4.6 Government Debt

Recall that the model features no government debt. This is done to conveniently isolate private bond transactions, which, unlike the government, are subject to borrowing costs. But the model can be relabeled to produce any time path of exogenously specified debt values without affecting the equilibrium (see Kotlikoff (1986, 1988, 1993, 2003), Auerbach and Kotlikoff (1987), and Green and Kotlikoff (2008)).[6]

Since with a different set of words the model can be described as featuring any amount of government debt (and associated taxes and transfers), the level of government debt per se does not affect the equity premium. What impacts the equity premium is the structure of the model, including borrowing costs as well as government spending and generational policies.

5 Sensitivity Analysis

5.1 Alternative Policy

The previous results were obtained with the fixed benefit policy in place. This section shows that an alterative policy—the proportional tax policy—does not affect the equity premium results.

[6]To see this, recall that the model features the intergenerational redistribution policy which takes resources from the workers and gives them to the elderly. Now, whatever amount, X_t, is taken from the workers in period t can be relabeled as a borrowing of X_t (or any fraction of it, including a fraction above 1) at time t by the government with repayment at time $t+1$ of $X_t \times (1+\bar{r}_t)$ plus a tax on the workers of $X_t \times (1+\bar{r}_t)$ at time $t+1$, where $\bar{r}_t$ is the return on the bond purchased at time t.

If X_t (the amount said to be borrowed) exceeds the amount of taxes, Z_t, being collected at time t (with the no-debt labeling), the government would describe this as borrowing X_t at time t, making a transfer payment of $X_t - Z_t$ to the worker at time t, having the worker receive X_t plus interest at $t+1$, but having the worker pay in taxes, at $t+1$, the amount Z_t plus interest, plus $X_t - Z_t$ plus interest. Note that with this as with any other relabeling, on balance and ignoring the labels, the worker hands over Z_t at time t to the government and gets back, on balance, zero at time $t+1$.

To have the elderly also holding bonds, the government would say that at time t the elderly are buying M_t in bonds and receiving a transfer payment at time t of M_t. At time $t+1$, they are receiving $M_t \times (1+\bar{r}_t)$ in principal plus interest, but also paying a tax of this same amount. So the government takes nothing extra on net from the elderly at time t and at time $t+1$, but gets to announce extra debt outstanding at time t of M_t.

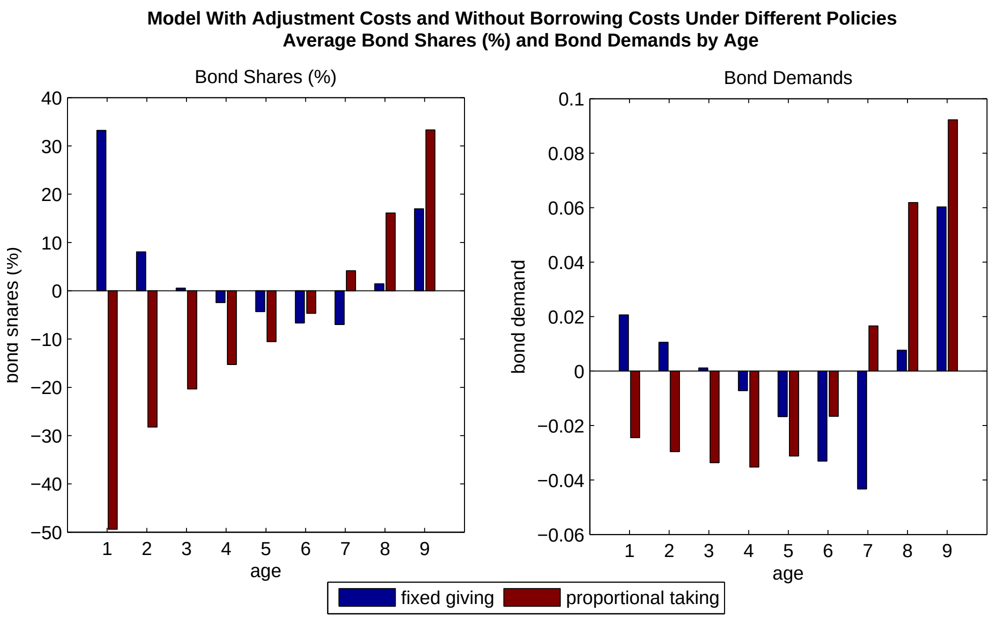

Figure 7: Average bond demands by age in the model with adjustment costs, fixed giving or proportional taking policy, and without borrowing costs.

Figure 7 plots the average bond shares and bond demands by age in the model with adjustment costs and without borrowing costs under each of the two policies. It shows that the choice of the policy matters for bond holdings. For example, the direction of bond positions of the young flips when the policy is changed—they are long bonds under the fixed benefit policy and short bonds under the proportional tax policy.

This pattern is intuitive since the two policies have different implications for the distribution of risk across generations. Under the fixed benefit policy, the workers pay a higher proportion of their wages when times are bad, which increases their demand for bonds. On the other hand, when taxes are proportional, the amount of transfers is higher when times are good. This reduces the workers' risks and leads them to supply more bonds. However, comparing Tables 1 and 2 shows that, quantitatively speaking, the sensitivity of the equity premium to the policy choice is negligible.

Statistic (Annual, in Percent)	Stochastic Depreciation	Adjustment Costs	Base Model
	No Borrowing Costs		
Equity Premium	0.027	0.011	0.001
Mean Stock Return	8.197	9.306	9.220
Mean Bond Return	8.170	9.295	9.219
	Flatter Borrowing Costs		
Equity Premium	3.275	3.081	3.069
Mean Stock Return	8.207	9.308	9.306
Mean Bond Return	4.932	6.227	6.237
	Steeper Borrowing Costs		
Equity Premium	5.107	4.800	4.785
Mean Stock Return	8.207	9.308	9.306
Mean Bond Return	3.100	4.508	4.521

Table 2: Equity premium and average rate of return on capital and bond returns in the models with stochastic depreciation or adjustment costs, and in the base model, with proportional taking policy, with and without borrowing costs.

5.2 Borrowing Costs on a Subset of Generations

Recall that the previous results were obtained with borrowing costs imposed on all generations. Figure 8 plots average bond demands by age for the model with stochastic depreciation where borrowing costs are imposed on different subsets of generations. It shows that if the costs are imposed only on the young, the middle-aged supply all the bonds that the old demand. And if the costs are imposed on everyone but the oldest generation, that generation becomes the supplier. Of course, the middle-aged, and especially the elderly, are not the natural suppliers of bonds in this model. Hence, the gross supply of bonds is more limited when they are the suppliers. However, it is not limited enough to significantly lower the risk-free rate and yield a sizable equity premium.

For example, in the model without borrowing costs, the gross supply of bonds is 0.201 on average. This value goes down to 0.164 if only the young face borrowing costs, further down to 0.068 if both the young and the middle-aged face the costs, and still further down to 0.0087 if everyone except the oldest faces the costs. The unlikely suppliers get compensated by a somewhat lower risk-free rate (higher bond price). However, at 0.038 percent, the equity premium remains two orders of magnitude too small.

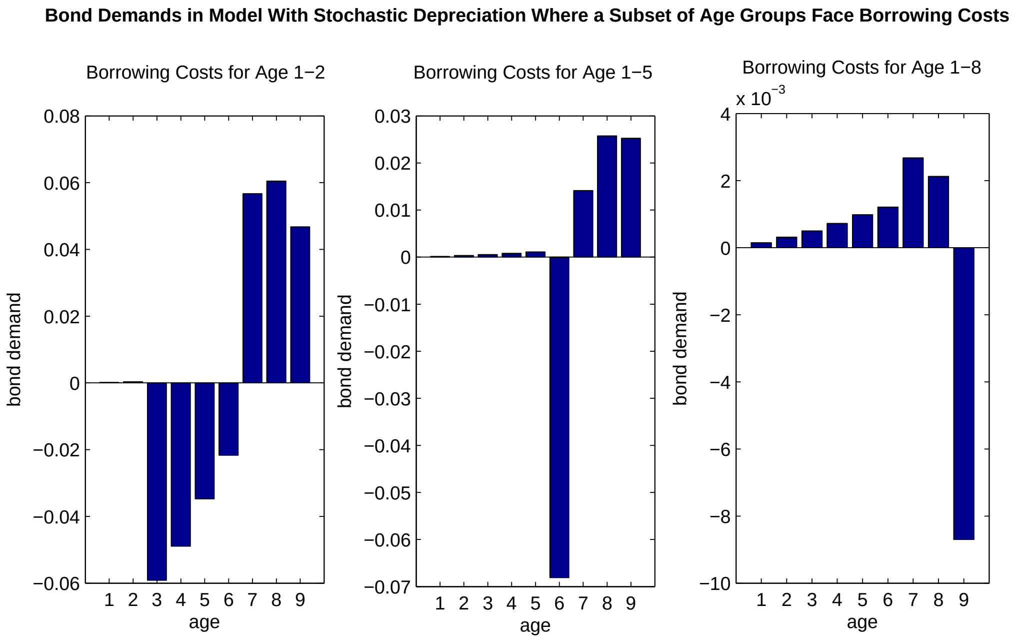

Figure 8: Bond demands by age in the model with stochastic depreciation, fixed giving policy, and steeper borrowing costs imposed on a subset of age groups.

These findings are in contrast with Constantinides, Donaldson, and Mehra (2002), where the equity premium emerges with borrowing constraints imposed only on the young.

6 Accuracy of Solutions

A satisfactory solution requires the generation-specific Euler equations (15) hold out of sample, i.e., on a set draws for the shocks not used to compute the equilibrium decision rules. Hence, for each model considered, the accuracy of solutions is tested on a fresh sequence of z's and δ's that is 60 times longer than the 640-period sequence used in the original simulation. This test entails simulating the model forward on the new path of shocks, using the original asset demand functions, θ_g, and clearing the bond market in each period.[7] The out-of-sample deviations from full satisfaction of the Euler equations,

[7]For details of the solution method, including the bond market clearing algorithm, see the Appendix.

	Fixed Benefit Policy			Proportional Tax Policy		
	Min	Mean	Max	Min	Mean	Max
	No Borrowing Costs					
Stochastic Depreciation	0.003	0.007	0.008	0.005	0.006	0.009
Adjustment Costs	0.002	0.002	0.004	0.005	0.006	0.007
Base Model	0.001	0.004	0.023	0.001	0.010	0.042
	Flatter Borrowing Costs					
Stochastic Depreciation	0.004	0.007	0.010	0.005	0.007	0.010
Adjustment Costs	0.002	0.004	0.005	0.004	0.006	0.007
Base Model	0.001	0.002	0.008	0.001	0.004	0.023
	Steeper Borrowing Costs					
Stochastic Depreciation	0.004	0.007	0.010	0.004	0.006	0.010
Adjustment Costs	0.003	0.003	0.005	0.003	0.005	0.006
Base Model	0.001	0.002	0.007	0.001	0.004	0.023
Borr. Costs on First 2	0.004	0.008	0.010	0.004	0.008	0.010
Borr. Costs on First 5	0.004	0.006	0.009	0.004	0.006	0.009
Borr. Costs on First 8	0.004	0.007	0.011	0.004	0.007	0.011
More Volatile Depreciation	0.005	0.009	0.014	0.005	0.009	0.014

Table 3: Minimum, mean, and maximum across generations of the average, across time, of the absolute value of the generation-specific, out-of-sample deviations from the perfect satisfaction of Euler equations.

$$\epsilon(s,z,\delta) = \beta \mathrm{E}_z\big[\big(1 + r(s',z',\delta') + \alpha_g(s,z,\delta)(\bar{r}(s,z,\delta) - r(s',z',\delta')) - f(\alpha_g(s,z,\delta))\big)\frac{u'(c_{g+1}(s',z',\delta'))}{u'(c_g(s,z,\delta))}\big] - 1, \tag{17}$$

are computed for each period in the newly simulated time path and for each generation $g \in 1,\dots,G-1$.[8] Finally, the average, across time, of the absolute value of the deviations from Euler equations is computed for each generation. Table 3 reports the summary statistics, across generations, of their average absolute deviations from Euler equations for each model considered.

The largest deviation—4 percentage points—is observed in the base model without borrowing costs and with proportional tax policy. Most other deviations are much less than 1 percent.

[8]The out-of-sample test does not apply to (16) since the inner loop is rerun, i.e. (16) will hold by construction.

7 Conclusion

Simulating a sizable equity premium in macroeconomic models has proved difficult, hence the "equity premium puzzle". To explain the puzzle, economists had to apply a lot of machinery. This paper shows that a sizable equity premium can easily be obtained in a standard, multi-period OLG setting. This is demonstrated in a ten-period, general equilibrium OLG model with aggregate uncertainty. The base model is quite simple: it features isoelastic preferences with modest risk aversion, Cobb-Douglas production technology, and realistic TFP shocks. On the fiscal side it includes government consumption, as well as an intergenerational redistribution policy which can be relabeled as government debt. The critical extra ingredient needed to produce a sizable equity premium is the increasing cost of supplying bonds, i.e. of borrowing. These costs are smooth but essentially zero when bond holdings are positive, and are rising as bond holdings become negative. Such representation is consistent with the empirical evidence of Scott (1996) and theoretical models of Milde and Riley (1988) and Chatterjee, Corbae, Nakajima, and Rios-Rull (2007). A sizable equity premium emerges immediately with the aforementioned features, but producing a pattern of increasing bond demands by age requires extra elements, namely modest stochastic depreciation or capital adjustment costs. The findings are robust to policy changes.

The model builds on Hasanhodzic and Kotlikoff (2013). As in that paper, it is solved using Marcet (1988) and Judd, Maliar, and Maliar (2009, 2011) to overcome the curse of dimensionality.

The results differ from those of Constantinides, Donaldson, and Mehra (2002), who use a three-period, partial equilibrium OLG model with pure exchange. In their model, hard borrowing constraints on the young suffice to limit the supply of bonds and yield a large equity premium. Here, only when all generations are subject to borrowing costs is the supply of bonds limited enough for the equity premium to emerge. This difference between this paper's results and theirs highlights the importance of more robust and realistic models in developing theories that match the real world.

A Computational Appendix

At the high level, the algorithm closely follows that of Hasanhodzic and Kotlikoff (2013). However, the low-level execution presents different issues because getting the model to converge is more challenging in the presence of borrowing costs, adjustment costs, and stochastic depreciation. For completeness, and to highlight where different equilibrium conditions are used, the high-level structure is outlined below.

The algorithm consists of an inner loop and an outer loop. The outer loop solves for the asset demand functions of each age group by porting Judd, Maliar, and Maliar's (2009, 2011) generalized stochastic simulation algorithm (GSSA) to the OLG setting. It starts by making an initial guess of generation-specific asset demand functions θ_g as polynomials in the state variables. Next it draws a path of the shocks for T periods and runs the model forward over those periods using the guessed asset demand functions to compute the state variables in each period. Then, for each age group, g, it evaluates the Euler equation (15) to determine what age group g's asset demand should be in each period t. Finally, it regresses these time series of generation-specific asset demands on the state variables, and uses the regression estimates to update the corresponding polynomial coefficients. It repeats these steps using the same path of shocks until asset demand functions converge.

The inner loop is the extension of GSSA by Hasanhodzic and Kotlikoff (2013) that allows for the bond market. It consists of a binary search algorithm which determines the risk-free rate $\bar{r}$ that satisfies (13). In this binary search, the evaluation of the net bond demand is achieved by using another binary search to determine the unique bond shares that satisfy the first order conditions (16).

The following is the step-by-step description.

Initialization:

- Set $\bar{z} = 1$, $\bar{\delta} = \mu_\delta$, and solve for the nonstochastic steady state asset demands of each age group without bond, $\bar{s} = (\bar{s}_1, \ldots, \bar{s}_{G-1})$. Let $(s_0, z_0, \delta_0) = (\bar{s}, \bar{z}, \bar{\delta})$ be the starting point of the simulation.

- Approximate $G - 1$ asset demand functions by polynomials in the state variables: $\theta_1(s, z, \delta) = \phi_1(s, z, \delta; b_1), \ldots, \theta_{G-1}(s, z, \delta) = \phi_{G-1}(s, z, \delta; b_{G-1})$, where $b_1, \ldots b_{G-1}$ are polynomial coefficients. We use degree 1 polynomials. To start iterations, we use the following initial guess for the coefficients: $b_1 = (0, 0.9, 0, \ldots, 0, 0.1\bar{s}_1, 0), \ldots, b_{G-1} = (0, 0, \ldots, 0, 0.9, 0.1\bar{s}_{G-1}, 0)$. Note that for all $g \in \{1, \ldots, G-1\}$, the initial b_g is such that $\bar{s}_g = \phi_g(\bar{s}, \bar{z}, \bar{\delta}; b_g)$.

Outer loop:

- Take draws of the path of z's and δ's for T years. We set T to 640.
- Simulate the model forward for $t = 0, \ldots, T$. More precisely, at time t, for each age group g, calculate its asset demand $\theta_g^{(p)}$ given the current guess for the coefficients $b_g^{(p)}$, where the subscript (p) denotes the current iteration of the outer loop. I.e., $\theta_{g,t}^{(p)}$ equals the inner product of the vector $(1, s_t, z_t, \delta_t)$ with the vector of coefficients $b_g^{(p)}$, where $s_t = (\theta_1^{(p)}(s_{t-1}, z_{t-1}, \delta_{t-1}), \ldots, \theta_{G-1}^{(p)}(s_{t-1}, z_{t-1}, \delta_{t-1}))$. Then the state at time $t+1$ and iteration p is given by $(s_{t+1}, z_{t+1}, \delta_{t+1}) = (\theta_1^{(p)}(s_t, z_t, \delta_t), \ldots, \theta_{G-1}^{(p)}(s_t, z_t, \delta_t), z_{t+1}, \delta_{t+1})$, where z_{t+1} given z_t is determined by (7).
- Inner loop:
  - Use binary search to solve (13) for $\bar{r}_t$, for all $t = 0, \ldots, T$. To start, make an (arbitrary) initial guess for the value of $\bar{r}_t$.
  - For all $t = 0, \ldots, T$, given $\bar{r}_t$, for all $g = 1, \ldots, G-1$, solve (16) for $\alpha_{g-1,t}$ using another binary search (evaluate the expectation in (16) using Gaussian quadrature).
  - Use $\alpha_{g-1,t}$ found above for all g and for all t to calculate (13) and update $\bar{r}_t$ for all t.

- Note that for each age group g and each state (s_t, z_t, δ_t), $t = 1, \ldots, T$, (15) implies

$$\begin{aligned}\theta_g(s_t, z_t, \delta_t) = \beta \mathrm{E}_z \big[\theta_g(s_t, z_t, \delta_t) \big(1 + r(s_{t+1}, z_{t+1}, \delta_{t+1}) \\ + \alpha_g(s_t, z_t, \delta_t)(\bar{r}(s_t, z_t, \delta_t) - r(s_{t+1}, z_{t+1}, \delta_{t+1})) \\ - f(\alpha_g(s_t, z_t, \delta_t)) \big) \frac{u'(c_{g+1}(s_{t+1}, z_{t+1}, \delta_{t+1}))}{u'(c_g(s_t, z_t, \delta_t))} \big]\end{aligned} \tag{A.1}$$

  for equilibrium asset demands θ_g. Denote the right-hand-side of (A.1) by $y_g(s_t, z_t, \delta_t)$ and evaluate the expectation using Gaussian quadrature.

- For each age group g, regress $y_g(s_t, z_t, \delta_t)$ on (s_t, z_t, δ_t) and a constant term using regularized least squares with Trikhonov regularization (see Judd, Maliar, and Maliar, 2011 for details). Denote the estimated regression coefficients by $\hat{b}_g^{(p)}$.

- Check for convergence: If

$$\frac{1}{G-1} \sum_{g=1}^{G-1} \frac{1}{T} \sum_{t=1}^{T} \left| \frac{\theta_g^{(p-1)}(s_t, z_t, \delta_t) - \theta_g^{(p)}(s_t, z_t, \delta_t)}{\theta_g^{(p-1)}(s_t, z_t, \delta_t)} \right| < \epsilon,$$

  end. Otherwise, for each age group g update the coefficients as $b_g^{(p+1)} = (1-\xi) b_g^{(p)} + \xi \hat{b}_g^{(p)}$, for $\xi = 0.01$, and return to the beginning of the outer loop.

Note that the solution method outlined above is globally valid. This is crucial, since the model generates a lot of variability in agents' decisions through time as they respond to and try to insure themselves against shocks. The decision rules of the poorest generation—the young—are particularly volatile. For example, their bond shares exhibit swings as large as 30 percent in the simplest model without stochastic depreciation or adjustment costs (where the young buy bonds). This makes perfect sense since when they are hit by adverse shocks, the young suffer heavy losses in the main source of their livelihood—their labor income. And, because of persistence in the process for the TFP shock, when a bad shock hits, the young insure themselves against further bad shocks coming in the near future by buying bonds and shorting stocks. Since the bond shares they hold in the period after the bad shock reflect the decisions they made in the period the shock hit, you'd expect the correlation between the bond shares and lagged productivity shocks to be highly negative—indeed, it is -97 percent in this model. There's also a lot of variability through time in the generation-specific asset demand rules. Here the young experience swings of almost 50 percent.

Because of this, a plain perturbation method, such as log-linearization, would not be appropriate. These methods produce a solution that is valid locally in the close neighborhood of the steady state. Approximations decay extremely fast away from the steady state even for higher order Taylor series expansions (see, e.g., Judd, 1998, p. 483). Moreover, when used globally, perturbation approximations exacerbate instabilities, and the solution can easily become explosive when some realizations of shocks drive the process outside the accuracy range (see, e.g., Maliar and Maliar, 2014). On the other hand, the sparse grid projection method of Krueger and Kubler (2004, 2006), would be an appropriate alternative.

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