Epistemic Foundations of Equilibria under Ambiguity

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Abstract

In this paper, we develop an interactive epistemology perspective justifying strategic ambiguity and various equilibrium concepts for games with non-additive beliefs. To accommodate strategic ambiguity in games, we introduce an extended version of interactive belief systems in which some types might not know the action they play. Yet, each type knows his theory, i.e., his probability distribution over the entire state space. It is shown that player's beliefs about his opponents' behavior are non-additive if he considers possible that his opponents are undetermined (i.e., his theory assigns a positive probability to opponents' types who do not know what actions they play). In this framework, we establish epistemic conditions under which beliefs constitute an equilibrium under ambiguity for games with two and more than two players, respectively. Our epistemic conditions for Nash-Equilibrium appear as a special case and thus generalize the celebrated results of Aumann and Brandenburger (1995).

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1 Introduction

Since the seminal contribution of Ellsberg (1961), economists acknowledged that individuals facing decision problems under ambiguity often fail to form additive beliefs. As response to the Ellsberg paradox, many ambiguity models have been developed triggering growing interests to exlore how ambiguous beliefs affect strategic behavior in games. As a result, various generalizations of the standard Nash-Equilibrium have been suggested including Dow and Werlang (1994), Lo (1996), Marinacci (2000), Eichberger and Kelsey (2000), Haller (2000), and Eichberger and Kelsey (2014), Dominiak and Eichberger (2017). In this paper, we provide an epistemic explanation for ambiguous beliefs in strategic games and various equilibrium concepts under ambiguity studied in the economics literature.

Equilibrium under Ambiguity (EUA), as Nash-Equilibrium, is formulated as an equilibrium in beliefs. That is, each player best responds to his beliefs (i.e., conjectures) about his opponents' behavior and these beliefs are consistent with actual behavior. In an EUA, any pure strategy of a player which his opponents believe to be played (i.e., any strategy in the support of the opponents' beliefs) constitutes his best respond given his beliefs. In contrast to Nash-Equilibrium, however, players' beliefs are represented by capacities (i.e., normalized and monotone but not-necessarily additive set functions). If equilibrium beliefs are additive, then EUA coincides with the standard Nash-Equilibrium concept.

An important achievement of epistemic approach to game theory was the identification of epistemic conditions leading to Nash-Equilibrium behavior. That is, conditions referring to what players know about the game, the players' conjectures, and their rationality so that the conjectures constitute a Nash-Equilibrium. In the celebretaed work of Aumann and Brandenburger (1995), Nash-Equilibrium is identified as a state of the world in which the game being played, the player's conjectures, and their rationality are mutually - or commonly - known.[1] A player is rational in the sense of choosing an action which is optimal given the expected payoff with respect to his conjectures about the opponents' behavior derived from his theory (i.e., a probability distribution) over the states of the world.

However, the epistemic foundation by Aumann and Brandenburger does not provide any explanation for the "critical" assumption underlying the Nash Equilibrium concept, namely, that players hold additive beliefs over all strategy combinations of their opponents.

The aim of this paper is develop an epistemic model which is suited to accommodate different forms of players' beliefs. For this purpose, we introduce an extended version interactive belief systems used by Aumann-Brandenburger. The novelty of the *extended interactive belief system* (EIBS) is its generality. Then, EIBS is a formal framework suited

(i) to provide epistemic explanation for additive vs. non-additive beliefs in games, and

[1]For games with two players mutual knowledge is sufficient. However, for games with more than two players, stronger epistemic conditions are required. If players' conjectures are derived from a common prior and they are commonly known across the players, then the conjectures constitute a Nash-Equilibrium.

(ii) to formulate epistemic conditions for various equilibrium notions under ambiguity.

The building block of interactive belief systems is set of types for each player. Roughly, a type describes all information that is relevant for interaction in a strategic situation. More precisely, a player's type is identified with a payoff-function, an action and a theory, i.e., a probability distribution over all states of the world (i.e., the Cartesian product of *all* players' type spaces). A player's theory "codifies" his conjectures about his opponents' behavior as well as his beliefs about other players' types; that is, their payoff-function, actions, theories, rationality and their beliefs about all these matters, and so on.

In our epistemic approach, we allow for the possibility that a player of a particular type might consider as possible that there are states of the world in which other types of himself exist who are similar to his own type with respect to his payoff function and his theory but are distinct with respect to the actions each of the types carries out. That is to say, an EIBS allows for the possibility that there are some types of a player who do not know of what type they are in the sense that they do not know what actions they carry out. However, each type knows his payoff-function and his theory, and thus each type knows his conjectures about the strategy combinations about the opponent players.

Our assumption is motivated by the fact that types with the same payoff function and the same theory (thus, inducing the same conjectures about opponents' behavior) may be indifferent between several actions and therefore, each such type may choose a different actions. While each type has to choose an action "at the end of the day", taking into account the possibility that some types of a player do not knot what actions they play might affect his opponents' beliefs about the player's behavior. Even if one assumes that the player chooses rationally, there may be ambiguity about which action he will choose.

When forming a player's beliefs over opponents' behavior, derived from his theory, we will assume that the player takes into account whether there are types of his opponents who know or do not know what actions they play. A player is said to consider possible that his opponent is *undetermined* if there are types in the support of his theory who do not know what actions they play, otherwise the player knows that his opponent is *determined*. It is then shown that each player's conjectures over his opponents' strategy combinations take the form of Möbius transforms of belief functions (a special form of convex capacities).[2]

The different equilibria concepts studied in economics literature mantioned previously, can be mainly distinguished by the underying support notion for players' capacities. Marinacci (2000) defines the support of a capacity as the set of opponents' strategy cominations to which the capacity ascribes a strictly positive value. Therefore, our reference equilibrium is an *Equilibrium under Ambiguity* (EUA) with the notion of support à la Marinacci. Dow and Werlang (1994) suggested an alternative support notion for capacities. A support of a capacity by Dow and Werlang is defined as a smallest set of strategy combinations whose complement has the capacity value of zero. It shall be noted that the two support

[2]Belief functions are well-understood capacities. A belief function on a state space admits an additive representation, the so-called Möbius transform, on the algebra of events generated by the state space.

notions are nested. That is, the Marinacci-support constitutes a Dow-Werlang support, but not *vice versa* (see Eichberger and Kelsey (2014); Dominiak and Eichberger (2016b)). It is also well-known that Marinacci-support might not exist while a Dow-Werlang support always exists. Therefore, to disentangle EUA from an equilibrium behavior under the support notion of Dow and Werlang, we refer to Dow-Werlang Equilibrium (DWE) as an equilibrium under ambiguity with capacities whose Marinacci-support is an empty set.[3]

For games with two players, it is shown that the following conditions lead to an EUA. If there is a state of the world at which each player considers possible that his opponent is determined, and at which the game being played (i.e., the payoff-functions), the players' conjectures and their rationality are mutually known, then the conjectures constitute an Equilibrium under Ambiguity. (That is, any action of a player that belongs to the (non-empty) Marrinacci-support of the opponent's conjectures constitutes his best respond with respect to his conjectures). In other words, an EUA is identified with a state of the world in which players best respond to their mutually known conjectures. The conjectures are derived from theories so that each player takes into account that there might types of his opponent who do not know what actions that play. Yet, at this state, each player consider possible that there is at least one type of his opponent who is determined. This warrants that the Marinacci-support of his conjecture contains at least one strategy of the opponent.

In the above epistemic foundation, the player's conjectures do not need to be additive. Therefore, an EUA might support behavior which is incompatible with Nash-Equilibrium. In the framework of extended interactive belief systems, one can clarify what players need to know about each other in order to behave as predicted by Nash-Equilibrium. For the mutually known conjecture to constitute a Nash-Equilibrium, it must be mutually known among the two players, in addition to their mutual knowledge of the game and of their rationality, that each of them is determined (i.e., each type in the support of each player's theory knows what action the types play. Due this additional epistemic condition, our result can be viewd as a generalization of the epistemic foundation of Nash-Equilibrium in normal-form games with two players derived by Aumann and Brandenburger (1995).

On the other hand, Dow-Werlang Equilibrium is identified as a state of the world in which it is mutually known that both players are undetermined in addition to their mutual knowledge of the game being played, of their conjectures and of their rationality. In contrast to EUA, conjectures constitute a DWE in states in which each player knows that all his opponent's types do not know what action he they will play. Each player best responds with an action that belongs to his opponent's Dow-Werlang support. Since every player knows that his opponent is undetermined, the Marinacci support is an empty set.

For games with n-players, additional conditions are required. In particular, one has to assume that conjectures are derived from a common prior over the states of the world. Since an EUA will be incompatible with the standard notion of stochastic independence,

[3]Thus, in our setup, a DWE cannot be depicted as an equilibrium under ambiguity with respect to Marinacci-support. This approach allows us to distinguish between the two equilibrium notions in terms of different epistemic condition leading to each equilibrium, respectively.

the common priors assumption warrants that player's conjectures are stochastically independent in the (weaker) sense of Möbius-products. Under the common prior assumption, an EUA is identified as a state of the world at which each players considers possible that *all* opponents are determined, at which the game being played, their rationality is mutually known, and at which their conjectures are commonly known. If additionally it is mutually known that all players are determined, then the EUA constitutes a Nash-Equilibrium.

The reminder of the paper is organized as follows. In Section 2, we present the extended interactive belief system. In this section, we derive player's conjectures, i.e., beliefs over the action profiles of the opponents and introduce the notion of (Choquet) rationality. In Section 3, we recall the various notions of equilibria under ambiguity studied in the literature. In Section 4, we derive epistemic conditions for equilibria under ambiguity with two players. In Section 5, we extend our epistemic analysis to games with n-players. In Section 6, we discuss the related literature and in Section 7, final remarks are provided.

2 Extended Interactive Belief System

Games in strategic form are considered. There is a finite set $I = \{1, \ldots, n\}$ of players indexed by i. We denote by $-i$ the opponent players of i. There is a finite set of *actions* (pure strategies) A_i for each player i. The set of actions of all players other than i is denoted by A_{-i}. An element $a = (a_i, a_{-i})$ of $A = A_i \times A_{-i}$ is called the action profile. A function $g_i : A \to \mathbb{R}$ represents player i's payoff function. That is, a player i's payoff $g_i(a_{i,}a_{-i})$ depends on i's action as well as the action profile of the opponent players $-i$. A function $g : A \to \mathbb{R}^2$ is called a *game*. Denote by $\mathcal{G}$ the set of all normal-form games.

In Aumann and Brandenburger (1995) (in short AB) the Nash equilibrium was viewed as a state of the world in which it is mutual - or common knowledge - that players are rational in the sense of choosing an action which is optimal given the expected payoff with respect to conjectures about the opponents' behavior derived from players' theories about the states of the world. A characteristic feature of the AB-epistemic approach is the assumption that a player's type is associated with a unique action and a notion of extended theory which identifies the player with a unique type. Hence, in any state of the world, a player not only knows the action which she chooses but also that no other action could have been chosen.

In contrast to Aumann and Brandenburger (1995), we allow for the possibility that a player of a particular type considers states of the world as possible where other types exist which are similar to the own type with respect to payoffs and theories but distinguished by the action. In particular, we will allow for the possibility that several types of player i exist which are identical with respect to their theories but distinct in the actions they choose. This assumption is motivated by the possibility that players with the same payoff and the same theory may be indifferent between actions and, hence, may choose different actions. Hence, even if one assumes that the other players choose rationally there may be ambiguity about which among several actions the opponents will choose.

If players consider it possible, however, that their opponents may choose among several actions then it is necessary to extend the notion of payoffs associated with single strategy combinations to a notion of payoff from an action in the face of several potential courses of action the other players may choose. We will assume that players will evaluate the payoff of an action by the Hurwicz (1951) criterion. According to this criterion a decision maker evaluates an action according to a weighted average of the best and worst outcome. In this case, the weight can be interpreted as the reflection of the player's attitude towards ambiguity. In our interpretation, this individual attitude towards ambiguity is an individual preference parameter, similar to a discount factor.

Given the payoff function $g_i(a_{i,}a_{-i})$ of player i and a parameter $\alpha_i \in [0,1]$, we define the *(ambiguity-weighted) payoff* of a player who chooses action a_i but considers action combinations of the opponent players in a set $E \subseteq A_{-i}$ as possible by

$$G_i(a_i, E) := \alpha_i \min_{a_{-i} \in E} g_i(a_{i,}a_{-i}) + (1-\alpha_i) \max_{a_{-i} \in E} g_i(a_{i,}a_{-i}). \tag{1}$$

The parameter α_i is a preferential parameter which measures the degree of pessimism of player i. If there is no ambiguity about the opponents' behavior, i.e., in case of $|E| = 1$, we have $G_i(a_i, \{a_{-i}\}; \alpha_i) = g_i(a_{i,}a_{-i})$, irrespective of the ambiguity attitude parameter α_i. The preference representation in Equation (1) is the well-known Hurwicz (1951) criterion applied to the set of the opponents' actions E which player i considers possible. Special cases include *"pure pessimism"* $\alpha_i = 1$, the attitude assumed most commonly in economic applications, and *"pure optimism"* $\alpha_i = 0$.

With this specification of *(ambiguity-weighted) payoff*, it is possible to stay with a notion of theories represented by a probability distribution over states.

2.1 Type Space and Theories

Given a strategic-form game in $\mathcal{G}$, we define an *extended interactive belief system.*

An *extended interactive belief system* for a game in $\mathcal{G}$ consists of the following elements:

(i) a finite set T_i of types for each player $i \in I$, and for each type $t_i \in T_i$ of i,

(ii) an action $a_i(t_i) \in A_i$, (t's *action*)

(iii) a function $G_i(t_i) : A_i \times 2^{A_{-i}} \to \mathbb{R}$ (t_i's *weighted payoff function*, and

(iv) a probability distribution $p_i(t_i)$ on the set T of all types (called t_i's *theory*),

A type is a formal description of a player i's action, payoff function, and theory, i.e., a probability distribution over his own types and the other players' types.

The Cartesian product of individual state spaces $T = T_i \times T_{-i}$, is called the *state space.* The elements $t = (t_i, t_{-i}) \in T$ are called *states of the world* or simple *states.*

In state $t = (t_i, t_{-i}) \in T$, player i chooses an action $a_i(t_i)$, has an (ambiguity-weighted) payoff function $G_i(t_i)$ and a theory $p_i(t_i)$.[4] A player i's theory $p_i(t_i)$ is the probability that i's type t_i ascribes to his own types and the types of his opponent players.

An *event* E is a subset of T. Denote by $p_i(t_i)(E)$ the probability that player i ascribes to E at state t. If $p_i(t_i)(E) > 0$, we say that player i considers E as a *possible event* at t. If player i ascribes probability 1 to E at state t (i.e., $p_i(t_i)(E) = 1$), he is said to *know* E at t, otherwise he does not know the event. The event that i knows E is denoted by $K_iE = \{t \in T : p_i(t_i)(E) = 1\}$. Thus, $t \in K_iE$ means that i knows E at t.

In general, a player i's theory may be an arbitrary probability distributions on T. For instance, a player i might have a theory according to which he does not know of what type he is, what action he plays and so on. In this regard, there is a substantial difference to the version of interactive belief system suggested by Aumann and Brandenburger (1995).

Remark 1 *In the epistemic approach of Aumann and Brandenburger (1995), a player i's theory, $p_i(t_i)$, is defined as a probability distribution on T_{-i}, the set types of the opponent players $-i$. Theories are then extended to a probability distribution $\tilde{p}(t_i)$ over the full state space T as follows. For an event $E \subseteq T$, $\tilde{p}_i(t_i)(E)$ is the probability that a type t_i's theory assigns to the event $\{t_i \in T_i : (t_i, t_{-i}) \in E)\}$. In our setup, theories are probability distributions on T. Hence, we do not need to extend theories to the full state space.*

In other words, Aumann and Brandenburger assume that every type t_i of player i knows to be the only (possible) type and, hence, all other types of player i have a probability of zero. This implies that player i knows $a_i(t_i)$ to be the only (possible) action at state $t \in T$.

In the extended version of an interactive belief system, we will allow for a set of types which hold the same theories but will differ in regard to the actions they choose. For this, we will tighten the player's theories by imposing the following two assumptions.

Assumption 1 *For every player $i \in I$ and every type $t_i \in T_i$, $p_i(t_i)(\{t_i\} \times T_{-i}) > 0$.*

The first assumption requires that every type of a player i regards himself as possible. Denote by $Q(t_i) := \left\{\tilde{t_i} \in T_i : p_i(t_{-i})(\{\tilde{t_i}\} \times T_{-i}) > 0\right\}$ the set player i's types that are considered possible by type t_i at state $t = (t_i, t_{-i})$. By Assumption 1, we have $t_i \in Q(t_i)$ for every $t_i \in T_i$ and every player $i \in I$.

Since types of a player who choose different strategies should hold identical beliefs, the second assumption states that every type $\tilde{t_i}$ that is regarded as possible by another type t_i has the same theory as t_i but plays a different action.

Assumption 2 *For every player $i \in I$ and all types $\tilde{t_i}, t_i \in T_i$, if $p_i(t_i)(\{\tilde{t_i}\} \times T_{-i}) > 0$ then $p_i(\widetilde{t_i}) = p_i(t_i)$ and $a(\widetilde{t_i}) \neq a_i(t_i)$.*

[4]For notational convenience, when referring to player i; at "at state $t = (t_i, t_{-i}) \in T$" means "at t_i". That is, player "i's theory, action, and payoff-function at t" means "t_i's theory, action and payoff-function", is denoted by $p_i(t_i)$, $a_i(t_i)$ and $G_i(t_i)$, respectively.

Assumption 2 implies that, for all types $t_i, \tilde{t}_i \in Q(t_i)$, we have $p(t_i)(\{Q(t_i) \times T_{-i}\}) = 1$. Assumptions 1 and 2 serve as "consistency" assumptions on players' theories and the induced knowledge.[5]

In the extended interactive belief system, we can define the set of actions of player i which, from the point of view of type t_i, are possible:

$$c_i(t_i) := \left\{a_i \in A_i : \; p_i(t_i)(\{\widetilde{t_i}\} \times T_{-i}) > 0 \;\; a(\widetilde{t_i}) = a_i\right\} \subseteq A_i. \tag{2}$$

The set $c_i(t_i)$ is referred to as the set of type t_i's *conceivable actions* at state $t \in T$. Thus, in our setup, we will distinguish between the set of t_i's conceivable actions $c_i(t_i)$ and $a_i(t_i)$, the action (actually) played by type t_i in state t.

Notice that Assumption 1 implies that $a_i(t_i) \subseteq c_i(t_i)$ for all types $t_i \in T_i$. By Assumption 2, we have that the set of types that consider each other as possible (i.e., all types in $Q(t_i)$) have the same set of conceivable actions.

Lemma 2.1 *Consider a type $t_i \in T_i$ of a player i. Then, $c_i(t_i) = c_i(\tilde{t}_i)$ for all $t_i, \tilde{t}_i \in Q(t_i)$.*

Remark 2 *In the setup of Aumann and Brandenburger, the extended theories $\tilde{p}(t_i)$ on T satisfy stronger condition than Assumption 1, i.e., $\tilde{p}_i(t_i)(\{t_i\} \times T_{-i}) = 1$ for every $t_i \in T_i$. The set of types that are considered possible by type t_i is a singleton set, i.e., $Q(t_i) = \{t_i\}$. In our setup, Aumann and Brandenburger's notion of extended theories corresponds to the special case where the set of type t_i's conceivable actions is always a singleton set and it coincides with his action $a_i(t_i) = c_i(t_i)$.*

At state t, a player i might be ignorant about his action $a_i(t_i)$. However, he always knows his theory $p_i(t_i)$ and he also knows his conceivable set of action $c_i(t_i)$.

Lemma 2.2 *Consider a state $t \in T$ and a player i. Let $p_i(t_i) = p$ be the player i's theory and $c_i(t_i) = H$ be his conceivable set of actions at state t. Then, $t \in K_i[p]$ and $t \in K_i[H]$.*

It should also be remarked that under Assumptions 1 and 2, player i's theories induce a partition of i's type space T_i. Let $Q_i^1, \ldots, Q_i^k, \ldots, Q_i^K$ be a collection of subsets of T_i such that for each Q_i^k and all types $t_i, \tilde{t}_i \in Q_i^k$, $p_i(t_i) = p_i(\tilde{t}_i)$. Thus, $Q_i^1, \ldots, Q_i^k, \ldots, Q_i^K$ is a partition of T_i. All types in the information cell Q_i^k are "indistinguishable" with respect to their theories (and thus with respect to their conceivable action sets) but are distinguishable with respect to the action each of them caries out (i.e., for all types $t_i, \tilde{t}_i \in$

[5]These assumptions are necessary for T to be well-defined and for the knowledge operator to satisfy the standard assumptions (such as the axioms of positive and negative introspection; the so-called S5-system of knowledge (see Kripke, Halpern and Aumann)). See Appendix for more details

Q_i^k, $a_i(t_i) \neq a_i(\tilde{t_i})$).[6] Further (epistemic) implications from Assumptions 1 and 2 are discussed in the Appendix.

The idea of an extended interactive belief system is illustrated in the example below.

Example 1 *There are two players called Alice (Player 1) and Bob (Player 2) with sets of actions $A_1 = \{L, R\}$ and $A_2 = \{U, D\}$. The players face the following payoff structure:*

		Player 2	
		L	R
Player 1	U	$1, 2$	$1, 1$
	D	$0, 0$	$2, 1$

There are three types of each player, i.e., $T_i = \{t_i^1, t_i^2, t_i^3\}$ where $i \in \{1, 2\}$. Suppose that each type plays the game above. That is, the game being played in commonly known. Every table below represents a type's theory, his action and his conceivable set of actions:

Player 1 (Alice):

$p_1(t_1^1)$	t_2^1	t_2^2	t_2^3
t_1^1	$\frac{1}{3}$	0	$\frac{2}{3}$
t_1^2	0	0	0
t_1^3	0	0	0

$a_1(t_1^1) = \{U\}$
$c_1(t_1^1) = \{U\}$

$p_1(t_1^2)$	t_2^1	t_2^2	t_2^3
t_1^1	0	0	0
t_1^2	$\frac{1}{6}$	0	$\frac{1}{3}$
t_1^3	$\frac{1}{6}$	0	$\frac{1}{3}$

$a_1(t_1^2) = \{D\}$
$c_1(t_1^2) = \{D, U\}$

$p_1(t_1^3)$	t_2^1	t_2^2	t_2^3
t_1^1	0	0	0
t_1^2	$\frac{1}{6}$	0	$\frac{1}{3}$
t_1^3	$\frac{1}{6}$	0	$\frac{1}{3}$

$a_1(t_1^3) = \{U\}$
$c_1(t_1^3) = \{D, U\}$

Player 2 (Bob):

[6] This argument can be shifted to Section 3.3. where the extended probability measure $p_i(\cdot; t_i)$ on T is defined. The equivalent definition would be the following: Let $Q_i^1, \ldots, Q_i^k, \ldots, Q_i^K$ be a partition of T_i so that for each $t_i \in Q_i^k$, $p_i(Q_i^k; t_i) = 1$.

$p_2(t_2^1)$	t_2^1	t_2^2	t_2^3
t_1^1	$\frac{1}{3}$	0	0
t_1^2	0	0	0
t_1^3	$\frac{2}{3}$	0	0

$a_2(t_2^1) = \{R\}$
$c_2(t_2^1) = \{R\}$

$p_2(t_2^2)$	t_2^1	t_2^2	t_2^3
t_1^1	0	$\frac{1}{6}$	$\frac{1}{6}$
t_1^2	0	0	0
t_1^3	0	$\frac{1}{3}$	$\frac{1}{3}$

$a_2(t_2^2) = \{L\}$
$c_2(t_2^2) = \{L, R\}$

$p_2(t_2^3)$	t_2^1	t_2^2	t_2^3
t_1^1	0	$\frac{1}{6}$	$\frac{1}{6}$
t_1^2	0	0	0
t_1^3	0	$\frac{1}{3}$	$\frac{1}{3}$

$a_2(t_2^3) = \{R\}$
$c_2(t_2^3) = \{L, R\}$

Consider state (t_1^1, t_2^1). In this state, Alice plays U while Bob's chooses R. Also, Alice assigns probability 1 to her type t_1^1 and Bob assigns probability 1 to his type t_2^1. Thus, the actions Alice and Bob play at state (t_1^1, t_2^1) coincide with her and his conceivable action set (i.e., U and R, respectively).

Consider state (t_1^3, t_2^3). Every players chooses the same action as previously. However, at (t_1^3, t_2^3), Alice's theory assigns a positive probability to both types t_1^2 and t_1^3. By Assumption 2, both types have the same theory and thus the same conceivable choice set, $\{U, D\}$. The same argument applies to Bob whose conceivable choice set at state (t_1^3, t_2^3) is $\{R, L\}$.

In other words, at state (t_1^3, t_2^3), Alice thinks that she might be of another type, t_1^2, who actually plays D instead of U. Both types have the same theory but each of them carries out a different action. That is, "at the beginning of the day", Alice cannot distinguish whether she is of type t_1^2 or t_1^3 according to her theory but she can distinguish her type by the action she plays "at the end of the day". Notice that Alice's theories induce partition $Q_1^1 = \{t_1^1\}$ and $Q_1^2 = \{t_1^2, t_1^3\}$ and Bob's theories induce partition $Q_2^1 = \{t_2^1\}$ and $Q_2^2 = \{t_2^2, t_2^3\}$.

Notational conventions. For any state $t \in T$, we write $a(t)$ to denote the tuple $(a_i(t_i), a_i(t_{-i}))$ of actions at t, $g(t)$ for the tuple $(g_i(t_i), g_i(t_{-i}))$ of payoff-functions at t, and $p(t)$ for the tuple $(p_i(t_i), p_i(t_{-i}))$ of theories at t. We will refer to $g(t)(a_i, a_{-i})$ as *"the game played at t"* (i.e., $g(t)(a_i(t_i), a_{-i}(t_{-i}))$).

Functions on T, such as a_i, a, g_i, g, and p_i define various sorts of events. For any function f on T and any $y \in f(T)$, $[f = y]$, or simply $[y]$, denotes the event $\{t \in T : f(t) = y\}$. For example, $[p_i] := \{t \in T : p_i(t) = p_i\}$ is the event that i's theory is p_i, $[a_i] := \{t \in T : a_i(t) = a_i\}$ is the event that i chooses a_i and so on.

The event that i knows that he knows E is denoted by $K_iK_iE := \{t \in T : p(t_i)(K_iE) = 1\}$. Set $K^1 := K_1 \cap \ldots \cap K_n$, the event that all players know event E. If $t \in K^1E$, we say that E is *mutually known* at t. Set $CKE := K^1E \cap K^1K^1E \cap \ldots$, the event that all players know E and that all players know that they know E and so on at infinitum. If $t \in CKE$, E is said to be *commonly known* at t.

2.2 Conjectures and Rationality

In this section, we derive the notion of players' conjectures and rationality in an extended interactive belief system.

A player i's *conjectures* represent his beliefs over the other players' behavior/actions. The conjectures are encoded in a theory that player i holds at state $t \in T$. More formally, we will denote by $\gamma_i(t_i)$ player i's conjectures on $2^{A_{-i}}$ induced by i's theory $p_i(t_i)$ at t.

When forming beliefs about his opponents' behavior, player i takes into account the sets of conceivable actions of his opponent players. That is, a player i's conjectures about the behavior of players $-i$ will be affected by his opponents' knowledge about themselves.

At state $t = (t_i, t_{-i}) \in T$, denote by

$$c_{-i}(t_{-i}) := \bigtimes_{j \neq i \in I} c_j(t_j) \subseteq A_{-i} \tag{3}$$

the set of conceivable actions of the opponent players where $c_j(t_j) \subseteq A_j$ for every $i \neq j$. For a single strategy profile $a_{-i} \in A_{-i}$, a player i's conjecture $\gamma_i(t_i)([a_{-i}])$ at state t that his opponents play the action profile a_{-i} is the probability that he ascribes to all the opponent types whose sets of conceivable actions are singletons, i.e., $c_j(t_j) = \{a_j\}$ for every $i \neq j$. In other words, the player i's conjecture that his opponents play action profile a_{-i} is the probability that his theory assigns to all the opponents' types who know that they play a_j. More generally, player i's conjectures over all subsets of A_{-i} are defined as follows.

Definition 2.1 *Let $t = (t_i, t_{-i}) \in T$ be a state and $p(t_i)$ be a player i's theory. The player i's conjectures $\gamma_i(t_i)$ on $2^{A_{-i}}$ in state t are defined as follows. For any subset $H \subseteq A_{-i}$:*

$$\gamma_i(t_i)(H) := \begin{cases} \sum\limits_{t_{-i} \in T_{-i}} p_i(t_i)(T_i \times \{t_{-i}\}) & , \textit{ if } \quad c_{-i}(t_{-i}) = H, \\ 0 & , \textit{ otherwise.} \end{cases} \tag{4}$$

We denote by $\gamma_i(t) := (\gamma_i(t_i), \gamma_i(t_{-i}))$ the tuple of all players' conjectures at state $t \in T$.

In the extended interactive belief system, a player i's conjectures, as defined in Definition 2.1, are represented by a probability distribution over the power set of his opponents' action profiles in A_{-i}. Hence, one can view the conjectures as the Möbius transform of a belief function, i.e., a totally monotone capacity on $2^{A_{-i}}$.[7]

[7]A real-valued set function $\phi : 2^{A_{-i}} \to \mathbb{R}$ is called a *capacity* on $2^{A_{-i}}$ if it is *normalized* (i.e, $\phi(\emptyset) = 0$

Lemma 2.3 *Let $t \in T$ be a state. A player i's conjecture $\gamma_i(t_i)$ at state t is the* Möbius transform of the belief function $\phi_i^\gamma(t_i)$ *on $2^{A_{-i}}$ defined as*

$$\phi_i^\gamma(t_i)(H) = \sum_{G \subseteq H} \gamma_i(t_i)(H) \quad \textit{for any } H \subseteq A_{-i}. \tag{6}$$

Given the complete description of a player i's type at state t, one can define his expected payoff from playing an action a_i given his conjectures γ_i in that state.

Definition 2.2 *Given a player i's action $a_i(t_i) = a_i$, conjecture $\gamma_i(t_i) = \gamma_i$, and payoff-function $G(t_i)$ at state $t = (t_i, t_{-i}) \in T$, the player i's expected payoff $V_i(a_i, \gamma_i)$ from playing action a_i with respect to γ_i is defined as*

$$V_i(a_i, \gamma_i) := \sum_{E \subseteq A_{-i}} \gamma_i(t_i)(E) G_i(t_i)(a_i, E). \tag{7}$$

Since a player's conjecture is the Möbius transform of his belief function ϕ_i^γ, defined on the algebra of his opponents' action profiles, his expected payoff can be equivalently seen as the weighted mean of the Choquet integral taken with respect to ϕ_i^γ the Choquet integral taken with respect to $\overline{\phi}_i$, the dual of ϕ_i^γ. The dual capacity of ϕ_i^γ is defined as $\overline{\phi}_i(E) := 1 - \phi_i^\gamma(E)$ for all $E \subseteq A_{-i}$.

Lemma 2.4 *Let $t \in T$ be a state. The expected payoff function $V_i(a_i, \gamma_i)$ is the Choquet integral with respect to a JP-capacity $\nu^{JP}(\alpha_i, \phi_i^\gamma) := \alpha_i \phi_i^\gamma + (1 - \alpha_i)\overline{\phi}_i^\gamma$ as defined in Eichberger and Kelsey (2014):*

$$\begin{aligned} V_i(a_i, \gamma_i) &= \int g_i(a_i, a_{-i}) d\nu^{JP}(\alpha_i, \phi_i^\gamma) \\ &= \alpha_i \int g_i(a_i, a_{-i}) d\phi_i^\gamma + (1 - \alpha_i) \int g_i(a_i, a_{-i}) d\overline{\phi}_i^\gamma. \end{aligned} \tag{8}$$

Given a player i's conjecture $\gamma_i(t_i) = \gamma_i$ and his payoff function $G(t_i)$ at state $t \in T$, one can define the set of his best responses. That is,

$$BR_i(\gamma_i) := \arg\max_{a_i \in A_i} V_i(a_i, \gamma_i).$$

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and $\phi(S) = 1$) and *monotone* (i.e., $\phi(F) \leq \phi(E)$ for all $F \subseteq E$). A capacity ϕ is called *totally monotone* if for any $K \geq 2$ and any collection of events $E_1, \ldots, E_K \in 2^{A_{-i}}$, it satisfies the following condition:

$$\phi(\bigcup_{k=1}^{K} E_k) \geq \sum_{\substack{J \subseteq \{1,\ldots,K\} \\ I \neq \emptyset}} (-1)^{|I|+1} \phi(\bigcap_{k \in I} E_k), \tag{5}$$

where $\mid J \mid$ is the cardinality of the index set J (see Dempster (1967) and Shafer (1976)).

A player i is said to be *rational* at state $t \in T$ if, the action $a_i(t_i)$ he chooses at t is a best response given his conjectures about the opponents' behavior.[8]

Definition 2.3 *Let $g(t)$ be a game and $a(t)$ an action profile played at state $t \in T$. A player i with conjectures $\gamma_i(t)$ is said to be rational at state t if, $a_i(t_i) \in BR_i(\gamma_i)$.*

Below we derive conjecture induced by the players' theories presented in Example 1.

Example 2 *Consider state (t_1^1, t_2^1). At this state, Alice assigns probability $\frac{1}{3}$ to Bob being of type t_2^1 and probability $\frac{2}{3}$ to his type t_2^3. Both types play R. However, only type t_2^1 knows (i.e., assigns probability 1 to) his own type and thus knows that he plays R. When Bob is of type t_2^3, he also considers possible that he is of type t_2^2 who plays L. Thus, when Bob is of type t_2^3, his set of conceivable actions is $\{R, L\}$. Hence, Alice's theory at state (t_1^1, t_2^1) induces the following conjecture $\gamma_1(t_1^1)$ (resp., belief function $\phi_1(t_1^1)$) on 2^{A_2}:*

$\cdot$	$\{R\}$	$\{L\}$	$\{R, L\}$
$\gamma_1(\cdot)$	$\frac{1}{3}$	0	$\frac{2}{3}$
$\phi_1^\gamma(\cdot)$	$\frac{1}{3}$	0	1

Notice that any of Alice's types holds the same conjectures and thus she has the same conjectures at every state $t \in T$.

Moreover, at state (t_1^1, t_2^1), Alice's best response given her conjecture is playing R (i.e., $BR(\gamma_1) = \{R\}$). Since Alice plays R in this state, she behaves rational (and she also behaves rational when her type is t_1^3). However, when Alice's type is t_1^2, she plays L and thus she behaves irrational with respect to her conjectures $\phi_1^\gamma(t_1^2)$.

Similar for Bob, his conjectures are ate every state are represented by

$\cdot$	$\{U\}$	$\{D\}$	$\{U, D\}$
$\gamma_2(\cdot)$	$\frac{1}{3}$	0	$\frac{2}{3}$
$\phi_2^\gamma(\cdot)$	$\frac{1}{3}$	0	1

3 Equilibrium Notions under Ambiguity

In this section, we recall notions of equilibrium under ambiguity. We focus on equilibrium notions in which

- ambiguity concerns the strategy choice of opponent players, i.e. $a_{-i} \in A_{-i}$, and

[8] Alternatively, one could require a player i to be *strongly rational* at state $t \in T$, if every action in his conceivable action set $c_i(t)$ constitutes a best response with respect to his conjecture $\gamma_i(t)$, i.e., $c_i(t) \subseteq BR_i(\gamma_i(t))$.

- equilibrium is defined in terms of beliefs represented by capacities (i.e., not necessarily additive probabilities) over $2^{A_{-i}}$, an algebra of pure strategy combinations.

The equilibrium concepts suggested in Dow and Werlang (1994), Marinacci (2000), Eichberger and Kelsey (2000), and Eichberger and Kelsey (2014) satisfy these two conditions. Dow and Werlang (1994), Marinacci (2000), and Eichberger and Kelsey (2000) consider the special case of *pure pessimism*, while Eichberger and Kelsey (2014) allow for *optimism* and *pessimism* as attitudes towards ambiguity. Since the treatment of ambiguity attitude in the latter article covers the other models as special cases, we will follow the general equilibrium notion presented by Eichberger and Kelsey (2014).

Suppose ambiguous beliefs about the opponents' strategy choice can be represented by a convex capacity μ on all subsets of A_{-i}. Denote by $\overline{\mu}$ the dual capacity which is defined by $\overline{\mu}(E) = 1 - \mu(A_{-i} \backslash E)$ for any $E \subseteq A_{-i}$. For $\alpha \in [0,1]$, a *Jaffray-Philippe (JP) capacity* is defined as a convex combination of the capacity μ and its dual $\overline{\mu}$, i.e.,

$$\nu^{JP}(\alpha, \mu) := \alpha\mu + (1-\alpha)\overline{\mu}. \tag{9}$$

It follows immediately from fundamental properties of the Choquet integral, that the Choquet expected payoff of $g_i(a_i, a_{-i})$ with respect to $\nu^{JP}(\alpha, \mu)$ is

$$\begin{aligned}\int g_i(a_i, a_{-i})\ d\nu^{JP}(\alpha, \mu)(a_{-i}) &= \alpha \int g_i(a_i, a_{-i})\ d\mu(a_{-i}) + (1-\alpha) \int g_i(a_i, a_{-i})\ d\overline{\mu}(a_{-i}) \\ &= \alpha \min_{p \in core(\mu)} \int g_i(a_i, a_{-i}) dp(a_{-i}) \\ &\quad + (1-\alpha) \max_{p \in core(\mu)} \int g_i(a_i, a_{-i})\ dp(a_{-i}).\end{aligned}$$

This justifies to take α as an ambiguity attitude parameter. When $\alpha = 1$, the JP-capacity is convex and one obtains the case of pure pessimism (or ambiguity aversion) axiomatized by Schmeidler (1989) and used in Dow and Werlang (1994), Marinacci (2000), and Eichberger and Kelsey (2000). The other extreme is when $\alpha = 0$ and the JP-capacity is concave. This case refers to pure optimism (or ambiguity seeking) as shown by Wakker (2001).

Given beliefs μ_i on A_{-i} and ambiguity attitude α_i, the best-reply correspondence of a player i given his beliefs is defined as

$$BR_i(\mu_i) := \arg \max_{a_i \in A_i} \int g_i(a_i, a_{-i})\ d\nu^{JP}(\alpha, \mu)(a_{-i}) \subseteq A_i.$$

Any equilibrium notion needs to relate equilibrium beliefs $\mu^* = (\mu_1^*, ..., \mu_n^*)$ to the best replies $BR(\mu^*) = (BR_1(\mu_1^*), ..., BR_n(\mu_n^*))$. Similarly, to the definition of a Nash equilibrium this consistency is achieved by the requirement that the support of a capacity μ_i, $supp(\mu_i)$, contains only best replies of the opponents.

Definition 3.1 *An n-tuple of capacities $\mu^* = (\mu_1^*, ..., \upsilon_n^*)$ constitutes an* Equilibrium under Ambiguity (EUA)*, if for all players $i \in I$,*

$$supp(\mu_i^*) \subseteq \times_{j \neq i} BR_j(\mu_j^*). \tag{10}$$

There are different notions of support for capacities. Apart from differences in ambiguity attitudes, the models by Dow and Werlang (1994), Marinacci (2000), Eichberger and Kelsey (2000), and Eichberger and Kelsey (2014) are distinct mainly by their respective support notions. For a comparison of various support notions for convex capacities in the context of game theory see Dominiak and Eichberger (2016b).

First, we focus on the support notion introduced by Marinacci (2000). The *Marinacci-support (M-support)* of a capacity μ_i, denoted by $supp_M(\mu_i)$, is defined as the set of all action profiles for which the capacity is strictly larger than zero.

Definition 3.2 *The* M-support *of μ_i on A_{-i}, $supp_M(\mu_i)$, is the set $\{a_{-i} \in A_{-i} : \mu_i(a_{-i}) > 0\}$.*

When defining the notion of EUA under JP-capacities, Eichberger and Kelsey (2014) suggest as definition of the support of $\nu^{JP}(\alpha, \mu)$ to consider the support of the convex capacity μ defined as the intersection of the supports of all probability distributions in the core of μ, i.e.,

$$supp(\mu) = \bigcap_{p \in core(\mu)} supp(p), \tag{11}$$

where $core(\mu) = \{p \in \Delta(A_{-i}) \mid p(E) \geq \mu(E) \text{ for all } E \in 2^{A_{-i}}\}$.

For the case of belief functions ϕ_i^γ, $supp(\phi_i^\gamma)$ and the M-support of ϕ_i^γ coincide with the set of all action profiles $a_{-i} \in A_{-i}$ to which the Möbious transform assigns strictly positive values (see Proposition 2.1 and 3.3 in Dominiak and Eichberger (2016b)). That is,

$$supp(\phi_i^\gamma) = supp_M(\phi_i^\gamma) = \{a_{-i} \in A_i \mid \gamma_i(\{a_i\}) > 0\}. \tag{12}$$

For convex capacities and pure pessimism, the existence of EUA has been established by Marinacci (2000) and Eichberger and Kelsey (2000). For JP-capacities defined as a convex combination of a belief function ϕ_i^γ and its dual $\overline{\phi_i^\gamma}$, the existence is proven by Dominiak and Eichberger (2016a). For existence of EUA under more general JP-capacities see Eichberger and Kelsey (2014).

An equilibrium under ambiguity is illustrated below.

Example 3 *Consider again the normal-form game presented in Example 1.*

When beliefs are additive, there are two pure Nash Equilibria (U, L) and (D, R), and one Nash Equilibrium with non-degenerate beliefs, i.e., $\phi_1(\{U\}) = \phi_1(\{D\}) = \frac{1}{2}$ and

$\phi_2(\{L\}) = \phi_2(\{R\}) = \frac{1}{2}$. *There five types of equilibria under ambiguity:*

$$(1)\ (\phi_1^*, \phi_2^*) = \begin{cases} \phi_1^*(L) > \frac{1}{2},\ \phi_1^*(r) = 0, & supp_M(\phi_1^*) = \{L\}, & BR_1(\phi_1^*) = \{U\}, \\ \phi_2^*(U) > 0,\ \phi_2^*(d) = 0, & supp_M(\phi_2^*) = \{U\}, & BR_2(\phi_2^*) = \{L\}, \end{cases}$$

$$(2)\ (\phi_1^*, \phi_2^*) = \begin{cases} \phi_1^*(l) = 0,\ \ \phi_1^*(r) > 0, & supp_M(\phi_1^*) = \{R\}, & BR_1(\phi_1^*) = \{D\}, \\ \phi_2^*(u) = 0,\ \phi_2^*(d) > \frac{1}{2}, & supp_M(\phi_2^*) = \{D\}, & BR_2(\phi_2^*) = \{R\}, \end{cases}$$

$$(3)\ (\phi_1^*, \phi_2^*) = \begin{cases} \phi_1^*(l) > 0,\ \ \ \phi_1^*(r) = \frac{1}{2}, & supp_M(\phi_1^*) = \{L, R\}, & BR_1(\phi_1^*) = \{U, D\}, \\ \phi_2^*(u) = \frac{1}{2},\ \ \phi_2^*(d) > 0, & supp_M(\phi_2^*) = \{U, D\}, & BR_2(\phi_2^*) = \{L, R\}, \end{cases}$$

$$(4)\ (\phi_1^*, \phi_2^*) = \begin{cases} \phi_1^*(l) = 0,\ \phi_1^*(r) \in [0, \frac{1}{2}), & supp(\phi_1^*) = \{R\}, & BR_1(\phi_1^*) = \{U\}, \\ \phi_2^*(u) \in [0, \frac{1}{2}),\ \phi_2^*(d) = 0, & supp(\phi_2^*) = \{U\}, & BR_2(\phi_2^*) = \{R\}. \end{cases}$$

$$(5)\ (\phi_1^*, \phi_2^*) = \begin{cases} \phi_1^*(l) = 0,\ \phi_1^*(r) = \frac{1}{2}, & supp_M(\phi_1^*) = \{R\}, & BR_1(\phi_1^*) = \{U, D\}, \\ \phi_2^*(u) = \frac{1}{2},\ \phi_2^*(d) = 0, & supp_M(\phi_2^*) = \{U\}, & BR_2(\phi_2^*) = \{R, L\}. \end{cases}$$

The behavior supported by EUA (1) *and* (2) *is the same as the pure Nash Equilibrium. The EUA* (3) *coincides with the Nash Equilibrium behavior where players uniformly randomize between their pure strategies. In the mixed-strategy Nash-equilibrium, as well as in* (3)*, both players are indifferent between any pure strategy in the support of the equilibrium capacities. The behavior captured by EUA* (4) *and* (5) *cannot be accommodate by the Nash Equilibrium. In these equilibria, players do not believe that coordination is possible. In EUA* (4)*, Player 1 chooses U and Player 2 chooses R yielding them a certain payoff of* 1*. This strategy insure each player against the ambiguity about his/her opponent behavior. In equilibrium* (5)*, the only difference is that both players are indifferent between playing the two pure strategies, while in the equilibrium* 4 *playing U and R will be always preferred.*

4 Equilibrium under Ambiguity with Two Players.

In this section, we will derive epistemic conditions for various notions of equilibrium under ambiguity. We first focus on strategic games with two players.

As it will be shown below, an important aspect that disentangles the various equilibrium notions is how players reason about their opponents. A player i is said to be *determined* in state $t \in T$ if his conceivable choice set in t is a singleton set, i.e., $c_i(t_i) = \{a_i\}$ for some $a_i \in A_i$, otherwise the player is *undetermined*. Notice that, by Lemma 2.2, a player i who is determined in state t knows $a_i(t_i)$, the action he plays in state t.

Theorem 4.1 *Let g be a game and $\gamma = (\gamma_i, \gamma_j)$ be a pair of conjectures. Suppose that at some state $t \in T$, every player considers possible that his opponent is determined, and it is* *mutually known that g is played, that the players' conjectures are γ, and that the players are rational. Then, $\gamma = (\gamma_i, \gamma_j)$ constitutes an Equilibrium under Ambiguity for game g.*

For a pair of conjectures to constitute an EUA, every player has to consider his opponent player as being determined. Yet, it is not required that player knows that his opponent is determined. That is, the player i might assign a strictly positive probability to the event that his opponent j is undetermined. This implies that his beliefs over j's actions are represented by a belief function ϕ_i^γ in 2^{A_j} with a non-empty M-support. If, however, every player knows that his opponent is determined, then his beliefs are represented by a standard additive probability measure. In the wake of this observation, the epistemic conditions for the Nash Equilibrium derived by Aumman and Brandenburger (1995, Theorem A), can be re-formulated as a corollary of Theorem 4.1. In the extended interactive beliefs system, the sufficient condition for the Nash Equilibrium is the fact that players mutually known that all players are determined, i.e., the players' conceivably choice stets are singleton sets.

Corollary 4.2 *Let g be a game and $\gamma = (\gamma_i, \gamma_j)$ be a pair of players' conjectures. Suppose at some state $t \in T$, it is mutually known that the players are determined, that g is played, that γ are the players' conjectures, and that the players are rational. Then, $\gamma = (\gamma_i, \gamma_j)$ constitutes a Nash Equilibrium for game g.*

In the case of pure pessimism (i.e., $\alpha = 1$), Dow and Werlang (1994) suggested an alternative notion of equilibrium behavior under a different notion of support. A *Dow-Werlang-support (DW-support)* of a capacity μ_i, denoted by $supp_{DW}(\mu_i)$, is the smallest set of action profiles whose complement has the capacity value zero. Formally, the support is defined es follows.

Definition 4.1 *A* DW-support *of μ_i on A_{-1}, $supp_{DW}(\mu_i)$, is a set $E \subseteq A_{-1}$ such that $\upsilon_i(A_{-1} \setminus E) = 0$ and $\upsilon_i(A_{-1} \setminus F) > 0$ for any $F \subset E$.*

It is important to remark that a DW-support of a capacity μ_i always exists while the M-support might be an empty set. Moreover, both support notions are "nested" in the sense that the M-support, provided it exists, is always a DW-support.[9] For this reason, we will consider conjectures that constitute an equilibrium under ambiguity for which the M-support is an empty set. That is, we are interested in equilibrium behavior that can be depicted by DW-supports but not by the Marinacci-support. We refer to such an equilibrium as Dow-Werlang Equilibrium (DWE).

For a pair conjecture to constitute a DWE, one has to assume that all players are undetermined and that this fact is mutually known. Formally, the following epistemic conditions lead to DWE.

[9] See Lemma 1 in Eichberger and Kelsey (2014).

Proposition 4.3 *Let g be a game and $\gamma = (\gamma_i, \gamma_j)$ a pair of conjectures. Suppose that at some state $t \in T$, it is mutually known that the players are undetermined, that g is played, that the players' conjectures are γ, and that the players are rational. Then, $\gamma = (\gamma_i, \gamma_j)$ constitutes a Dow-Werlang Equilibrium for game g.*

The notion of DWE includes as special type of equilibrium under ambiguity where the players' belief functions take the form of "complete" ignorance capacities. That is, if $\upsilon_i^\gamma(E) = 0$ for all $E \subset A_i$ and $\upsilon_i^\gamma(E) = 1$ for $E = A_i$. If υ_i^γ on $2^{A_{-i}}$ is a complete ignorance capacity, then every singleton set $\{a_{-i}\}$, where $a_{-i} \in A_{-i}$, is a DW-support.

However, the epistemic conditions for DWE derived in Proposition 4.3, do not encompass the complete ignorance case. Or, putting it differently, the conditions are too strong; the mutual knowledge of rationality rules out the complete ignorance capacities. The reason is that if a player has a theory inducing a incomplete ignorance capacity, then the player might not know that he is rational (unless he is indifferent between all actions). This is due to the fact that any action $a_i \in A_i$ constitutes a Dow-Werlang support of player j's complete ignorance capacity.

An epistemic justification for DWE with complete ignorance capacities requires a separate treatment. Therefore, we refer to equilibrium under complete ignorance as a Complete Ignorance Equilibrium (for short, CIE).

One epistemic condition from Proposition 4.3 has to be weakened. It is assumed that player i knows that his opponent player j is rational play, but this fact is not mutually known. That is, player j cannot not know that he is rational (unless he is indifferent between all his actions, i.e. $BR_j(\upsilon_j) = A_j$).

Proposition 4.4 *Let g be a game and $\gamma = (\gamma_i, \gamma_j)$ be a pair of conjectures. Suppose that at some state $t \in T$, every player knows that his opponent is rational, it is mutually known that at t that the players are "completely" undetermined, and that the conjectures are γ. Then, $\gamma = (\gamma_i, \gamma_j)$ constitutes a Complete Ignorance Equilibrium for game g.*

5 Equilibria under Ambiguity with N Players.

In this section, we extend our analysis to to strategic games with more than two players.

For strategic game whose number of players exceeds two, epistemic foundation of equilibria under ambiguity requires a special treatment. In particular, one has to assume that players theories (and thus their conjectures) are derived form a common prior over the entire state space.

A probability distribution p on T is called a *common prior* if for each player $i \in I$ and all their types $t_i \in T_i$, the conditional distribution of p given t_i is $p_i(t_i)$, i.e., i's theory. In words, players are said to have a common prior on T if all differences between their probability assessments are due only to differences in their information.

In our setup, the common priors assumption have two important implications. First, it guarantees players' behavior is stochastically independent in the sense of Möbius products. Second, the common prior assumption warrants that player j's conjecture about behavior about player i agree with player k's conjectures about i's behavior.

Under the common prior assumption, the following conditions lead to an EUA in strategic games with more than two players.

Theorem 5.1 *Let g be a game and $\gamma = (\gamma_i, \ldots, \gamma_N)$ be an N-tuple of conjectures. Suppose that players have a common prior p on T. Suppose that at some state $t \in T$, at which each player i considers possible that all his opponents $-i$ are determined, at which it is mutually known that g is played and that all players are rational, and at which that it is commonly known that the players' conjectures are γ. Then, for each player i, all his opponents $-i$ agree on the same conjecture about i and the N-tuple of conjectures $\gamma = (\gamma_i, \ldots, \gamma_N)$ constitutes an Equilibrium under Ambiguity for game g.*

At states of the world at which is it mutually known that all players are determined, an EUA reduces to the standard Nash-Equilibrium for games with more than two players.

Corollary 5.2 *Let g be a game and $\gamma = (\gamma_i, \ldots, \gamma_N)$ be an N-tuple of conjectures. Suppose that players have a common prior p on T. Suppose that at some state $t \in T$, at whichat which it is mutually known that all players are determined, that g is played, that all players are rational, and at which that it is commonly known that the players' conjectures are γ. Then, for each player i, all his opponents $-i$ agree on the same conjecture about i and the N-tuple of conjectures $\gamma = (\gamma_i, \ldots, \gamma_N)$ constitutes a Nash-Equilibrium for game g.*

6 Related Literature

The developments of ambiguity models triggered keen interests in providing an (epistemic) interpretation of non-additive beliefs. In this section, we provide a brief overview about this research program and explain how our approach is related to the existing literature.

The literature on interpretation of non-additive beliefs is as old as the ambiguity models themselves. The existing contributions can be mainly classified into two categories. One category deals with epistemic justification of non-additive beliefs in the context of individual decision making. The other type of literature concerns epistemic interpretation of interactive behavior with ambiguous beliefs. Our approach contributes to both categories.

From the point of view of individual decision making, non-additive beliefs have been often viewed, roughly speaking, as a result of a "misspecified" or "coarse" decision problem. In the context of belief functions, the Möbius weight $p(E)$ that is assigned to an event E is often interpreted as "direct evidence" for E. However, this evidence cannot be further "split" across the states that constitute E. As Gilboa and Schmeidler (1994, p. 52) argues:

"*One of the reasons one gets evidence for E but not for any subset thereof may be model misspecifications, i.e., that the states of the world included in the model do not exhaust "the actual" ones. In other words, the non-additivity [...] may be explained by "omitted" states of the world* . Although the notion of "omitted" states is vague, in our setup, one could view such states as states in which player's types are determined, i.e., states with types whose theories unequivocally specify the action the player actually plays.

Mukerji (1997) is more specific about the origin of non-additivity of subjective beliefs. Ambiguous beliefs are attributed to limitations in reasoning about uncertain contingencies. In his setup, a decision maker faces two state spaces; a "primitive" state space and a "payoff-relevant" state space. The latter states determine the final consequences of actions. A decision maker holds two types of beliefs. His "primitive" beliefs are additive. However, the form of the decision maker's beliefs over the payoff-relevant states depends on his - possibly limited - understanding (knowledge) about the link between the primitive states and the payoff-relevant ones. His understanding is captured via an *implication mapping* (i.e., a correspondence from primitive states to sets of payoff-relevant states). If the range of the implication mapping comprises solely singleton sets, the decision maker has an unlimited understanding about all the logical connections between both spaces spaces. In this case, his beliefs over payoff-relevant contingencies are additive. However, if the image of the implication mapping assigns non-singletons, his understanding about the logical connections between states is limited and his beliefs take the form of belief functions.

There are a few parallels to our approach. In the language of extended interactive belief systems, the primitive state space refers to the set of types whereas the payoff-relevant space refers to the set of action profiles. A player's theory represent his primitive beliefs which induce conjectures over the payoff-relevant space, i.e., the algebra of all possible strategy combinations of the opponent players.

Ghirardato (2001) interpreters non-additive beliefs as a decision maker's limited understanding about objects of choice, i.e., acts. Instead of viewing (Savage) acts as functions from states to consequences, Ghirardato takes for granted that the decision maker chooses across "coarse" acts, i.e, correspondences between states and sets of consequences. In an axiomatic fashion, he derives a preference representation across such acts. If acts are seen as function, the decision maker beliefs are additive, otherwise his beliefs are represented by beliefs functions. Also in our setup, player's actions can be seen as correspondences.

The second category of literature deals with epistemic intepretation of equilibrium behavior in games with players displaying ambiguity-sensitive preferences. The closest to our approach is Lo (1996) who provides epistemic condition for an equilibrium notion for strategic games in which players' beliefs take the form of set of priors in the spirit of Gilboa and Schmeidler (1989). However, there are substantial differences in at least two respects. First of all, Lo identifies types directly with sets of priors without providing any explanation for this particular form of player's beliefs. In our setup, the non-additive player' beliefs are due the possibility that opponents are ignorant about their own behavior. Second, Lo assumes (pure) pessimism whereas we allow for various ambiguity attitudes in the family of Choquet expected utility preferences.

Another related work is by Mukerji and Shin (2002). They attribute equilibrium with ambiguous beliefs in two-players normal form games to the lack of the players' common knowledge of the game being played. Similar to us, they focus on behavior in the context of belief functions. Different to our approach, however, is their interpretation of such beliefs. Mukerji and Shin's approach relies on embedding of normal-form games under ambiguity into a standard game of incomplete information. They demonstrate that equilibrium with belief functions can be reinterpreted as an equilibrium in an associated Bayesian game with additive beliefs over types in which common knowledge of the game does not apply.

7 Conclusion

A Proofs

The proof of Theorem 4.1 relies on the following lemma.

Lemma A.1 *Let g be a game and $\gamma = (\gamma_i, \gamma_j)$ be a pair of conjectures. Suppose that at some state $t = (t_i, t_j) \in T$, each player considers possible that his opponent is determined, and it is mutually known that g is played, that γ are the players' conjectures, and that the players are rational. Let a_i be a player i's action to which j's conjecture assigns a positive value. Then, a_i is the player i's best response for g_i with respect to γ^i at state t.*

Proof Lemma A.1. By Lemma 2.3, j's conjecture γ_j at state t is γ_j is the Möbius transform of ϕ_j^{γ}, i.e., a probability distribution on 2^{A_i}. By assumption, player j attributes a positive probability to $[a_i]$, the event that player i plays a_i. Thus, by Definition 2.1, the player j's theory $p_j(t_j)$ at state t assigns a positive probability to the set of i's types whose conceivable choice set is a_i. That is,

$$\gamma_j(t)([a_i]) = \sum_{\{t_i \in T_i \;:\; c_i(t_i) = \{a_i\}\}} p_j(t)(\{t_i\} \times T_j) > 0, \tag{13}$$

or equivalently,

$$\gamma_j(t)([a_i]) = \sum_{\{t_i \in T_i \;:\; p_i(t_i)([a_i]) = 1\}} p_j(t)(\{t_i\} \times T_j) > 0. \tag{14}$$

At state t, player j also attributes probability 1 to each of the three events: [i is rational], [i's conjecture is ϕ_i] and [i's payoff is g_i]. Since j's theory assigns a positive probability to the i's types who know that they play a_i (and in fact they play a_i, by Lemma), there must be a state $\tilde{t}$ at which all four events obtain: [i plays a_i], [i's conjecture is γ_i], [i's payoff function is g_i], and [i is rational]. Thus, at state $\tilde{t}$, action a_i maximizes the Choquet integral of g_i with respect to i's conjecture γ^i which is a belief function ϕ_i^γ by Lemma 2.3. That is, $a_i \in BR(\gamma_i)$. ■

Proof of Theorem 4.1. Fix a player i. Let $t \in T$ be a state in which player i considers possible that his opponent is determined, i.e., $t \in [j$ is determined$]$. This means that i's theory at t assigns a positive probability to $S_j = \{t_i \in T_j : c_j(t_j) = \{a_j\},\ a_j \in A_j\}$, the set of j's types whose set of conceivable actions is a singleton set. Let $B_j = \{a_j \in A_j : p_i(t_i)([a_j]) > 0\}$ be a set of j's actions for which there exist a type whose are considered possible according to player i's theory at t and whose conceivable choice set a singleton set in H_j, i.e., for every H_j^- By Definition 2.1, i's conjecture that j plays $a_j \in H_j$ is $\phi([a_j]) = p([a_j^-]; t_i)$. Thus, the M-support associated with ϕ^i, supp(ϕ^i), is nonempty and equals to H_j. By Lemma A.1, any $a_j \in$ supp(ϕ^i) is j's best response to j's conjectures about i's behavior in game g. Thus, the tuple of conjectures $\phi = (\phi^i, \phi^j)$ constitutes a $\mathcal{M}$ equilibrium under ambiguity. ■

Corollary A.2 *Let g be a game and $\gamma = (\gamma_i, \gamma_j)$ be a pair of players' conjectures. Suppose at some state $t \in T$, it is mutually known that the players are determined, that g is played, that γ are the players' conjectures, and that the players are rational. Then, $\gamma = (\gamma_i, \gamma_j)$ constitutes a Nash Equilibrium for game g.*

Proof of Corollary A.2. Consider a player i. Let $S_j \subseteq T_j$ be the set player j's types to which the player i's theory $p_i(t_i)$ at state $t = (t_i, t_j)$ assigns a strictly positive probability, i.e. $p_i(t_i)(T_i \times \{t_j\}) > 0$ for any $t_j \in S_j$, and thus

$$\sum_{t_j \in S_j} p_i(t_i)(T_i \times \{t_j\}) = 1. \tag{15}$$

At state t, player i knows that his opponent - player j - is determined. That is, for each type $t_j \in S_j$, type t_j's set of conceivable actions is a singleton set (i.e., $c_j(t_j) = \{a_j\}$ for some $a_j \in A_j$). By Definition 2.1, for any $H_j \subset A_j$:

$$\gamma_i(t_i)([H_j]) := \begin{cases} \sum\limits_{t_j \in T_j} p_i(t_i)(T_i \times \{t_j\}) & \text{, if} \quad c_j(t_j) = H_j, \\ 0 & \text{, otherwise.} \end{cases} \tag{16}$$

By Lemma 2.3, $\gamma_i(t_i)$ is the Möbius transform of i's belief function ϕ^γ on 2^{A_j}. Thus, for any $H_j \subset A_j$

$$\phi_i^\gamma(t_i)([H_j]) = \sum_{G_j \subseteq H_j} \gamma_i(t_i)(G_j). \tag{17}$$

Since for each type $t_j \in S_j$, $c_j(t_j)$ is a singleton set, Equation 19 implies that

$$\sum_{a_j \in A_j} \phi_i(t_j)([a_j]) = 1. \tag{18}$$

Thus, i's belief function $\phi_i^\gamma(t_i)$ is a probability measure on A_j. ■

The proof of Proposition 4.3 relies on the following lemma.

Lemma A.3 *Let g be a game and $\gamma = (\gamma_i, \gamma_j)$ a pair of conjectures. Suppose that at some state $t \in T$, it is mutually known that the players are undetermined, that g is played, that the players' conjectures are γ, and that the players are rational. Let $H_i \subseteq A_i$ be a set of actions of player i to which j's conjectures assigns a positive value, i.e., $\gamma_j(t_j)(H_i) > 0$. Then, $a_i \in H_i$ is the player i's best response for $g_i(t_i)$ with respect to his conjecture γ_i.*

Proof of Lemma A.3. Consider a player i. Let $S_j \subseteq T_j$ be the set player j's types to which the player i's theory $p_i(t_i)$ at state $t = (t_i, t_j)$ assigns a strictly positive probability, i.e. $p_i(t_i)(T_i \times \{t_j\}) > 0$ for any $t_j \in S_j$, and thus

$$\sum_{t_j \in S_j} p_i(t_i)(T_i \times \{t_j\}) = 1. \tag{19}$$

At state t, player i knows that his opponent j is undetermined. Thus, for any type $t_j \in S_j$, t_j's set of conceivable actions is a non-singleton set (i.e., $c_j(t_j) = H_j \subseteq A_j$ where $|H_j| > 1$). By Definition 2.1, for any $H_j \subset A_j$:

$$\gamma_i(t_i)([H_j]) := \begin{cases} \sum\limits_{t_j \in T_j} p_i(t_i)(T_i \times \{t_j\}) & \text{, if} \quad c_j(t_j) = H_j, \\ 0 & \text{, otherwise.} \end{cases} \tag{20}$$

Now, consider .... ■

Proof of Proposition 4.3.

■

Proof of Proposition 4.4. At state t, player i's theory ascribes probability 1 to the event [j is "completely" undetermined]. Thus, player i's theory assigns probability 1 to the set of j's types, $S_j \subseteq T_j$, whose conceivable choice set $c_j(t_j)$ is A_j, the set of all actions. Thus, by Definition 2.1, one has

$$\gamma_i(t_j)([H_j]) = \sum_{t_j \in S_j} p_i(t_i)(\{T_i\} \times \{t_j\}) = 1, \tag{21}$$

whenever $H_j = A_j$, and $\gamma_i(t_j)([H_j]) = 0$ for any $H_j \subset A_j$. By Lemma 2.3, $\gamma_i(t_i)$ is the Möbius transform of i's belief function ϕ^γ on 2^{A_j}. Thus, for any $H_j \subset A_j$

$$\phi_i^\gamma(t_i)([H_j]) = \sum_{G_j \subseteq H_j} \gamma_i(t_i)(G_j) = 0, \tag{22}$$

whenever $H_j = A_j$, and $\phi_i^\gamma(t_i)([H_j]) = 0$ for any $H_j \subset A_j$. Hence, i's conjecture is the Möbius transform of the complete ignorance capacity. The Dow-Werlang support of $\phi_i^\gamma(t_i)$ is any action of player j, i.e., $\mathcal{DW}(\phi_i^\gamma) = \{\{a_j\} : a_j \in A_j\}$. By the same argument as above, ϕ_j^γ's conjecture at t is a complete ignorance capacity and $\mathcal{DW}(\phi_j^\gamma) = \{\{a_i\} : a_i \in A_i\}$. At state t, player j ascribes probability 1 to the event $[g^i]$, $[\gamma^i]$ and [i is rational]. Thus, there is a state $\tilde{t}$ at which i's payoff is $g^i(\tilde{t})$, i's conjecture is $\gamma^i(\tilde{t}_i)$, i plays $a_i(\tilde{t}_i) = a_i$, and a_i maximizes g^i with respect to $\gamma^i(\tilde{t}_i)$ (i.e., $a_i \in \arg\max\{m(a_i) : a_i \in A_i\}$), where $m(a_i) = \{g(a_i, a_j) : a_j \in A_j\}$ is the worst payoff player i may obtain from playing a_i. Since $\{a_i\} \in \mathcal{DW}(\phi_j^\gamma)$ and the same argument holds true for player j, the tuple of conjectures (γ_i, γ_j) constitutes a Complete Ignorance Equilibrium. ■

Proof of Theorem 5.1.

■

Proof of Corollary 5.2.

■

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