Identification and Linear Estimation of General Dynamic Programming Discrete Choice Models

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Abstract

This paper studies the nonparametric identification and estimation of the structural parameters, including the per period utility functions, discount factors, and state transition laws, of general dynamic programming discrete choice (DPDC) models. I show an equivalence between the identification of general DPDC model and the identification of a linear GMM system. Using such an equivalence, I simplify both the identification analysis and the estimation practice of DPDC model. First, I prove a series of identification results for the DPDC model by using rank conditions. Previous identification results in the literature are based on normalizing the per period utility functions of one alternative. Such normalization could severely bias the estimates of counterfactual policy effects. I show that the structural parameters can be nonparametrically identified without the normalization. Second, I propose a closed form nonparametric estimator for the per period utility functions, the computation of which involves only least squares estimation. In the presence of parametric specification of utility functions, a two-step minimum distance estimator is proposed for the unknown parameters in the parametric specification given the nonparametric estimates of the per period utility functions. The existing estimation procedures rely on assuming that the dynamic programming (DP) problem is stationary or on solving the DP problem numerically with the aid of terminal conditions. Neither the identification nor the estimation requires terminal conditions, the DPDC model to be stationary, or having a sample that covers the entire decision period. The Monte Carlo studies show that our minimum distance estimator has smaller mean squared error and is 34, 600 and 850 times faster than the pseudo-maximum likelihood (PML), nested pseudo-likelihood (NPL) and the nested fixed point (NFXP) estimators.

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1 Introduction

The dynamic programming discrete choice (DPDC) model is an empirical framework for studying the intertemporal discrete choices in fields as labor economics and empirical industrial organization (see Ackerberg, Benkard, Berry, and Pakes, 2007, and Keane, Todd, and Wolpin, 2011, for surveys of applications). The DPDC model extends the static discrete choice model by allowing an individual's current choice to affect not only her current utility but also her future state or utility. Taking occupational choice as an example (Keane and Wolpin, 1997), starting at some age, an individual can choose among different occupations/activities: attend school/college, work in either a blue or white collar occupation, start her own business, enlist in the military or stay at home (unemployed). Such an occupational choice can be made repeatedly through her lifetime. Her current occupational choice will affect not only her current utility but also her future human capital, and hence future income. For example, attending college is costly, but a college graduate is more likely to find a job. It is reasonable to view her current occupational choice as a result of intertemporal maximization of her expected lifetime utility. In DPDC models, this intertemporal optimization is solved by dynamic programming. The econometric problems in DPDC models are the identification and estimation of structural parameters, including the per period utility functions, discount factors, and state transition laws. If the structural parameters of a DPDC model are obtained, counterfactual policy interventions, such as the effect of subsidizing tuition on college enrollment, can be simulated.

The existing identification results and estimation methods for (non)stationary DPDC models are both conceptually complicated and numerically difficult due to the complexity of (non)stationary dynamic programming that is a recursive solution method. This paper will show that the identification of (non)stationary DPDC models and their estimation can be greatly simplified, because I will show that the identification of general DPDC models is equivalent to the identification of a linear GMM system. So the identification of DPDC models can be understood from the familiar rank conditions in linear models. Moreover, the per period utility functions and discount factors can be estimated by a closed form linear estimator.

The idea of linear identification and estimation is inspired by the econometric literature on dynamic game models. Pesendorfer and Schmidt-Dengler (2008), Bajari, Chernozhukov, Hong, and Nekipelov (2009), Bajari, Hong, and Nekipelov (2010) show that the Markovian equilibria of dynamic games with discrete choices can be equivalently written as a system of equations linear in the per period utility functions. Hence the identification of per period utility functions in dynamic game models is similar to the identification of a linear GMM system. Moreover, the per period utility functions can then be estimated by least squares. As a special case of the dynamic game with discrete choices, the identification and estimation of infinite horizon stationary single agent DPDC models can also be addressed using the equivalence to a linear GMM system (Pesendorfer and Schmidt-Dengler, 2008; Srisuma and Linton, 2012). Because the equivalence to a linear GMM has greatly simplified our understanding of the identification of stationary DPDC models and their estimation, a natural question is if such an equivalence exists for general DPDC models, especially finite horizon nonstationary DPDC models. Finite horizon models are common in labor economics, since households live for a finite time. This paper addresses this question.

The DPDC model studied in this paper is general in three ways. First, the decision horizon can be finite or infinite. Second, all structural parameters, including per period utility functions, discount factors and transition laws, are allowed to be time varying. Third, I do not assume that the per period utility function associated with one particular alternative is known, or is normalized to be a known constant. This feature is important, because normalization of the per period utility function will bias counterfactual policy predictions.

The normalization derives from the analogy between dynamic and static choice. In static discrete choice the conditional choice probabilities (CCP) only depend on the differences between the payoffs of alternatives. So we can change payoffs of alternatives so long as their differences are not changed. This ambiguity motivates the normalization of the payoff of one alternative (Magnac and Thesmar, 2002; Bajari, Benkard, and Levin, 2007; Pesendorfer and Schmidt-Dengler, 2008; Bajari, Chernozhukov, Hong, and Nekipelov, 2009; Blevins, 2014). However, normalization in dynamic discrete choice models is not innocuous for counterfactual policy predictions. This point has been mentioned recently by some authors in a variety of settings, e.g. Norets and Tang (2014); Arcidiacono and Miller (2015); Aguirregabiria and Suzuki (2014); Kalouptsidi, Scott, and Souza-Rodrigues (2015).[1] The intuition is that in a dynamic discrete choice model, a forward-looking individual's current choice depends on future utility. This future utility depends on the per period utility functions of all alternatives. Consider the normalization of setting the per period utility of the first alternative to be zero for all states. Such a normalization will distort the effects of the current choice on future utility, because the per period utility of the first alternative does not depend on the state. When we consider counterfactual interventions, the effects of the current choice on counterfactual future payoff will be also distorted, hence the counterfactual choice probability will be biased.

Without imposing a normalization, I provide two alternative ways to identify the per period utility functions and discount factors. One is to assume that there are excluded state variables that do not affect per period utilities but affect state transitions. When excluded state variables are not available, another way is to assume that per period utility function is time invariant but that state transition laws are time varying. The excluded variables restriction has been used to identify discount factors in exponential discounting (Ching and Osborne, 2015) and hyperbolic discounting (Fang and Wang, 2015), but it has not been used to identify per period utility functions in general DPDC models. The closest work is Aguirregabiria and Suzuki's (2014) study of market entry and exit decisions, where the per period utility function is equal to the observable revenue net of unobservable cost. Assuming that the firms' dynamic programming problem is stationary, and the discount factor is known, they use exclusion restrictions to identify the cost function. However they do not consider the identification of the discount factor and of nonstationary DPDC models. Let us consider a binary choice model to explain the intuition why the exclusion restrictions can identify the per period utility function without

[1] I provide two propositions in the appendix showing the misleading consequence of normalization for counterfactual analysis.

normalization. The observable CCP is determined by the difference between the payoffs of the two alternatives. In DPDC model, such a payoff difference is the sum of the difference between per period utility functions and the difference between the discounted continuation value functions. Exclusion restrictions create "exogenous" variation that can identify the value functions from the CCP. The identification of the per period utility functions follows from the Bellman equation.

Using the equivalence to linear GMM, the estimation of DPDC models becomes so simple that the per period utility functions and discount factors can be estimated by a closed form linear estimator after estimating the conditional choice probabilities (CCP) and the state transition distributions. The implementation of our linear estimator is simple because only basic matrix operations are involved. Our linear estimator can be applied to situations where the agent's dynamic programming problem is nonstationary, the panel data do not cover the whole decision period, and there are no terminal conditions available. Such simplicity in computation and flexibility in modeling are desirable in practice, because the existing estimation algorithms (Rust, 1987; Hotz and Miller, 1993; Aguirregabiria and Mira, 2002; Su and Judd, 2012) depend on complicated numerical optimization and/or iterative updating algorithms, and many of them cannot be applied when the dynamic programming problem is nonstationary and no terminal conditions are available.

1.1 Literature review

We now survey the literature. If the agent's dynamic programming problem is stationary,[2] Rust (1994, section 3.5) shows that *the structural parameters* of the DPDC models, including the per period utility functions and the discount factor, are nonparametrically unidentified. However the exact degree of underidentification is not clear. Magnac and Thesmar (2002) extend Rust's underidentification argument in two ways. First, they determine the exact degree of underidentification in a two periods DPDC model and discuss the identifying power of various restrictions (section 2 to 4 of their paper). Their conclusion is that the alternative specific per period utility functions cannot be nonparametrically identified if the distribution of the unobserved payoff shocks, the discount factor, and the per period utility function and the *alternative specific value function* (ASVF) associated with one specific alternative are not *all* known. The precise definition of the ASVF will be given later; at this moment, one just needs to understand that the ASVF is the best expected remaining lifetime payoff if that particular alternative is chosen. Second, they study the identification of DPDC models with unobserved heterogeneity (section 5). The unobserved heterogeneity in their paper is discrete and affects per period utility functions but does not affect the law of state transitions. Their conclusion is that the DPDC models with unobservable heterogeneity are nonparametrically unidentified even under strong restrictions, such as that the current and future payoffs of one alternative are assumed to be known.

[2]The agent's dynamic programming problem is stationary if the per period utility functions, the law of state transitions and the discount factor of future payoff are all time invariant, and the decision horizon is infinite.

The identification and estimation of the stationary single agent DPDC model is closely related to the identification of dynamic game models with Markov perfect equilibria. The crucial observation is that if the dynamic programming problem is stationary, the Bellman equation becomes of a Fredholm integral equation of type 2 from which the value function can be solved. This implies that we have a closed form representation of the value function in terms of the per period utility functions, the discount factor and the observable CCP. Moreover, the observable CCP is determined by the difference between the payoffs of the alternatives, which is the sum of the difference between the per period utility functions and the difference between the discounted continuation value functions. We then have a closed form representation of CCP in terms of the per period utility functions, the discount factor and the value function. Replacing the value function in the CCP representation with its closed form representation from solving the Bellman equation, we have an equation that involves only the observable CCP, the unknown per period utility functions and the unknown discount factor. Assuming that the discount factor and the per period utility function associated with one alternative are known, Pesendorfer and Schmidt-Dengler (2008), Bajari, Chernozhukov, Hong, and Nekipelov (2009), Bajari, Hong, and Nekipelov (2010) use this equation to study the identification of per period utility functions in dynamic game models, and Srisuma and Linton (2012) study the identification and estimation of stationary single agent DPDC model when some of the state variables are continuous.

Blevins (2014) studies the nonparametric identification of the stationary dynamic programming decision process when the decisions involve both discrete and continuous choice. In the first stage, an agent makes a discrete choice; in the second stage, the agent makes a continuous choice given her previous discrete choice. Bajari, Benkard, and Levin (2007) also use such a two-stage specification. Bajari, Benkard, and Levin focus on the estimation issues, and their analysis allows for dynamic games. The advantage of using such a two-stage specification is that once the policy function of continuous choice is identified, the optimal continuous choice can be viewed as an observable state variable. Thus, Blevins' model becomes a stationary DPDC model. Blevins' conclusion is that when the discount factor, the per period utility function of one specific alternative, and the distribution of preference shocks are known, the per period utility functions of the other alternatives are nonparametrically identifiable. This conclusion corresponds to the earlier observation in Magnac and Thesmar (2002). When the distribution of preference shocks is unknown, he provides some exclusion restrictions (Assumption 12 on p. 546 of his paper) that can lead to identification of the distribution of the differences between payoff shocks. His method for identifying the distribution is similar to the control function approach used in the nonparametric instrumental variable literature (e.g.. Blundell and Powell, 2004; Imbens and Newey, 2009). His identification arguments depend crucially on the stationarity assumption, without which the functional mapping in his proof does not exist. When the distribution of preference shocks is unknown, Norets and Tang (2014) provide a partial identification approach to analyze the stationary DPDC models when all observable state variables are discrete with finite support.

Heckman and Navarro (2007) and Aguirregabiria (2010) study the identification of nonstationary DPDC models with a finite decision horizon. Both papers assume that researchers can observe the "outcomes" of agents' choices. For example, the outcome is one's earnings in Heckman and Navarro's schooling decisions study. An agent's per period utility is assumed to be the outcome net of the unobservable cost of the choice, and hence the identification of utility function is then equivalent to the identification of the cost function. Heckman and Navarro identify the period utility function under several restrictions. The most substantial two restrictions are (1) the continuation value associated with one specific alternative is known, and (2) the transition between the observed states does not depend on the agent's decisions. These assumptions are restrictive in practice.

Without these assumptions, Aguirregabiria aims to identify the effects of certain counterfactual policy interventions rather than the structural parameters in the case that the policy effects on the agents' per period utility functions are completely known. There are two limitations of his approach. First, his method applies only to counterfactual policy interventions that affect the per period utility functions and the effects of which on current utility are completely known. If the intervention effects are unknown or the interventions are on state transitions, his method cannot be applied. Second, identification and estimation statements are based on backward induction, and the estimation requires data about decisions in the final decision period. It is not clear if this method can be extended to deal with infinite horizon DPDC model, for which the panel data cannot cover the entire decision process.[3]

The estimation of a DPDC model is usually complicated since the model is based on dynamic programming that is a recursive solution method. Researchers usually adopt the maximum likelihood method to estimate the structural parameters, although it is not clear if the log likelihood function has a unique global maximizer. The first estimation method was the nested fixed-point (NFXP) algorithm proposed by Rust (1987). To alleviate the computational burden, Hotz and Miller (1993) developed a semiparametric two-step estimator of the structural parameters. The first step is to estimate the CCP nonparametrically. The second step uses the famous Hotz and Miller inversion proposition that gives a representation of the ASVF in terms of the CCP, per period utility functions, and the discount factor. Consequently, one has a closed form representation of the CCP in terms of the CCP itself and the structural parameters (see equation (3.12) of Hotz and Miller's paper). Substituting nonparametric estimates obtained in the first step for the CCP in the closed form representation, one has the CCP for each value of structural parameters. Equating these expressions with its nonparametric estimates, one can develop a GMM estimator of the structural parameters, and this is the second step of Hotz and Miller's estimation method. Hotz and Miller's idea also applies to nonstationary DPDC models. There are two potential limitations of the Hotz and Miller two-step estimator. First, the computational gain comes at the expense of efficiency. This drawback has been addressed by Aguirregabiria and Mira (2002). Second, the closed form representation of the ASVF becomes complicated when there are many future periods before the decision horizon. The complication

[3] In the working paper version (Aguirregabiria, 2005), he did study the identification and estimation when decision horizon is infinite. But there he has to assume that the dynamic programming problem is stationary, and the estimation procedure becomes computationally difficulty because some contraction mappings are involved in his procedure.

comes from the fact that the representation of the ASVF, see equation (3.12) of Hotz and Miller's paper, requires the evaluation of the probabilities of all possible future paths and the expected utilities associated with these paths. This has not been noticed in the literature because the existing estimators focus on the stationary DPDC models, and under stationarity it is easier to express the ASVF in terms of the CCP and structural parameters (see equation (8) of Aguirregabiria and Mira, 2002, for example). Aguirregabiria and Mira (2002) provide a new approach called the nested pseudo likelihood (NPL) algorithm to estimate stationary DPDC models when the state variables are discrete. Their estimator could be as efficient as Rust's NFXP estimator but computationally easier. When the dynamic programming process is stationary, Aguirregabiria and Mira establish a contraction mapping for the CCP. Using this contraction mapping, Aguirregabiria and Mira's NPL estimator can improve the estimate of the CCP used in the second step of Hotz and Miller's two-step estimator. Recently, Su and Judd (2012) provide another estimation approach called the mathematical program with equilibrium constraints (MPEC) for the stationary DPDC model. The equilibrium constraint in stationary DPDC models corresponds to the integrated Bellman equation. Their idea is to treat the ex ante value function, which becomes a vector when the observable states are discrete, as a parameter in maximizing the log likelihood function subject to the constraint that the ex ante value function must solve the integrated Bellman equation. However, their method works only with discrete state variables, and the number of points in the support has to be small. It is also not clear whether their method can be used to estimate nonstationary DPDC models.

1.2 Structure of the paper and notation rules

In section 2, we develop the dynamic programming discrete choice model of which identification and estimation will be studied. The model's set up follows the literature, except that we allow per period utility functions and discount factors to be time varying. In section 4, we show that the identification of the DPDC model is equivalent to the identification of a linear GMM system, and provide a list of identification results under various restrictions. In particular, we show two ways to identify the DPDC models without normalizing per period utility functions. After clarifying the identification of the model, we show that the DPDC model can be estimated by simple linear estimators. Numerical experiments are conducted to check the performance and to highlight some issues with our estimator. The last section concludes the paper with a discussion of some extensions.

Notation. Let X, Y and Z be three random variables. We write $X \perp\!\!\!\perp Y$ to denote that X and Y are independent. And write $X \perp\!\!\!\perp Y|Z$ to denote that X and Y are independent conditional on Z. If the random variable X can take only a finite number of values. The support of X is $\mathcal{X} \equiv (x_1, \ldots, x_{d_x})$. Let $f(X) : \mathcal{X} \mapsto \mathbb{R}$ be a real function. We use f to denote the d_x-dimensional vector $(f(x_1), \ldots, f(x_{d_x}))^\intercal$. For a real number $a \in \mathbb{R}$, let $a_n \equiv (a, \ldots, a)^\intercal$ be an n-dimensional vector with entries all equal to a. Let I_n be the $n \times n$ identity matrix. Because diagonal matrix and diagonal block matrix will be frequently used, we define

$$\operatorname{diag}(a_1,\ldots,a_n) \equiv \begin{bmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{bmatrix},$$

where a_i can be either a number or a matrix for each $i = 1, \ldots, n$.

2 Dynamic Programming Discrete Choice Model

2.1 The model

We first set up the dynamic programming discrete choice model. A female labor force participation example then follows the abstract setup to illustrate the notation.

We restrict our attention to the binary choice case. The extension to multinomial choice is straightforward at the expense of more cumbersome notation (see Remark 4 in section 4). In each period t, an agent makes a binary choice $D_t \in \{\,\mathtt{0},\mathtt{1}\,\}$ based on a vector of state variables $\Omega_t = (S_t, \varepsilon_t^{\mathtt{0}}, \varepsilon_t^{\mathtt{1}})$. Researchers only observe the choice D_t and the state variable S_t. The choice in period t affects both the agent's instantaneous utility in period t and the distribution of the next period state variable Ω_{t+1}. Assumption 1 restricts the instantaneous utility to be additive in the unobserved state variables. Assumption 2 assumes that the state variable Ω_t is a controlled first-order Markov process. Both are standard assumptions in the literature.

Assumption 1. *The agent receives instantaneous utility $u_t(\Omega_t, D_t)$ in period t. In particular, let*

$$u_t(\Omega_t, D_t) = D_t(\mu_t^{\mathtt{1}}(S_t) + \varepsilon_t^{\mathtt{1}}) + (1 - D_t)(\mu_t^{\mathtt{0}}(S_t) + \varepsilon_t^{\mathtt{0}}),$$

so that $u_t(\Omega_t, D_t = \mathtt{d})$ is additive in the unobserved state variable $\varepsilon_t^{\mathtt{d}}$. We call $\mu_t^{\mathtt{d}}(S_t)$ the (structural) per period utility function in period t associated with alternative $\mathtt{d}$.

Assumption 2. *The choice in period t affects the distribution of the next period state variable Ω_{t+1}. Given the current state variable Ω_t and choice D_t, the next period state variable Ω_{t+1} is independent of all previous state variables and choices, that is $\Omega_{t+1} \perp\!\!\!\perp (\Omega_{t'}, D_{t'})|(\Omega_t, D_t)$ for any $t' < t$.*

Let $T_* \leq \infty$ be the last decision period. In each period t, the agent makes a sequence of choices $\{D_t, \ldots, D_{T_*}\}$ to maximize the expected remaining lifetime utility in period t,

$$u_t(\Omega_t, D_t) + \sum_{r=t+1}^{T_*} \left(\prod_{j=t}^{r-1} \delta_j \right) \mathrm{E}_{\Omega_r}[u_r(\Omega_r, D_r)|\Omega_t, D_t],$$

where $\delta_t \in [0, 1)$ is the discount factor in period t. The agent's problem is a Markov decision process, which can be solved by dynamic programming. Let $V_t(\Omega_t)$ be the value function in period t. The optimal choice D_t solves the Bellman equation,

$$\begin{aligned} V_t(\Omega_t) &= \max_{\mathtt{d}\in\{\,\mathtt{0},\mathtt{1}\,\}} u_t(\Omega_t, D_t = \mathtt{d}) + \delta_t \, \mathrm{E}_{\Omega_{t+1}}[V_{t+1}(\Omega_{t+1})|\Omega_t, D_t = \mathtt{d}] \\ &= \max_{\mathtt{d}\in\{\,\mathtt{0},\mathtt{1}\,\}} \mu_t^{\mathtt{d}}(S_t) + \varepsilon_t^{\mathtt{d}} + \delta_t \, \mathrm{E}_{\Omega_{t+1}}[V_{t+1}(\Omega_{t+1})|S_t, \varepsilon_t^{\mathtt{0}}, \varepsilon_t^{\mathtt{1}}, D_t = \mathtt{d}]. \end{aligned} \tag{2.1}$$

In other words, the agent's decision rule is as follows,

$$D_t = \begin{cases} \mathtt{1}, & \mu_t^{\mathtt{1}}(S_t) + \delta_t \, \mathrm{E}_{\Omega_{t+1}}[V_{t+1}(\Omega_{t+1})|S_t, \varepsilon_t^{\mathtt{0}}, \varepsilon_t^{\mathtt{1}}, D_t = \mathtt{1}] + \varepsilon_t^{\mathtt{1}} \\ & \qquad > \mu_t^{\mathtt{0}}(S_t) + \delta_t \, \mathrm{E}_{\Omega_{t+1}}[V_{t+1}(\Omega_{t+1})|S_t, \varepsilon_t^{\mathtt{0}}, \varepsilon_t^{\mathtt{1}}, D_t = \mathtt{0}] + \varepsilon_t^{\mathtt{0}}, \\ \mathtt{0}, & \text{otherwise.} \end{cases} \tag{2.2}$$

Without further restriction about the state transition distribution, the continuation value $\mathrm{E}_{\Omega_{t+1}}[V_{t+1}(\Omega_{t+1})|S_t, \varepsilon_t^{\mathtt{0}}, \varepsilon_t^{\mathtt{1}}, D_t = \mathtt{d}]$ is non-separable from the unobserved state variables $\varepsilon_t^{\mathtt{0}}$ and $\varepsilon_t^{\mathtt{1}}$. To avoid dealing with non-separable models, we make the following assumption.

Assumption 3. (i) *Let $\varepsilon_t = (\varepsilon_t^{\mathtt{0}}, \varepsilon_t^{\mathtt{1}})^{\intercal}$. The sequence of unobserved state variables $\{\varepsilon_t\}$ is independent and identically distributed.*

(ii) *For each period t, $S_t \perp\!\!\!\perp (\varepsilon_t, \varepsilon_{t+1})$.*

(iii) *For each period t, $S_{t+1} \perp\!\!\!\perp \varepsilon_t|(S_t, D_t)$.*

The assumption is standard in the literature, but we want to emphasize the implied limitations. Assumption 3.(i) implies that the unobserved state variable ε_t does not include the unobserved heterogeneity that is constant or serially correlated over time. For example, suppose $\varepsilon_t^{\mathtt{d}} = \alpha + \eta_t^{\mathtt{d}}$, where α is time invariant unobserved heterogeneity, and $\eta_t^{\mathtt{d}}$ is a serially independent random utility shock. Then the unobserved state variable $\varepsilon_t^{\mathtt{d}}$ becomes serially correlated. Moreover, if the unobserved heterogeneity α is fixed effect that is correlated with the observed state variable S_t, Assumption 3.(ii) is violated. If conditional on (S_t, D_t), the unobserved heterogeneity α can still affect the distribution of the next period state variable S_{t+1}, Assumption 3.(iii) is violated.

Applying Assumption 3, it can be verified that for each alternative $\mathtt{d} \in \{\,\mathtt{0}, \mathtt{1}\,\}$,

$$\mathrm{E}_{\Omega_{t+1}}[V_{t+1}(\Omega_{t+1})|S_t, \varepsilon_t^{\mathtt{0}}, \varepsilon_t^{\mathtt{1}}, D_t = \mathtt{d}] = \mathrm{E}_{S_{t+1}}[v_{t+1}(S_{t+1})|S_t, D_t = \mathtt{d}], \tag{2.3}$$

where

$$v_{t+1}(S_{t+1}) \equiv \mathrm{E}_{\varepsilon_{t+1}}[V_{t+1}(S_{t+1}, \varepsilon_{t+1})|S_{t+1}] \tag{2.4}$$

is called the *ex ante value function* in the literature. Because the conditional expectations $\mathrm{E}_{S_{t+1}}(\cdot|S_t, D_t = \mathtt{0})$ and $\mathrm{E}_{S_{t+1}}(\cdot|S_t, D_t = \mathtt{1})$ as well as their difference will be frequently used, define the following new notation for expositional simplicity,

$$\begin{aligned}
&\mathrm{E}^{\mathtt{d}}_{t+1}(\cdot|S_t) \equiv \mathrm{E}_{S_{t+1}}(\cdot|S_t, D_t = \mathtt{d}),\ \mathtt{d} \in \{\mathtt{0}, \mathtt{1}\},\\
&\mathrm{E}^{\mathtt{1/0}}_{t+1}(\cdot|S_t) \equiv \mathrm{E}_{S_{t+1}}(\cdot|S_t, D_t = \mathtt{1}) - \mathrm{E}_{S_{t+1}}(\cdot|S_t, D_t = \mathtt{0}).
\end{aligned} \tag{2.5}$$

It should be remarked that $\mathrm{E}^{\mathtt{1/0}}_{t+1}(c|S_t) = 0$ for any real constant c, so the conditional expectation difference $\mathrm{E}^{\mathtt{1/0}}_{t+1}(\cdot|S_t)$ viewed as a linear operator is not invertible.

Define the *alternative specific value function* (ASVF) $v^{\mathtt{d}}_t(S_t)$ for each alternative $\mathtt{d} \in \{\mathtt{0}, \mathtt{1}\}$,

$$\begin{aligned}
v^{\mathtt{d}}_t(S_t) &= \mu^{\mathtt{d}}_t(S_t) + \delta_t\, \mathrm{E}_{\Omega_{t+1}}[V_{t+1}(\Omega_{t+1})|S_t, \varepsilon^{\mathtt{0}}_t, \varepsilon^{\mathtt{1}}_t, D_t = \mathtt{d}]\\
&= \mu^{\mathtt{d}}_t(S_t) + \delta_t\, \mathrm{E}^{\mathtt{d}}_{t+1}[v_{t+1}(S_{t+1})|S_t].
\end{aligned} \tag{2.6}$$

The second line of the above display follows from equation (2.3). Using the notation of the ASVF, the Bellman equation (2.1) becomes

$$V_t(S_t, \varepsilon_t) = \max_{\mathtt{d} \in \{\mathtt{0},\mathtt{1}\}} v^{\mathtt{d}}_t(S_t) + \varepsilon^{\mathtt{d}}_t, \tag{2.7}$$

and the decision rule (2.2) now has a simpler expression,

$$D_t = \begin{cases} \mathtt{1}, & \text{if } v^{\mathtt{1}}_t(S_t) + \varepsilon^{\mathtt{1}}_t > v^{\mathtt{0}}_t(S_t) + \varepsilon^{\mathtt{0}}_t,\\ \mathtt{0}, & \text{otherwise.} \end{cases} \tag{2.8}$$

By the decision rule (2.8), the CCP $p_t(S_t) = \mathrm{P}(D_t = \mathtt{1}|S_t)$ equals the following,

$$p_t\,(S_t) = \mathrm{P}(\varepsilon^{\mathtt{0}}_t - \varepsilon^{\mathtt{1}}_t < v^{\mathtt{1}}_t(S_t) - v^{\mathtt{0}}_t(S_t)).$$

Let $G(\cdot,\cdot)$ be the cumulative distribution function (CDF) of the vector of unobserved state variables $\varepsilon_t = (\varepsilon^{\mathtt{0}}_t, \varepsilon^{\mathtt{1}}_t)^\intercal$, and let $\tilde{G}(\cdot)$ be the CDF of $\tilde{\varepsilon}_t = \varepsilon^{\mathtt{0}}_t - \varepsilon^{\mathtt{1}}_t$. In terms of the CDF $\tilde{G}(\cdot)$, the CCP is written as follows,

$$\begin{aligned}
p_t(S_t) &= \tilde{G}(v^{\mathtt{1}}_t(S_t) - v^{\mathtt{0}}_t(S_t))\\
&= \tilde{G}(\mu^{\mathtt{1}}_t(S_t) - \mu^{\mathtt{0}}_t(S_t) + \delta_t\, \mathrm{E}^{\mathtt{1/0}}_{t+1}[v_{t+1}(S_{t+1})|S_t]).
\end{aligned} \tag{2.9}$$

When the CDF $\tilde{G}(\cdot)$ is unknown, even the ASVF difference $v^{\mathtt{1}}_t(S_t) - v^{\mathtt{0}}_t(S_t)$ cannot be identified, let alone the structural per period utility functions $\mu^{\mathtt{0}}_t$ and $\mu^{\mathtt{1}}_t$. Suppose that the CDF $\tilde{G}(\cdot)$ is known, the absolute level the per period utility functions $\mu^{\mathtt{0}}_t(S_t)$ and $\mu^{\mathtt{1}}_t(S_t)$ cannot be identified. Take $\delta = 0$ for example, for any constant $c \in \mathbb{R}$,

$$p_t(S_t) = \tilde{G}(\mu^{\mathtt{1}}_t(S_t) - \mu^{\mathtt{0}}_t(S_t)) = \tilde{G}([\mu^{\mathtt{1}}_t(S_t) + c] - [\mu^{\mathtt{0}}_t(S_t) + c]).$$

To address these concerns, we make the following assumption.

Assumption 4. (i) *The CDF $G(\cdot,\cdot)$ of the unobserved state variables $\varepsilon_t = (\varepsilon^{\mathtt{0}}_t, \varepsilon^{\mathtt{1}}_t)^\intercal$ and the*

CDF $\tilde{G}(\cdot)$ *of* $\tilde{\varepsilon}_t = \varepsilon_t^{\mathtt{0}} - \varepsilon_t^{\mathtt{1}}$ *are known. Moreover,* $\tilde{\varepsilon}_t$ *is a continuous random variable with support* $\mathbb{R}$*, and the CDF* $\tilde{G}(\cdot)$ *is strictly increasing.*

(ii) *The observable state variable* S_t *is discrete with time invariant support* $\mathcal{S} = \{ s_1, \ldots, s_{d_s} \}$*.*

(iii) *(Normalization). For every period* t*, let* $\mu_t^{\mathtt{0}}(s_1) = 0$*.*

Note that besides the presence of the unknown ex ante value function $v_{t+1}(S_{t+1})$, the CCP formula (2.9) is similar to the CCP in the binary static discrete choice model studied by Matzkin (1992), in which the CDF $\tilde{G}(\cdot)$ can be nonparametrically identified. In the presence of the "special regressor" and the median assumption as assumed in Matzkin (1992), the CDF $\tilde{G}(\cdot)$ of $\tilde{\varepsilon}_t$ can be identified by following Matzkin's arguments (see also Aguirregabiria, 2010).

The normalization in Assumption 4.(iii) differs from the commonly used normalization by letting

$$\mu_t^{\mathtt{0}}(s_1) = \mu_t^{\mathtt{0}}(s_2) = \cdots = \mu_t^{\mathtt{0}}(s_{d_s}) = 0, \quad \forall t. \tag{2.10}$$

The normalization (2.10) implies that the per period utility of alternative $\mathtt{0}$ does not vary with respect to the values of the state variable S_t. It has been gradually realized that the normalization (2.10) is not innocuous for predicting counterfactual policy effects (see e.g. Norets and Tang, 2014; Arcidiacono and Miller, 2015; Aguirregabiria and Suzuki, 2014; Kalouptsidi, Scott, and Souza-Rodrigues, 2015). In Appendix B, we show two things. First, the normalization (2.10) will bias the counterfactual policy predictions, if the per period utility of alternative $\mathtt{0}$ depends on the value of the observed state variable S_t. Second, the normalization of Assumption 4.(iii) will not bias the counterfactual policy predictions.

By assuming discrete state space (Assumption 4.(ii)), the structural per period utility functions $\mu_t^{\mathtt{0}}(S_t)$ and $\mu_t^{\mathtt{1}}(S_t)$, the CCP $p_t(S_t)$, the ASVF $v_t^{\mathtt{0}}(S_t)$ and $v_t^{\mathtt{1}}(S_t)$, and the ex ante value functions $v_t(S_t)$ are all finitely dimensional. Denote $\mu_t^{\mathtt{0}} = (\mu_t^{\mathtt{0}}(s_1), \ldots, \mu_t^{\mathtt{0}}(s_{d_s}))^{\mathsf{T}}$, and $\mu_t^{\mathtt{1}}$, p_t, $v_t^{\mathtt{0}}$, $v_t^{\mathtt{1}}$ and v_t are defined similarly. It should be remarked that our identification results below hold for any finite number of states d_s. Let $f_{t+1}(S_{t+1}|S_t, D_t)$ be the conditional probability function of S_{t+1} given S_t and D_t. Let $F_{t+1}^{\mathtt{d}}$ be the state transition matrix describing the transition probabilities from state S_t to S_{t+1} when choice $D_t = \mathtt{d} \in \{\mathtt{0}, \mathtt{1}\}$:

$$F_{t+1}^{\mathtt{d}} \equiv \begin{bmatrix} f_{t+1}(s_1|s_1, D_t = \mathtt{d}) & \ldots & f_{t+1}(s_{d_s}|s_1, D_t = \mathtt{d}) \\ \vdots & \vdots & \vdots \\ f_{t+1}(s_1|s_{d_s}, D_t = \mathtt{d}) & \ldots & f_{t+1}(s_{d_s}|s_{d_s}, D_t = \mathtt{d}) \end{bmatrix}.$$

In the sequel, the following notation about the state transition matrices will be used. We collect them here for the ease of reference. Denote

$$F_t^{\mathtt{1/0}} \equiv F_t^{\mathtt{1}} - F_t^{\mathtt{0}},$$

and define

$$f_{t+1}^{\mathtt{1/0}}(S_{t+1}|S_t) \equiv f_{t+1}(S_{t+1}|S_t, D_t = \mathtt{1}) - f_{t+1}(S_{t+1}|S_t, D_t = \mathtt{0}).$$

For $s \geq t$, define

$$F^{1/0}_{t:s} = \operatorname{diag}\left(F^{1/0}_t, F^{1/0}_{t+1}, \ldots, F^{1/0}_s\right),$$

$$\check{F}^{0}_{t:s} = \begin{bmatrix} I_{d_s} & -F^{1/0}_t & & & \\ & I_{d_s} & -F^{1/0}_{t+1} & & \\ & & \ddots & \ddots & \\ & & & I_{d_s} & -F^{0}_s \end{bmatrix}.$$

Example (Female labor force participation model)**.** Our particular model is based on Keane, Todd, and Wolpin (2011, section 3.1). In each year t, a married woman makes a labor force participation decision $D_t \in \{ \mathtt{0}, \mathtt{1} \}$, where 1 is "to work" and 0 is "not to work", to maximize the expected lifetime utility.

The per period utility depends on the household consumption ($cons_t$) and the number of young children (kid_t) in the household.[4] Consumption equals the household's income net of child-care expenditures. The household income is the sum of the husband's income ($husb_t$) and the wife's income ($wage_t$) if she works. The per-child child-care cost is β if she works, and zero if she stays at home. So consumption is

$$cons_t = husb_t + wage_t \times D_t - \beta kid_t \times D_t.$$

Suppose the wage offer function takes the following form

$$wage_t = \alpha_1 + \alpha_2 xp_t + \alpha_3 (xp_t)^2 + \alpha_4 edu + \omega_t,$$

where xp_t is the working experience (measured by the number of prior periods the woman has worked) of the woman in year t, edu is her education level, ω_t is a random shock, which is independent of the wife's working experience and education. The wife's working experience xp_t evolves by

$$xp_{t+1} = xp_t + D_t.$$

Assume the period utility functions associated with the two alternatives are

$$\begin{aligned} u^{\mathtt{1}}_t(S_t, \varepsilon^{\mathtt{1}}_t) &= \mu^{\mathtt{1}}_t(husb_t, xp_t, edu, kid_t) + \varepsilon^{\mathtt{1}}_t \\ &= husb_t + \alpha_1 + \alpha_2 xp_t + \alpha_3 (xp_t)^2 + \alpha_4 edu - \beta kid_t + \varepsilon^{\mathtt{1}}_t, \qquad (2.11) \\ u^{\mathtt{0}}_t(S_t, \varepsilon^{\mathtt{0}}_t) &= \mu^{\mathtt{0}}_t(husb_t, kid_t) + \varepsilon^{\mathtt{0}}_t. \end{aligned}$$

Besides the observable state variables about the woman, we also observe her husband's working experience xp^H_t and education level edu^H. Given husband's income $husb_t$, these two state variables, xp^H_t and edu^H, do not affect the period utility but affect the state transitions by affecting the husband's future income. These two state variables excluded from the period utility function will be useful for identification of the structural parameters. Let

[4] We do not model the fertility decision, and assume the arrival of children as an exogenous stochastic process.

$S_t = (husb_t, xp_t, edu, kid_t, xp_t^H, edu^H)$ be the vector of observable state variables.

The problem is dynamic because the woman's current working decision D_t affects her working experience in the next period: $xp_{t+1} = xp_t + D_t$. As in the general model, the woman's choice D_t maximizes the value function

$$D_t = \arg \max_{\mathtt{d} \in \{0,1\}} v_t^{\mathtt{d}}(S_t) + \varepsilon_t^{\mathtt{d}},$$

where the ASVF $v_t^{\mathtt{d}}(S_t)$ is defined by equation (2.6) with the period utility functions being substituted by equation (2.11).

We are interested in predicting the labor supply effects of some counterfactual policy intervention, such as child-care subsidy, tax reduction or the introduction of contraceptive techniques to households. In terms of the CCP, this means we would like to know the new CCP after imposing these counterfactual policy interventions. To answer these questions, we first need to identify and estimate the structural parameters.

2.2 Data and structural parameters of the model

Assume that researchers only observe T *consecutive* decision periods, rather than the whole decision process. Denote the T sampling periods by $1, 2, \ldots, T$. It should be remarked that the first sampling period 1 does not need to correspond to the first decision period, nor does the last sampling period T correspond to the terminal decision period T_*. Denote the data by D:

$$\mathsf{D} = (D_1, S_1, D_2, S_2, \ldots, D_T, S_T),$$

whose support is $\mathcal{D} = (\{\mathtt{0}, \mathtt{1}\} \times \mathcal{S})^T$. Let θ denote the vector of structural parameters of this model including per period utility functions $(\mu_t^{\mathtt{0}}, \mu_t^{\mathtt{1}})$, discount factors (δ_t) and transition matrices $(F_t^{\mathtt{0}}, F_t^{\mathtt{1}})$ in each period t. It will be useful to reparameterize $(\mu_t^{\mathtt{0}}, \mu_t^{\mathtt{1}})$ as $(\mu_t^{\mathtt{0}}, \mu_t^{\mathtt{1/0}})$, where $\mu_t^{\mathtt{1/0}} = (\mu_t^{\mathtt{1/0}}(s_1), \ldots, \mu_t^{\mathtt{1/0}}(s_{d_s}))^\intercal$ with $\mu_t^{\mathtt{1/0}}(S_t) = \mu_t^{\mathtt{1}}(S_t) - \mu_t^{\mathtt{0}}(S_t)$. Let $\theta_t = (\mu_t^{\mathtt{0}}, \mu_t^{\mathtt{1/0}}, \delta_t, F_t^{\mathtt{0}}, F_t^{\mathtt{1}})$ for $t = 1, \ldots, T-1$. And let $\theta_T = (v_T^{\mathtt{0}}, v_T^{\mathtt{1}}, F_T^{\mathtt{0}}, F_T^{\mathtt{1}})$ instead of $\theta_T = (\mu_T^{\mathtt{0}}, \mu_T^{\mathtt{1/0}}, \delta_T, F_T^{\mathtt{0}}, F_T^{\mathtt{1}})$, because the CCP $p_T(S_T)$ cannot be determined by the per period utility functions $\mu_T^{\mathtt{0}}$ and $\mu_T^{\mathtt{1/0}}$ alone when $T < T_*$. Let $\theta = (\theta_1, \ldots, \theta_T)$, and let Θ be the parameter space.

We consider identification for such data that we call a short panel not only because short panel data are common in empirical studies, but also because the number of time periods turns out to play an important role in the identification of DPDC models. As shown below, when the discount factors are known, one needs at least three consecutive periods to identify nonstationary DPDC models without the terminal conditions, e.g. $T = T_*$ (so researchers observe the decision in the terminal period) or $\mathrm{E}_{T+1}[v_{T+1}(S_{T+1})|S_T, D_T = \mathtt{d}] = 0$ for both $\mathtt{d} = \mathtt{0}$ and $\mathtt{1}$. In the presence of terminal conditions, we can identify the model with two consecutive periods data, when the discount factors are known. If the discount factors are unknown, we need one additional period data to identify the discount factors. It is remarkable that such dependence of identification of DPDC models on the number of periods has not been noticed in the current literature.[5]

3 An example with four-period dynamic discrete choice

To develop some intuition for the general results that will be presented in section 4 and 5, we consider the identification and estimation of structural parameters in a *four* period dynamic discrete choice model.

The goal is to show that with the Exclusion Restriction below, we can identify the per period utility functions without assuming that $\mu_t^0(S_t) = 0$ for all S_t, and the per period utility functions can be estimated by a closed form linear estimator. To keep the example concrete and simple, we maintain the following three assumptions in this section. First, assume that the unobserved state variables ε_t^0 and ε_t^1 are independent and follow the type-1 extreme value distribution. So the CDF $\tilde{G}(\cdot)$ of $\tilde{\varepsilon}_t = \varepsilon_t^0 - \varepsilon_t^1$ in this section is the logistic distribution function. Second, the state transition matrices are time invariant. Let $F_t^0 = F^0$ and $F_t^1 = F^1$ for each t. We also omit the time subscript "t" in the conditional expectations E_t^0, E_t^1 and $\mathrm{E}_t^{1/0}$, and simply write E^0, E^1 and $\mathrm{E}^{1/0}$. Third, assume that the discount factor is constant over the decision periods and is denoted by δ.

We will study three cases below. In the first case (subsection 3.1), assume that researchers observe the decisions in the last two decision periods, that is the data there are (D_3, S_3, D_4, S_4). In the second case (subsection 3.2), we have data of only the first two decision periods, (D_1, S_1, D_2, S_2). In the last case (subsection 3.3), researchers observe the decisions in the first *three* decision periods, $(D_1, S_1, D_2, S_2, D_3, S_3)$. Since period 4 is the terminal decision period, there is no continuation value for the choice in period 4. So we have "terminal condition" in the first case, but not in the second and third cases. The comparison between case 1 and 2 clarifies the role of "terminal conditions" in the identification of dynamic discrete choice models. We will assume that the discount factor δ is known in the first two cases. In the third case, in which we observe one additional period than case 2, we show how to identify the discount factor.

3.1 Identification and estimation with the data of the last two decision periods

Consider the dynamic discrete choices backwardly. In period 4 (terminal period), there is no continuation value for the choice. Hence the decision rule in period 4,

$$D_4 = \begin{cases} 1, & \text{if } \mu_4^1(S_4) + \varepsilon_4^1 > \mu_4^0(S_4) + \varepsilon_4^0, \\ 0, & \text{otherwise,} \end{cases}$$

[5] In the literature of the identification of the CCP with unobserved discrete types, the identification also depends on the number of time periods in panel data. Interestingly, $T \geq 3$ is also required to identify type specific CCP (e.g. Kasahara and Shimotsu (2009); Hu and Shum (2012); Bonhomme, Jochmans, and Robin (2013, 2014)).

which is a logit model. We then have

$$p_4(S_4) = \tilde{G}(\mu_4^{1/0}(S_4)). \tag{3.1}$$

Here $\tilde{G}(\cdot)$ is the logistic distribution function. Also, the ex ante value function $v_4(S_4)$ equals the following,

$$\begin{aligned} v_4(S_4) &= \mathrm{E}_{\varepsilon_4}[V_4(S_4, \varepsilon_4)|S_4] \\ &= \mathrm{E}_{\varepsilon_4}\left[\max_{\mathtt{d}\in\{\mathtt{0},\mathtt{1}\}} \mu_4^{\mathtt{d}}(S_4) + \varepsilon_4^{\mathtt{d}} \middle| S_4\right] \\ &= \mu_4^{\mathtt{0}}(S_4) + [\gamma - \ln(1 - p_4(S_4))], \end{aligned} \tag{3.2}$$

where $\gamma \approx 0.5772$ is Euler's constant. The last line of the above display follows from the properties of the logit model. Because we will refer the term $\gamma - \ln(1 - p_4(S_4))$ frequently, define

$$\psi(p_4(S_4)) = \gamma - \ln(1 - p_4(S_4)).$$

It follows from the CCP formula (2.9) that the CCP in period 3 is

$$\begin{aligned} p_3(S_3) &= \tilde{G}(\mu_3^{1/0}(S_3) + \delta\, \mathrm{E}^{1/0}[v_4(S_4)|S_3]) \\ &= \tilde{G}\left(\mu_3^{1/0}(S_3) + \delta\, \mathrm{E}^{1/0}[\mu_4^{\mathtt{0}}(S_4)|S_3] + \delta\, \mathrm{E}^{1/0}[\psi(p_4(S_4))|S_3]\right). \end{aligned} \tag{3.3}$$

The second line follows from replacing $v_4(S_4)$ with equation (3.2).

Let $\phi(p) = \ln p - \ln(1-p)$ be the inverse of the logistic distribution function. It follows from equation (3.1) and (3.3) that

$$\phi(p_4(S_4)) = \mu_4^{1/0}(S_4), \tag{3.4a}$$

$$\phi(p_3(S_3)) = \mu_3^{1/0}(S_3) + \delta\, \mathrm{E}^{1/0}[\mu_4^{\mathtt{0}}(S_4)|S_3] + \delta\, \mathrm{E}^{1/0}[\psi(p_4(S_4))|S_3]. \tag{3.4b}$$

From data (D_3, S_3, D_4, S_4), we can identify and estimate the state transition matrices $F^{\mathtt{0}}$ and $F^{\mathtt{1}}$, and the CCP $p_3(S_3)$ and $p_4(S_4)$. The per period utility functions difference in the terminal period $\mu_4^{1/0}(S_4)$ is then identified from equation (3.4a) without further restriction. However, $\mu_3^{1/0}(S_3)$ and $\mu_4^{\mathtt{0}}(S_4)$ cannot be identified from equation (3.4b) without further restriction even when the discount factor δ is known. To see this, we can identify only

$$\mu_3^{1/0}(S_3) + \delta\, \mathrm{E}^{1/0}[\mu_4^{\mathtt{0}}(S_4)|S_3] = \phi(p_3(S_3)) - \delta\, \mathrm{E}^{1/0}[\psi(p_4(S_4))|S_3],$$

when the discount factor δ is known. Because both $\mu_3^{1/0}(S_3)$ and $\mathrm{E}^{1/0}[\mu_4^{\mathtt{0}}(S_4)|S_3]$ are unknown functions of S_3, we cannot identify $\mu_3^{1/0}(S_3)$ and $\mu_4^{\mathtt{0}}(S_4)$ separately. Moreover, we cannot identify the discount factor δ, because

$$\phi(p_3(S_3)) = \mu_3^{1/0}(S_3) + (\delta + c)\, \mathrm{E}^{1/0}[\mu_4^{\mathtt{0}}(S_4) - \psi(p_4(S_4))|S_3] + (\delta + c)\, \mathrm{E}^{1/0}[\psi(p_4(S_4))|S_3].$$

So the new discount factor $\tilde{\delta} = \delta + c$ and the new per period utility $\tilde{\mu}_4^0(S_4) = \mu_4^0(S_4) - \psi(p_4(S_4))$ will also satisfy equation (3.4b).

We will show how to identify and estimate $\mu_3^{1/0}(S_3)$ and $\mu_4^0(S_4)$ using equation (3.4b) and the following Exclusion Restriction, when the discount factor δ is known. Note that $\mu_4^1(S_4) = \mu_4^0(S_4) + \mu_4^{1/0}(S_4)$. Given that $\mu_4^{1/0}(S_4)$ is identified, the per period utility function $\mu_4^1(S_4)$ is identified as long as $\mu_4^0(S_4)$ is identified.

Exclusion Restriction. *The vector of observable state variables S_t has two parts X_t and Z_t. Let $S_t = (X_t, Z_t)$, where $X_t \in \mathcal{X} = \{ x_1, \ldots, x_{d_x} \}$ and $Z_t \in \mathcal{Z} = \{ z_1, \ldots, z_{d_z} \}$. Assume that*

$$\mu_t^1(X_t, Z_t) = \mu_t^1(X_t) \qquad \text{and} \qquad \mu_t^0(X_t, Z_t) = \mu_t^0(X_t)$$

for any (X_t, Z_t). For expositional simplicity, assume that $\mathcal{S} = \mathcal{X} \times \mathcal{Z}$, so that $d_s = d_x \cdot d_z$. In particular, let

$$\mathcal{S} = \operatorname{vec} \begin{bmatrix} (x_1, z_1) & (x_2, z_1) & \ldots & (x_{d_x}, z_1) \\ \vdots & \vdots & \vdots & \vdots \\ (x_1, z_{d_z}) & (x_2, z_{d_z}) & \ldots & (x_{d_x}, z_{d_z}) \end{bmatrix}.$$

For $d_x = d_z = 2$, this means $\mathcal{S} = \{ (x_1, z_1), (x_1, z_2), (x_2, z_1), (x_2, z_2) \}$.

For simplicity, suppose that X_t and Z_t in the Exclusion Restriction can take two values in this section, that is the support $\mathcal{X} = \{ x_1, x_2 \}$ and $\mathcal{Z} = \{ z_1, z_2 \}$. Applying the Exclusion Restriction and evaluating equation (3.4b) at each $(x_i, z_j) \in \mathcal{X} \times \mathcal{Z}$, we have

$$\phi(p_3(x_i, z_j)) = \mu_3^{1/0}(x_i) + \delta\, \mathrm{E}^{1/0}[\mu_4^0(X_4)|x_i, z_j] + \delta\, \mathrm{E}^{1/0}[\psi(p_4(S_4))|x_i, z_j]., \tag{3.5}$$

for $i, j = 1, 2$. For each $x_i \in \mathcal{X}$, the difference $\phi(p_3(x_i, z_1)) - \phi(p_3(x_i, z_2))$ depends on $\mu_4^0(X_4)$, but not on $\mu_3^{1/0}(X_3)$. We are going to identify $\mu_4^0(X_4)$ using the differences $\{ \phi(p_3(x_i, z_1)) - \phi(p_3(x_i, z_2)) : i = 1, 2 \}$ first. Then $\{ \mu_3^{1/0}(x_i) : i = 1, 2 \}$ is identified by the above the display. For $i = 1, 2$, considering the difference $\phi(p_3(x_i, z_1)) - \phi(p_3(x_i, z_2))$, we have

$$\begin{aligned} b_i &= \mathrm{E}^{1/0}[\mu_4^0(X_4)|x_i, z_1] - \mathrm{E}^{1/0}[\mu_4^0(X_4)|x_i, z_2] \\ &= \sum_{j=1,2} f^{1/0}(x_j|x_i, z_1)\mu_4^0(x_j) - \sum_{j=1,2} f^{1/0}(x_j|x_i, z_2)\mu_4^0(x_j), \quad i = 1, 2, \end{aligned} \tag{3.6}$$

with

$$b_i = \left(\frac{\phi(p_3(x_i, z_1))}{\delta} - \mathrm{E}^{1/0}[\psi(p_4(S_4))|x_i, z_1] \right) - \left(\frac{\phi(p_3(x_i, z_2))}{\delta} - \mathrm{E}^{1/0}[\psi(p_4(S_4))|x_i, z_2] \right).$$

Equation (3.6) can be organized as the following system of equations that is linear in $\mu_4^0 = (\mu_4^0(x_1), \mu_4^0(x_2))^\intercal$,

$$A\mu_4^0 = b, \tag{3.7}$$

where

$$b = (b_1, b_2)^\intercal$$

and

$$A = \begin{bmatrix} f^{1/0}(x_1|x_1,z_1) - f^{1/0}(x_1|x_1,z_2) & f^{1/0}(x_2|x_1,z_1) - f^{1/0}(x_2|x_1,z_2) \\ f^{1/0}(x_1|x_2,z_1) - f^{1/0}(x_1|x_2,z_2) & f^{1/0}(x_2|x_2,z_1) - f^{1/0}(x_2|x_2,z_2) \end{bmatrix}.$$

Using the notation $F^{1/0} = F^1 - F^0$, the matrix A can be written alternatively as follows,

$$A = MF^{1/0}(I_2 \otimes 1_2),$$

where I_2 is the 2×2 identity matrix, "$\otimes$" is Kronecker product, $1_2 = (1, 1)^\intercal$, and

$$M \equiv \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}. \tag{3.8}$$

The linear system of equations like equation (3.7) will be frequently encountered in the sequel. The matrix A will always depend only on the state transition matrices and the discount factors; the vector b will always depend only on the CCP and the discount factors. However, their explicit definitions will change with respect to different model specifications. Note that $A\,1_2 = 0_2$. Hence 1_2 is a non-zero eigenvector of matrix A associated with the eigenvalue 0. The matrix A cannot be of full rank. If $\operatorname{rank} A = 1$, the solution set of equation (3.7) is

$$\left\{ A^+b + c \cdot 1_2 : c \in \mathbb{R} \right\},$$

where A^+ is the Moore-Penrose pseudoinverse of A (see lemma A.1 for proof). So the solution for μ_4^0 is unique up to a constant that does not change with respect to the states. Note that if $X_4 \perp\!\!\!\perp Z_3|X_3, D_3$, both columns of A are zero, hence $\operatorname{rank} A = 0$. Though the solution for μ_4^0 is not unique, we have a unique solution for the per period utility functions difference $\mu_3^{1/0} = (\mu_3^{1/0}(x_1), \mu_3^{1/0}(x_2))^\intercal$. Let $\mu_4^0 = A^+b + c \cdot 1_2$ be an arbitrary solution of equation (3.7), it follows from equation (3.5) that

$$\begin{aligned} \mu_3^{1/0}(x_i) &= \phi(p_3(x_i, z_j)) - \delta \begin{bmatrix} f^{1/0}(x_1|x_i, z_j) \\ f^{1/0}(x_2|x_i, z_j) \end{bmatrix}^\intercal \left(A^+b + c \cdot 1_2\right) - \delta\, \mathrm{E}^{1/0}[\psi(p_4(S_4))|x_i, z_j]. \\ &= \phi(p_3(x_i, z_j)) - \delta \begin{bmatrix} f^{1/0}(x_1|x_i, z_j) \\ f^{1/0}(x_2|x_i, z_j) \end{bmatrix}^\intercal \left(A^+b\right) - \delta\, \mathrm{E}^{1/0}[\psi(p_4(S_4))|x_i, z_j], \end{aligned}$$

for both $j = 1$ and 2. Note that the above display does not depend on the unknown constant c, so $\mu_3^{1/0}(x_i)$ is identified for $i = 1, 2$. It should be remarked that $\mu_3^{1/0}(x_i)$ is linear in the discount factor δ, and such linearity will be used to identify the discount factor in subsection 3.3.

The per period utility function $\mu_4^0 = (\mu_4^0(x_1), \mu_4^0(x_2))^\intercal$ is identified with the normalization

$\mu_4^0(x_1) = 0$ (Assumption 4.(iii)). With such normalization, we can identify μ_4^0 as

$$\mu_4^0 = \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix} A^+ b.$$

To estimate $\mu_3^{1/0}$ and μ_4^0, we only need to estimate the difference between the state transition matrices F^0 and F^1, and the CCP $p_3(S_3)$ and $p_4(S_4)$, with which A and b are then estimated. The per period utility functions $\mu_3^{1/0}$ and μ_4^0 can be estimated by the above displays after substituting the unknowns with their estimates.

Remark 1 (Identification of discount factor using parametric specification). This remark shows that the parametric specification of the per period utility functions helps identify the discount factor with the terminal conditions. Suppose

$$\mu_t^{1/0}(X_t) = \alpha_{t,0} + X_t^\intercal \alpha_{t,1} \qquad \text{and} \qquad \mu_t^0(X_t) = \beta_{t,0} + X_t^\intercal \beta_{t,1}. \tag{3.9}$$

Under the above linear specification, equation (3.5) becomes

$$\phi(p_3(S_3)) = \alpha_{3,0} + X_3^\intercal \alpha_{3,1} + \mathrm{E}^{1/0}\left(X_4^\intercal | X_3, Z_3\right)(\delta\beta_{4,1}) - \mathrm{E}^{1/0}\left[\psi(p_4(S_4))|X_3, Z_3\right]\delta. \tag{3.10}$$

Note that the intercept term $\beta_{4,0}$ disappears because $\mathrm{E}^{1/0}(\beta_{4,0}|X_3, Z_3) = 0$, and this corresponds to our earlier conclusion that the per period utility function $\mu_4^0(X_4)$ is identified up to a constant. It follows from equation (3.10) that $(\alpha_{3,0}, \alpha_{3,1}, \delta\beta_{4,1}, \delta)$ can be identified if the three terms X_3, $\mathrm{E}^{1/0}(X_4|X_3, Z_3)$ and $\mathrm{E}^{1/0}[\psi(p_4(S_4))|X_3, Z_3]$ are not linearly dependent.

Remark 2. In general the discount factor is not identifiable with two periods data even with the Exclusion Restriction. Without parametric specification about the per period utility functions, we have

$$\phi(p_3(X_3, Z_3)) = \mu_3^{1/0}(X_3) + \delta\, \mathrm{E}^{1/0}[\mu_4^0(X_4)|X_3, Z_3] + \delta\, \mathrm{E}^{1/0}[\psi(p_4(S_4))|X_3, Z_3].$$

Let $\mathcal{U}$ be the space of the per period utility function $\mu_4^0(X_4)$. The linear specification $\mu_t^0 = \beta_{t,0} + X_t^\intercal \beta_{t,1}$ in Remark 1 assumes that $\mathcal{U}$ is the set of all linear functions of X_4. The discount factor δ may not be identified, because $\delta\, \mathrm{E}^{1/0}[\mu_4^0(X_4)|X_3, Z_3]$ could be any function of (X_3, Z_3). If the equation of unknown function $g(X_4)$,

$$\mathrm{E}^{1/0}[g(X_4)|X_3, Z_3] = \mathrm{E}^{1/0}[\psi(p_4(X_4, Z_4))|X_3, Z_3],$$

has a solution in $\mathcal{U}$, the discount factor cannot be identified.[6] (In this particular case, there is always a solution because the CCP $p_4(S_4)$ in the terminal period depends only on X_4 by the

[6] In Remark 1, in order to identify the discount factor δ, we require that $\mathrm{E}^{1/0}(X_4|X_3, Z_3)$ are $\mathrm{E}^{1/0}[\psi(p_4(S_4))|X_3, Z_3]$ are not linearly dependent. Note that this is equivalent to the condition here that the equation $\mathrm{E}^{1/0}[g(X_4)|X_3, Z_3] = \mathrm{E}^{1/0}[\psi(p_4(S_4))|X_3, Z_3]$ has no solution in $\mathcal{U}$, which is the set of all linear functions of X_4 given the linear specification in Remark 1.

Exclusion Restriction.) Suppose $g(X_4)$ is one solution, then let

$$\tilde{\mu}_4^{0}(X_4) = \mu_4(X_4) - g(X_4).$$

We have

$$\phi(p_3(X_3, Z_3)) = \mu_3^{1/0}(X_3) + (\delta + c)\,\mathrm{E}^{1/0}[\tilde{\mu}_4^{0}(X_4)|X_3, Z_3] + (\delta + c)\,\mathrm{E}^{1/0}[\psi(p_4(S_4))|X_3, Z_3],$$

for any c such that $0 < \delta + c < 1$, and the discount factor is not identified.

Remark 3. Without the Exclusion Restriction or if the excluded variable Z_t does not affect the transition of the state variables affecting per period utilities ($X_4 \perp\!\!\!\perp Z_3|(X_3, D_3)$), the per period utility functions are not identifiable in general even with the linear specification (3.9) and the terminal conditions (the continuation value associated with each alternative is zero in period 4). Suppose there is no excluded variable Z_t, and the state variable $S_t = X_t$ is a scalar. Assume that X_t follows an autoregressive process,

$$X_t = \rho_0 + \rho_1^{\mathtt{d}} X_{t-1} + \omega_t, \qquad \mathtt{d} \in \{\,\mathtt{0}, \mathtt{1}\,\},$$

with $\mathrm{E}(\omega_t|X_{t-1}) = 0$. Let $\rho_1^{1/0} = \rho_1^{1} - \rho_1^{0}$. Equation (3.10) becomes

$$\begin{aligned}
\phi(p_3(X_3)) &= \alpha_{3,0} + X_3\alpha_{3,1} + \mathrm{E}^{1/0}(X_4|X_3)(\delta\beta_{4,1}) - \mathrm{E}^{1/0}[\ln(1 - p_4(X_4))|X_3]\delta \\
&= \alpha_{3,0} + X_3\alpha_{3,1} + (X_3\rho_1^{1/0})(\delta\beta_{4,1}) - \mathrm{E}^{1/0}[\ln(1 - p_4(X_4))|X_3]\delta \\
&= \alpha_{3,0} + X_3(\alpha_{3,1} + \rho_1^{1/0}\delta\beta_{4,1}) - \mathrm{E}^{1/0}[\ln(1 - p_4(X_4))|X_3]\delta.
\end{aligned}$$

We can only identify $(\alpha_{3,1} + \rho_1^{1/0}\delta\beta_{4,1})$ as a whole. However, if one is willing to assume that the per period utility functions are time invariant, so that $\alpha_{3,1} = \alpha_{4,1}$, we then can identify $\beta_{4,1}$ separately from the sum $(\alpha_{3,1} + \rho_1^{1/0}\delta\beta_{4,1})$, because $\alpha_{3,1} = \alpha_{4,1}$, $\rho_1^{1/0}$ and δ are identified ($\alpha_{4,1}$ is identified because $\mu_4^{1/0}(S_4)$ is identified from equation (3.4a)). This observation will be generalized to Proposition 5 of section 4.

3.2 Identification and estimation with data of the first two decision periods

Suppose now researchers observe the decisions in the *first two* periods only, hence there is no terminal condition in this case. The decision rule in period 2 is

$$D_2 = \begin{cases} \mathtt{1}, & \text{if } \mu_2^{1}(S_2) + \delta\,\mathrm{E}^{1}[v_3(S_3)|S_2] + \varepsilon_2^{1} > \mu_2^{0}(S_2) + \delta\,\mathrm{E}^{0}[v_3(S_3)|S_2] + \varepsilon_2^{0}, \\ \mathtt{0}, & \text{otherwise.} \end{cases}$$

Without the terminal condition, we now have the unknown ex ante value function $v_3(S_3)$. The CCP in period 2 (last sampling period) is

$$p_2(S_2) = \tilde{G}(\mu_2^{1/0}(S_2) + \delta\,\mathrm{E}^{1/0}[v_3(S_3)|S_2]). \tag{3.11}$$

The ex ante value function $v_2(S_2)$ equals the following,

$$\begin{aligned} v_2(S_2) &= \mathrm{E}_{\varepsilon_2}[V_2(S_2, \varepsilon_2)|S_2] \\ &= \mathrm{E}_{\varepsilon_2}\left\{ \max_{\mathtt{d}\in\{0,1\}} \mu_2^{\mathtt{d}}(S_2) + \delta\,\mathrm{E}^{\mathtt{d}}[v_3(S_3)|S_2] + \varepsilon_2^{\mathtt{d}} \middle| S_2 \right\} \\ &= (\mu_2^{0}(S_2) + \delta\,\mathrm{E}^{0}[v_3(S_3)|S_2]) + \psi(p_2(S_2)). \end{aligned} \tag{3.12}$$

Here $\psi(p_2(S_2)) = \gamma - \ln(p_2(S_2))$ is as defined before. The CCP in period 1 is

$$p_1(S_1) = \tilde{G}(\mu_1^{1/0}(S_1) + \delta\,\mathrm{E}^{1/0}[v_2(S_2)|S_1]), \tag{3.13}$$

with $v_2(S_2)$ satisfying equation (3.12).

Similar to equation (3.4), we have the following equations from equation (3.11), (3.12) and (3.13),

$$\phi(p_1(S_1)) = \mu_1^{1/0}(S_1) + \delta\,\mathrm{E}^{1/0}[v_2(S_2)|S_1], \tag{3.14a}$$

$$\phi(p_2(S_2)) = \mu_2^{1/0}(S_2) + \delta\,\mathrm{E}^{1/0}[v_3(S_3)|S_2], \tag{3.14b}$$

$$v_2(S_2) = \mu_2^{0}(S_2) + \delta\,\mathrm{E}^{0}[v_3(S_3)|S_2] + \psi(p_2(S_2)). \tag{3.14c}$$

Without terminal condition, the per period utility functions difference $\mu_2^{1/0}$ cannot be identified from equation (3.14b). As in case 1 of subsection 3.1, without restriction, the per period utility functions $\mu_1^{1/0}$ and μ_2^{0}, and the discount factor δ are not identified. So we conclude that without further restriction, the terminal condition only helps identify the difference between the per period utility functions in the last sampling period.

Applying the Exclusion Restriction and evaluating equation (3.14) at each $(x_i, z_j) \in \mathcal{X} \times \mathcal{Z} \equiv \{x_1, x_2\} \times \{z_1, z_2\}$, we have

$$\begin{aligned} \phi(p_1(x_i, z_j)) &= \mu_1^{1/0}(x_i) + \delta\,\mathrm{E}^{1/0}[v_2(S_2)|x_i, z_j], && i,j = 1,2, \\ \phi(p_2(x_i, z_j)) &= \mu_2^{1/0}(x_i) + \delta\,\mathrm{E}^{1/0}[v_3(S_3)|x_i, z_j], && i,j = 1,2, \\ v_2(x_i, z_j) &= \mu_2^{0}(x_i) + \delta\,\mathrm{E}^{0}[v_3(S_3)|x_i, z_j] + \psi(p_2(x_i, z_j)), && i,j = 1,2. \end{aligned} \tag{3.15}$$

We want to identify $\mu_1^{1/0}$, $\mu_2^{1/0}$ and μ_2^{0} by solving the unknowns $\left(\mu_1^{1/0}, \mu_2^{1/0}, \mu_2^{0}, v_2, v_3\right)$ explicitly from equation (3.15). Note that the per period utility function μ_1^{0} does not appear in the above equations, hence cannot be identified. Also note that the terminal payoff v_3 is identified (up to a constant). We solve equation (3.15) by following the steps below.

Step 1: Eliminate the per period utility functions $\mu_1^{1/0}$, $\mu_2^{1/0}$ and μ_2^0 from equation (3.15). Let

$$\phi_t(i,j) \equiv \phi(p_t(x_i,z_j)), \qquad \psi_t(i,j) \equiv \psi(p_t(x_i,z_j)), \qquad \text{and} \qquad \bar{v}_3(S_3) \equiv \delta v_3(S_3).$$

We have the following equations,

$$\frac{\phi_1(1,1)-\phi_1(1,2)}{\delta} = \mathrm{E}^{1/0}[v_2(S_2)|x_1,z_1] - \mathrm{E}^{1/0}[v_2(S_2)|x_1,z_2], \tag{3.16a}$$
$$\frac{\phi_1(2,1)-\phi_1(2,2)}{\delta} = \mathrm{E}^{1/0}[v_2(S_2)|x_2,z_1] - \mathrm{E}^{1/0}[v_2(S_2)|x_2,z_2], \tag{3.16b}$$

$$\phi_2(1,1)-\phi_2(1,2) = \mathrm{E}^{1/0}[\bar{v}_3(S_3)|x_1,z_1] - \mathrm{E}^{1/0}[\bar{v}_3(S_3)|x_1,z_2], \tag{3.16c}$$
$$\phi_2(2,1)-\phi_2(2,2) = \mathrm{E}^{1/0}[\bar{v}_3(S_3)|x_2,z_1] - \mathrm{E}^{1/0}[\bar{v}_3(S_3)|x_2,z_2], \tag{3.16d}$$

$$\psi_2(1,1)-\psi_2(1,2) = v_2(x_1,z_1) - v_2(x_1,z_2) - \mathrm{E}^0[\bar{v}_3(S_3)|x_1,z_1] + \mathrm{E}^0[\bar{v}_3(S_3)|x_1,z_2], \tag{3.16e}$$
$$\psi_2(2,1)-\psi_2(2,2) = v_2(x_2,z_1) - v_2(x_2,z_2) - \mathrm{E}^0[\bar{v}_3(S_3)|x_2,z_1] + \mathrm{E}^0[\bar{v}_3(S_3)|x_2,z_2]. \tag{3.16f}$$

When the discount factor δ is known, the above system is equivalent to

$$A \begin{bmatrix} v_2 \\ \bar{v}_3 \end{bmatrix} = b_2, \tag{3.17}$$

where the unknown is

$$\begin{bmatrix} v_2 \\ \bar{v}_3 \end{bmatrix} = \mathrm{vec} \begin{bmatrix} v_2(x_1,z_1) & \bar{v}_3(x_1,z_1) \\ v_2(x_1,z_2) & \bar{v}_3(x_1,z_2) \\ v_2(x_2,z_1) & \bar{v}_3(x_2,z_1) \\ v_2(x_2,z_2) & \bar{v}_3(x_2,z_2) \end{bmatrix},$$

the coefficient matrix A is a 6×8 matrix,

$$A \equiv \begin{bmatrix} MF^{1/0} & 0 \\ 0 & MF^{1/0} \\ M & -MF^0 \end{bmatrix},$$

with the M matrix as defined by equation (3.8), and b_2 is a 6-dimensional vector consisting of the terms on the left-hand-side of equation (3.16),

$$b_2 \equiv \mathrm{vec} \begin{bmatrix} (\phi_1(1,1)-\phi_1(1,2))/\delta & \phi_2(1,1)-\phi_2(1,2) & \psi_2(1,1)-\psi_2(1,2) \\ (\phi_1(2,1)-\phi_1(2,2))/\delta & \phi_2(2,1)-\phi_2(2,2) & \psi_2(2,1)-\psi_2(2,2) \end{bmatrix}. \tag{3.18}$$

Step 2: Solve v_2 and $\bar{v}_3$ from equation (3.17). Let A^+ be the Moore-Penrose pseudoinverse of matrix A, then

$$\begin{bmatrix} v_2^+ \\ \bar{v}_3^+ \end{bmatrix} = A^+ b_2$$

solves equation (3.15). Because we need to use v_t^+ and $\bar{v}_{t+1}^+$ separately, it is useful to split the matrix A^+ into two parts:

$$A^+ = \begin{bmatrix} A_u^+ \\ A_l^+ \end{bmatrix},$$

where A_u^+ and A_l^+ are the 4×6 matrices formed by the first and last 4 rows of matrix A^+, respectively. Then

$$\begin{bmatrix} v_2^+ \\ \bar{v}_3^+ \end{bmatrix} = \begin{bmatrix} A_u^+ b_2 \\ A_l^+ b_2 \end{bmatrix}.$$

If rank $A = 6$, we know from lemma A.2 that the solution set of equation (3.17) is that

$$\left\{ \begin{bmatrix} v_2^+ + c_2 \cdot 1_4 \\ \bar{v}_3^+ + c_3 \cdot 1_4 \end{bmatrix} : c_2, c_3 \in \mathbb{R} \right\}.$$

Step 3: Identify the per period utility functions $\mu_1^{1/0}$, $\mu_2^{1/0}$ and μ_2^0. Suppose rank $A = 6$, and let $v_2 = v_2^+ + c_2 \cdot 1_4$ and $\bar{v}_3 = \bar{v}_3^+ + c_3 \cdot 1_4$ be arbitrary solutions of equation (3.17) and let

$$f^{1/0}(i,j) = \begin{bmatrix} f^{1/0}(x_1, z_1 | x_i, z_j) \\ f^{1/0}(x_1, z_2 | x_i, z_j) \\ f^{1/0}(x_2, z_1 | x_i, z_j) \\ f^{1/0}(x_2, z_2 | x_i, z_j) \end{bmatrix}.$$

Then associated with $v_2 = v_2^+ + c_2 \cdot 1_4$ and $\bar{v}_3 = \bar{v}_3^+ + c_3 \cdot 1_4$, we have the following from equation (3.15): for $j = 1, 2$,

$$\mu_1^{1/0}(x_i) = \phi(p_1(x_i, z_j)) - \delta f^{1/0}(i,j)^\intercal A_u^+ b_2, \tag{3.19}$$

$$\mu_2^{1/0}(x_i) = \phi(p_2(x_i, z_j)) - f^{1/0}(i,j)^\intercal A_l^+ b_2, \tag{3.20}$$

$$\mu_2^0(x_i) = v_2^+(x_i, z_j) - f^{1/0}(i,j)^\intercal A_l^+ b_2 - \psi(p_2(x_i, z_j)) + (c_2 - c_3).$$

The constant $c_2 - c_3$ in $\mu_2^0(x_i)$ can be determined by the normalization condition that $\mu_2^0(x_1) = 0$. So we conclude that the per period utility functions $\mu_1^{1/0}$, $\mu_2^{1/0}$ and μ_2^0 are identified.

Given the explicit formulas for the per period utility functions, their estimation is easy. We again estimate the CCP and the state transition matrices, then plug their estimates into the above formulas to estimate the per period utility functions.

3.3 Identification of the discount factor with three-period data

Suppose we have data $(D_1, S_1, D_2, S_2, D_3, S_3)$. Applying the identification arguments of case 2 (subsection 3.2) with data (D_1, S_1, D_2, S_2) and data (D_2, S_2, D_3, S_3), respectively, we will have two formulas for $\mu_2^{1/0}(x_i)$:

$$\mu_2^{1/0}(x_i) = \phi(p_2(x_i, z_j)) - \delta f^{1/0}(i,j)^\intercal A_u^+ b_3, \tag{3.21}$$

$$\mu_2^{1/0}(x_i) = \phi(p_2(x_i, z_j)) - f^{1/0}(i,j)^\intercal A_l^+ b_2, \tag{3.22}$$

where equation (3.21) follows from equation (3.19) with data (D_2, S_2, D_3, S_3), and equation (3.22) follows from (3.20) with data (D_1, S_1, D_2, S_2).

So we have the following equation,

$$\delta f^{1/0}(i,j)^\intercal A_u^+ b_3 = f^{1/0}(i,j)^\intercal A_l^+ b_2, \tag{3.23}$$

about the discount factor δ. In the remainder of this section, we derive the solution of the discount factor δ.

For $t = 2, 3$, define two vectors $b_{t,u}$ and $b_{t,l}$,

$$b_{t,u} \equiv \text{vec} \begin{bmatrix} \phi_{t-1}(1,1) - \phi_{t-1}(1,2) & 0 & 0 \\ \phi_{t-1}(2,1) - \phi_{t-1}(2,2) & 0 & 0 \end{bmatrix},$$

$$b_{t,l} \equiv \text{vec} \begin{bmatrix} 0 & \phi_t(1,1) - \phi_t(1,2) & \psi_t(1,1) - \psi_t(1,2) \\ 0 & \phi_t(2,1) - \phi_t(2,2) & \psi_t(2,1) - \psi_t(2,2) \end{bmatrix},$$

where $\phi_t(i,j) \equiv \phi(p_t(x_i, z_j))$ and $\psi_t(i,j) \equiv \psi(p_t(x_i, z_j))$, so that

$$b_t = \frac{1}{\delta} b_{t,u} + b_{t,l},$$

according to the definition of b_t in equation (3.18). Denote

$$h_u(i,j) \equiv f^{1/0}(i,j)^\intercal A_u^+, \qquad \text{and} \qquad h_l(i,j) \equiv f^{1/0}(i,j)^\intercal A_l^+.$$

Then equation (3.23) becomes

$$\delta h_u(i,j) b_{3,l} - \frac{1}{\delta} h_l(i,j) b_{2,u} + (h_u(i,j) b_{3,u} - h_l(i,j) b_{2,l}) = 0. \tag{3.24}$$

Let

$$r_{3,l}(i,j) \equiv h_u(i,j) b_{3,l} \qquad\qquad r_{3,u}(i,j) \equiv h_u(i,j) b_{3,u},$$

$$r_{2,u}(i,j) \equiv h_l(i,j) b_{2,u}, \qquad\qquad r_{2,l}(i,j) \equiv h_l(i,j) b_{2,l}.$$

Letting $i = 1, 2$, we have two equations about δ:

$$\delta r_{3,l}(1,j) - \frac{1}{\delta} r_{2,u}(1,j) + (r_{3,u}(1,j) - r_{2,l}(1,j)) = 0,$$
$$\delta r_{3,l}(2,j) - \frac{1}{\delta} r_{2,u}(2,j) + (r_{3,u}(2,j) - r_{2,l}(2,j)) = 0.$$

Hence we have

$$\delta = \frac{r_{2,u}(2,j)(r_{2,l}(1,j) - r_{3,u}(1,j)) - r_{2,u}(1,j)(r_{2,l}(2,j) - r_{3,u}(2,j))}{r_{3,l}(1,j) r_{2,u}(2,j) - r_{3,l}(2,j) r_{2,u}(1,j)},$$

provided that

$$\tilde{r} \equiv r_{3,l}(1,j) r_{2,u}(2,j) - r_{3,l}(2,j) r_{2,u}(1,j) \neq 0.$$

It is instructive to see when $\tilde{r} \neq 0$. We write

$$\begin{aligned}
\tilde{r} &= h_u(1,j) b_{3,l} h_l(2,j) b_{2,u} - h_u(2,j) b_{3,l} h_l(1,j) b_{2,u} \\
&= h_u(1,j) b_{3,l} b_{2,u}{}^\intercal h_l(2,j)^\intercal - h_u(2,j) b_{3,l} b_{2,u}{}^\intercal h_l(1,j)^\intercal \\
&= (h_u(1,j) - h_u(2,j))^\intercal b_{3,l} b_{2,u}{}^\intercal h_l(2,j)^\intercal + h_u(2,j)^\intercal b_{3,l} b_{2,u}{}^\intercal (h_l(2,j) - h_l(1,j))^\intercal.
\end{aligned}$$

To ensure that $\tilde{r} \neq 0$, that is to identify the discount factor, it is necessary that

(i) the choice D_t changes the state transition distributions, that is $f^{1/0}(S_{t+1}|S_t) \neq 0$ for some S_t, S_{t+1}. Otherwise, we have $f^{1/0}(i,j) = 0$, hence $h_u(i,j) = 0$ and $h_l(i,j) = 0$, hence $\tilde{r} = 0$;

(ii) the state variable X_t should affect the difference between the state transition distributions under the two alternatives given the excluded variable Z_t, that is $f^{1/0}(1,j) \neq f^{1/0}(2,j)$ for some $j = 1, 2$. Otherwise, $h_u(1,j) = h_u(2,j)$ and $h_l(1,j) = h_l(2,j)$, hence $\tilde{r} = 0$;

(iii) for each period t, the excluded variable Z_t should still change the CCP conditional on the state variables X_t that enter into the per period utility functions, that is $p_t(x_i, z_j) \neq p_t(x_i, z_{j'})$ for some $i, j \neq j'$. Otherwise, $b_{2,u} = 0$ or $b_{3,l} = 0$, hence $\tilde{r} = 0$.

4 Identification of Structural Parameters

We first show that the identification of DPDC models is equivalent to the identification of a linear GMM model. Then, applying this equivalence, we prove a list of identification results. Several important remarks are added at the end of this section.

4.1 Linear GMM representation of DPDC models

Our DPDC model maps its structural parameters θ to a joint probability function $f(\mathsf{D};\theta)$ of data $\mathsf{D} = (D_1, S_1, \ldots, D_T, S_T)$. Define $\mathcal{F} \equiv \{ f(\mathsf{D};\theta) : \theta \in \Theta \}$. Two sets of structural parameters θ and $\tilde{\theta}$ are observationally equivalent if and only if (iff) $f(\mathsf{D};\theta) = f(\mathsf{D};\tilde{\theta})$ for all

$\mathsf{D} \in \mathcal{D}$, where $\mathcal{D}$ is the support of data D. Given data D, the structural parameters θ are identified in the parameter space Θ iff any two observationally equivalent parameters in Θ are identical. In other words, the structural parameters θ are identified if for any $f(\mathsf{D}) \in \mathcal{F}$, the system of equations

$$f(\mathsf{D}) = f(\mathsf{D}; \theta), \ \forall \mathsf{D} \in \mathcal{D}, \tag{4.1}$$

has a unique solution for θ in the parameter space Θ. Due to limited data and/or weak restrictions on the parameter space, we sometimes can only identify one component of the structural parameters, which turns out to be our case. Let $\theta = (\theta_a, \theta_b) \in \Theta$ be the vector of parameters. We say that θ_a is identified in Θ iff for any pair $\theta = (\theta_a, \theta_b), \tilde{\theta} = (\tilde{\theta}_a, \tilde{\theta}_b)$, the condition that θ and $\tilde{\theta}$ are observationally equivalent implies $\theta_a = \tilde{\theta}_a$. Again, this statement can be rephrased in terms of equation (4.1) as follows: θ_a is identified in Θ iff for any $f(\mathsf{D}) \in \mathcal{F}$, equation (4.1) has a unique solution for θ_a.

Any joint probability function $f(\mathsf{D}) = f(D_1, S_1, \ldots, D_T, S_T) \in \mathcal{F}$ can be decomposed as the following product

$$\begin{aligned} f(\mathsf{D}) = \mathrm{P}(S_1)\mathrm{P}(D_1|S_1)\mathrm{P}(S_2|S_1, D_1)\mathrm{P}(D_2|S_2, S_1, D_1) \times \\ \mathrm{P}(S_3|S_2, S_1, D_2, D_1)\mathrm{P}(D_3|S_3, S_2, S_1, D_2, D_1) \times \cdots \times \\ \mathrm{P}(S_T|S_{T-1}, \ldots, S_1, D_{T-1}, \ldots, D_1)\mathrm{P}(D_T|S_T, S_{T-1}, \ldots, S_1, D_{T-1}, \ldots, D_1). \end{aligned} \tag{4.2}$$

Because the joint probability function $f(\mathsf{D}) \in \mathcal{F}$ satisfies the Markovian assumptions 2 and 3, we have

$$\begin{aligned} \mathrm{P}(S_t|S_{t-1}, \ldots, S_1, D_{t-1}, \ldots, D_1) &= \mathrm{P}(S_t|S_{t-1}, D_{t-1}) \\ &= f_t(S_t|S_{t-1}, D_{t-1}), \\ \mathrm{P}(D_t|S_t, \ldots, S_1, D_{t-1}, \ldots, D_1) &= \mathrm{P}(D_t|S_t) \\ &= (p_t(S_t))^{D_t}(1 - p_t(S_t))^{1-D_t}. \end{aligned}$$

So equation (4.2) equals the following,

$$f(\mathsf{D}) = f_1(S_1)\left[\prod_{t=1}^{T-1} \mathrm{P}(D_t|S_t) f_{t+1}(S_{t+1}|S_t, D_t)\right]\mathrm{P}(D_T|S_T).$$

Similarly, we can decompose $f(\mathsf{D}; \theta)$ by

$$f(\mathsf{D}; \theta) = f_1(S_1; \theta)\left[\prod_{t=1}^{T-1} \mathrm{P}(D_t|S_t; \theta) f_{t+1}(S_{t+1}|S_t, D_t; \theta)\right]\mathrm{P}(D_T|S_T; \theta),$$

where $\mathrm{P}(D_t|S_t; \theta) = (p_t(S_t; \theta))^{D_t}(1 - p_t(S_t; \theta))^{1-D_t}$. Because of the above decomposition of

$f(\mathsf{D})$ and $f(\mathsf{D};\theta)$, it can be verified that equation (4.1) is equivalent to the following[7]

$$f_1(S_1) = f_1(S_1;\theta),\ \forall S_1 \in \mathcal{S}, \tag{4.3a}$$

$$f_{t+1}(S_{t+1}|S_t, D_t) = f_{t+1}(S_{t+1}|S_t, D_t;\theta),\ t = 1,\dots,T-1,\ \forall(S_{t+1}, S_t, D_t) \in \mathcal{S}^2 \times \{\,\mathsf{0},\mathsf{1}\,\}, \tag{4.3b}$$

$$p_t(S_t) = p_t(S_t;\theta),\ t = 1,\dots,T,\ \forall S_t \in \mathcal{S}. \tag{4.3c}$$

From equation (4.3a) and (4.3b), we conclude that the state transition probabilities are identified. In the remainder of the identification analysis, we assume that the state transition probabilities are known and focus on the identification of per period utility functions (μ_t^{0} and $\mu_t^{\mathsf{1/0}}$) and discount factors (δ_t).

The attention now is equation (4.3c), which requires the explicit form of the CCP $p_t(S_t;\theta)$ in terms of the structural parameters θ. It follows from the CCP formula of equation (2.9) that

$$\begin{aligned} p_t(S_t;\theta) &= \tilde{G}(v_t^{\mathsf{1}}(S_t) - v_t^{\mathsf{0}}(S_t)) \\ &= \tilde{G}(\mu_t^{\mathsf{1/0}}(S_t) + \delta_t \operatorname{E}_{t+1}^{\mathsf{1/0}}[v_{t+1}(S_{t+1})|S_t]),\ t = 1,\dots,T-1, \end{aligned} \tag{4.4}$$

and

$$p_T(S_T;\theta) = \tilde{G}(v_T^{\mathsf{1}}(S_T) - v_T^{\mathsf{0}}(S_T)),$$

where $\tilde{G}$ is the CDF of $\tilde{\varepsilon} = \varepsilon_t^{\mathsf{0}} - \varepsilon_t^{\mathsf{1}}$. Because the CDF $\tilde{G}$ is known and strictly increasing (Assumption 3.(i)), its inverse $\tilde{G}^{-1}$ is known. Let $\phi(\cdot) = \tilde{G}^{-1}(\cdot)$ denote the inverse. So that we have

$$\phi(p_t(S_t;\theta)) = v_t^{\mathsf{1}}(S_t) - v_t^{\mathsf{0}}(S_t),\ t = 1,\dots,T. \tag{4.5}$$

It should be noted that the ex ante value functions $\{\, v_{t+1}(S_{t+1}) : t = 1,\dots,T-1 \,\}$ in equation (4.4) are not structural parameters. So we express $v_t(S_t)$ in terms of the structural parameters. It follows from the definition of $v_t(S_t)$ in equation (2.4) and the Bellman equation (2.7)

[7] Take $T = 2$ for example, so that $\mathsf{D} = (D_1, S_1, D_2, S_2)$. If equation (4.3) holds, we clearly have $f(\mathsf{D}) = f(\mathsf{D};\theta)$. Suppose $f(\mathsf{D}) = f(\mathsf{D};\theta)$, and we will show equation (4.3). We first have $f_1(S_1) = \sum_{D_1,D_2,S_2} f(\mathsf{D})$ and $f_1(S_1;\theta) = \sum_{D_1,D_2,S_2} f(\mathsf{D};\theta)$. From $f(\mathsf{D}) = f(\mathsf{D};\theta)$, we conclude $f_1(S_1) = f_1(S_1;\theta)$. The notation $\sum_{D_1,D_2,S_2}$ means sum over all values of (D_1, D_2, S_2) in their support. We next have $f(S_1, D_1) = \sum_{D_2,S_2} f(\mathsf{D})$ and $f(S_1, D_1;\theta) = \sum_{D_2,S_2} f(\mathsf{D};\theta)$, hence $f(S_1, D_1) = f(S_1, D_1;\theta)$. Because $f(S_1, D_1) = f_1(S_1)\mathrm{P}(D_1|S_1)$, $f(S_1, D_1;\theta) = f_1(S_1;\theta)\mathrm{P}(D_1|S_1;\theta)$ and $f_1(S_1) = f_1(S_1;\theta)$, we conclude $\mathrm{P}(D_1|S_1) = \mathrm{P}(D_1|S_1;\theta)$, which is equivalent to $p_1(S_1) = p_1(S_1;\theta)$. Following the same strategy, we conclude $f(S_1, D_1, S_2) = f(S_1, D_1, S_2;\theta)$, which implies that $f_2(S_2|S_1, D_1) = f_2(S_2|S_1, D_1;\theta)$ as $f(S_1, D_1) = f(S_1, D_1;\theta)$. We conclude $p_2(S_2) = p_2(S_2;\theta)$ by $f(\mathsf{D}) = f(\mathsf{D};\theta)$ and $f(S_1, D_1, S_2) = f(S_1, D_1, S_2;\theta)$.

that

$$\begin{aligned}
v_t(S_t) &= \int V_t(S_t, \varepsilon_t)\, \mathrm{d}G(\varepsilon_t) \\
&= \int \max\{v_t^{0}(S_t) + \varepsilon_t^{0}, v_t^{1}(S_t) + \varepsilon_t^{1}\}\, \mathrm{d}G(\varepsilon_t) \\
&= v_t^{0}(S_t) + \int \max\{\varepsilon_t^{0}, v_t^{1}(S_t) - v_t^{0}(S_t) + \varepsilon_t^{1}\}\, \mathrm{d}G(\varepsilon_t) \\
&= v_t^{0}(S_t) + \int \max\{\varepsilon_t^{0}, \phi(p_t(S_t;\theta)) + \varepsilon_t^{1}\}\, \mathrm{d}G(\varepsilon_t) \\
&= v_t^{0}(S_t) + \psi(p_t(S_t;\theta)),
\end{aligned}$$

where ψ depends only on the CDF G of the utility shocks $\varepsilon_t = (\varepsilon_t^{0}, \varepsilon_t^{1})^{\intercal}$. Replacing v_t^{0} in the above display with its definition, $v_t^{0}(S_t) = \mu_t^{0}(S_t) + \delta_t \mathrm{E}_{t+1}^{0}[v_{t+1}(S_{t+1})|S_t]$, we have a recursive expression of the ex ante value function,

$$\begin{aligned}
v_t(S_t) &= \mu_t^{0}(S_t) + \delta_t \mathrm{E}_{t+1}^{0}[v_{t+1}(S_{t+1})|S_t] + \psi(p_t(S_t;\theta)), \quad t < T, \\
v_T(S_T) &= v_T^{0}(S_T) + \psi(p_T(S_T;\theta)).
\end{aligned} \tag{4.6}$$

Note that the ASVF $v_T^{0} \in \theta_T = (v_T^{0}, v_T^{1}, F_T^{0}, F_T^{1})$, and $p_T(S_T;\theta) = \tilde{G}(v_T^{1}(S_T) - v_T^{0}(S_T))$ is determined by θ_T. So that $v_T(S_T)$ is completely determined by θ_T, which is a part of the structural parameters.

Given the above results, equation (4.3c) is equivalent to the following system of equations,

$$\begin{aligned}
p_t(S_t) &= p_t(S_t;\theta) \\
&= \tilde{G}(\mu_t^{1/0}(S_t) + \delta_t \mathrm{E}_{t+1}^{1/0}[v_{t+1}(S_{t+1})|S_t]),\ t = 1, \dots, T-1,\ \forall S_t \in \mathcal{S}, \\
p_T(S_T) &= p_T(S_T;\theta) \\
&= \tilde{G}(v_T^{1}(S_T) - v_T^{0}(S_T)),\ \forall S_T \in \mathcal{S},
\end{aligned}$$

with

$$\begin{aligned}
v_t(S_t) &= \mu_t^{0}(S_t) + \delta_t \mathrm{E}_{t+1}^{0}[v_{t+1}(S_{t+1})|S_t] + \psi(p_t(S_t;\theta)),\ t = 2, \dots, T-1,\ \forall S_t \in \mathcal{S}, \\
v_T(S_T) &= v_T^{0}(S_T) + \psi(p_T(S_T;\theta)),\ \forall S_T \in \mathcal{S}.
\end{aligned}$$

In this system of equations, the *known* objects are the CCP $\{\, p_t(S_t) : t = 1, \dots, T \,\}$ and state transition matrices $\{\, F_2^{\mathtt{d}}, \dots, F_T^{\mathtt{d}} : \mathtt{d} = 0, 1 \,\}$ hidden in the conditional expectation operators $\mathrm{E}_{t+1}^{1/0}(\cdot|S_t)$ and $\mathrm{E}_{t+1}^{0}(\cdot|S_t)$; the *unknowns* are per period utility functions $\{\mu_1^{1/0}, \dots, \mu_{T-1}^{1/0}, \mu_2^{0}, \dots, \mu_{T-1}^{0}\}$, ex ante value functions $\{v_2, \dots, v_T\}$, the two ASVF v_T^{0} and v_T^{1}, and the discount factors $\{\delta_1, \dots, \delta_{T-1}\}$. One component of the structural parameters is identified iff the above system of equations has a unique solution for it.

Two remarks help to simplify the above system of equations. First, using the invertibility of the CDF $\tilde{G}$ and the identities $\{\, p_t(S_t;\theta) = p_t(S_t) : t = 1, \dots, T \,\}$, the above system has the same solutions as

$$\phi(p_t(S_t)) = \mu_t^{1/0}(S_t) + \delta_t \,\mathrm{E}_{t+1}^{1/0}[v_{t+1}(S_{t+1})|S_t],\ t = 1,\dots,T-1,\ \forall S_t \in \mathcal{S}, \tag{4.7a}$$

$$\phi(p_T(S_T)) = v_T^1(S_T) - v_T^0(S_T),\ \forall S_t \in \mathcal{S} \tag{4.7b}$$

$$v_t(S_t) = \mu_t^0(S_t) + \delta_t \,\mathrm{E}_{t+1}^0[v_{t+1}(S_{t+1})|S_t] + \psi(p_t(S_t)),\ t = 2,\dots,T-1,\ \forall S_t \in \mathcal{S}, \tag{4.7c}$$

$$v_T(S_T) = v_T^0(S_T) + \psi(p_T(S_T)),\ \forall S_T \in \mathcal{S}. \tag{4.7d}$$

Second, equation (4.7b) and (4.7d) simply state that v_T^0 and v_T^1 are uniquely determined by v_T. Hence, in order to solve for $\theta = (\theta_1,\dots,\theta_T)$ from equation (4.7), we can solve for $\theta_1,\dots,\theta_{T-1}$ and v_T. Moreover, the solutions of $(\theta_1,\dots,\theta_{T-1},v_T)$, which appears only in equation (4.7a) and (4.7c), do not depend on equation (4.7b) and (4.7d). So equation (4.7) has the same solution for $(\theta_1,\dots,\theta_{T-1},v_T)$ as the following system,

$$\begin{cases} \phi(p_t(s_t)) = \mu_t^{1/0}(s_t) + \delta_t \,\mathrm{E}_{t+1}^{1/0}[v_{t+1}(S_{t+1})|s_t], & t = 1,\dots,T-1, \forall S_t \in \mathcal{S}, \\ \psi(p_t(s_t)) = v_t(s_t) - \mu_t^0(s_t) - \delta_t \,\mathrm{E}_{t+1}^0[v_{t+1}(S_{t+1})|s_t], & t = 2,\dots,T-1, \forall S_t \in \mathcal{S}. \end{cases} \tag{ID}$$

The identification analysis below will be based on checking if there is a unique solution for (some parts of) $(\theta_1,\dots,\theta_{T-1},v_T)$ by solving $(\theta_1,\dots,\theta_{T-1},v_T)$ from equation (ID).

Equation (ID) has the feature that given the discount factors δ_t, it is linear in all the other unknowns; meanwhile, equation (ID) is linear in the discount factors, given the other unknowns. When the discount factors δ_t are known, the uniqueness of solution is very easy to check because equation (ID) is linear in all the other unknowns. More explicitly, using the notation of F_{t+1}^0 and $F_{t+1}^{1/0}$, equation (ID) can be written as follows,

$$\begin{cases} \phi(p_t) = \mu_t^{1/0} + \delta_t F_{t+1}^{1/0} v_{t+1}, & t = 1,\dots,T-1, \\ \psi(p_t) = v_t - \mu_t^0 - \delta_t F_{t+1}^0 v_{t+1}, & t = 2,\dots,T-1, \end{cases} \tag{ID'}$$

where

$$\phi(p_t) = (\phi(p_t(s_1)),\dots,\phi(p_t(s_{d_s})))^\intercal \qquad \text{and} \qquad \psi(p_t) = (\psi(p_t(s_1)),\dots,\psi(p_t(s_{d_s})))^\intercal.$$

In this sense, we claim that the identification of DPDC models is equivalent to identification of a linear GMM system, henceforth a familiar problem. The necessary condition for identification is that the number of equations should be greater than the number of unknowns (order condition). If the order condition fails, we shall consider restrictions that can eliminate certain number of unknowns, or add more equations by increasing the number of time periods T in panel data.

4.2 Identification of DPDC models by the linear GMM representation

A sequence of identification results will be derived by using the linear GMM representation of the DPDC model in equation (ID). The unknowns in equation (ID) are

$$\{ \mu_1^{1/0}, \ldots, \mu_{T-1}^{1/0}, \mu_2^{0}, \ldots, \mu_{T-1}^{0}, v_2, \ldots, v_T, \delta_1, \ldots, \delta_{T-1} \}.$$

Without restriction, the system of equations (ID) has $(2T-3)\cdot d_s$ equations with $(3T-4)\cdot d_s + (T-1)$ unknowns. This implies that the structural parameters are not identified even when all discount factors are known (removing $T-1$ unknowns). The non-identification of the DPDC model has long been known in the literature (Rust, 1994; Magnac and Thesmar, 2002). The problem of interests is what restrictions shall we use? We focus on the identification using the Exclusion Restriction stated in section 3, which is copied below for the convenience of reading.

Exclusion Restriction. *The vector of observable state variables S_t has two parts X_t and Z_t. Let $S_t = (X_t, Z_t)$, where $X_t \in \mathcal{X} = \{ x_1, \ldots, x_{d_x} \}$ and $Z_t \in \mathcal{Z} = \{ z_1, \ldots, z_{d_z} \}$. Assume that*

$$\mu_t^{1}(X_t, Z_t) = \mu_t^{1}(X_t) \qquad \textit{and} \qquad \mu_t^{0}(X_t, Z_t) = \mu_t^{0}(X_t)$$

for any (X_t, Z_t). For expositional simplicity, assume that $\mathcal{S} = \mathcal{X} \times \mathcal{Z}$, so that $d_s = d_x \cdot d_z$. In particular, let

$$\mathcal{S} = \mathrm{vec} \begin{bmatrix} (x_1, z_1) & (x_2, z_1) & \ldots & (x_{d_x}, z_1) \\ \vdots & \vdots & \vdots & \vdots \\ (x_1, z_{d_z}) & (x_2, z_{d_z}) & \ldots & (x_{d_x}, z_{d_z}) \end{bmatrix}.$$

For $d_x = d_z = 2$, this means $\mathcal{S} = \{ (x_1, z_1), (x_1, z_2), (x_2, z_1), (x_2, z_2) \}$.

Notice that under the above restriction, $\mu_t^{\mathsf{d}} = (\mu_t^{\mathsf{d}}(x_1), \ldots, \mu_t^{\mathsf{d}}(x_{d_x}))^{\intercal}$ is a d_x-dimensional vector. The above restriction is satisfied in our female labor force participation example, where $S_t = (husb_t, xp_t, edu, kid_t, xp_t^H, edu^H)$ with $X_t = (husb_t, xp_t, edu, kid_t)$ and $Z_t = (xp_t^H, edu^H)$. In general, given a set of state variables X_t that affect per period utilities, one searches for Z_t by looking for the variables that affect X_{t+1} but not affect per period utilities given X_t. For example, in Rust's (1987) bus engine replacement application, X_t is the mileage of the bus. Then Z_t could be characteristics of the bus' route, which will affect the bus' mileage in the next period, but not the current maintenance cost given the mileage.

We have shown the identification power of the Exclusion Restriction in the previous section. Below, we provide more general identification results. It is instructive to consider first the stationary dynamic programming problem with known discount factor. By "stationary dynamic programming problem", we mean that the decision horizon T_* is infinite, and the per period utility functions, the discount factors and the state transition distributions are time invariant. When the agent's dynamic programming problem is stationary, and the Exclusion Restriction holds, (ID) becomes

$$\begin{cases} \phi(p(X,Z)) = \mu^{1/0}(X) + \delta\, \mathrm{E}^{1/0}[v(X',Z')|X,Z], & \forall (X,Z) \in \mathcal{S}, \\ \psi(p(X,Z)) = v(X,Z) - \mu^{0}(X) - \delta\, \mathrm{E}^{0}[v(X',Z')|X,Z], & \forall (X,Z) \in \mathcal{S}, \end{cases} \tag{4.8}$$

or equivalently

$$\begin{cases} \phi(p) = \mu^{1/0} \otimes 1_{d_z} + \delta F^{1/0} v, \\ \psi(p) = v - \mu^{0} \otimes 1_{d_z} - \delta F^{0} v. \end{cases} \tag{4.9}$$

For $d_x = d_z = 2$,

$$\mu^{1/0} \otimes 1_{d_z} = (\mu^{1/0}(x_1), \mu^{1/0}(x_1), \mu^{1/0}(x_2), \mu^{1/0}(x_2))^{\intercal},$$
$$\mu^{0} \otimes 1_{d_z} = (\mu^{0}(x_1), \mu^{0}(x_1), \mu^{0}(x_2), \mu^{0}(x_2))^{\intercal}.$$

We will need to recover $\mu^{1/0}$ and μ^0 from $\mu^{1/0} \otimes 1_{d_z}$ and $\mu^0 \otimes 1_{d_z}$, respectively. To this end, define the $d_x \times d_s$ matrix W by

$$W \equiv I_{d_x} \otimes \left(\frac{1}{d_z} \cdot 1_{d_z} \right)^{\intercal}. \tag{4.10}$$

So that $\mu^{1/0} = W(\mu^{1/0} \otimes 1_{d_z})$ and $\mu^0 = W(\mu^0 \otimes 1_{d_z})$.

The linear system of equations (4.8) or (4.9) has $2d_s$ equations with $2d_x + d_s$ unknowns. So if $d_s \geq 2d_x$, we may be able to identify the structural parameters. In particular, when $d_s = d_x \cdot d_z$, the order condition $d_s \geq 2d_x$ would be satisfied as long as $d_z \geq 2$. The identification of the per period utility functions $(\mu^{1/0}, \mu^0)$ will be based on solving $(\mu^{1/0}, \mu^0)$ explicitly from equation (4.8) or (4.9). We solve equation (4.8) by following the steps below, which are similar to the steps in subsection 3.2.

Step 1: Eliminate the per period utility functions $\mu^{1/0}$ and μ^0 from equation (4.8) by considering the followings differences,

$$\phi(p(x_i, z_j)) - \phi(p(x_i, z_{j+1})) \qquad \text{and} \qquad \psi(p(x_i, z_j)) - \psi(p(x_i, z_{j+1})),$$

for $i = 1, \ldots, d_x$ and $j = 1, \ldots, d_z - 1$. For expositional simplicity, denote

$$\phi(i,j) = \phi(p(x_i, z_j)) \qquad \text{and} \qquad \psi(i,j) = \psi(p(x_i, z_j)).$$

It follows from equation (4.8) that

$$\begin{cases} \phi(i,j) - \phi(i,j+1) = \delta\, \mathrm{E}^{1/0}[v(X',Z')|x_i, z_j] - \delta\, \mathrm{E}^{1/0}[v(X',Z')|x_i, z_{j+1}], \\ \psi(i,j) - \psi(i,j+1) = v(x_i, z_j) - v(x_i, z_{j+1}) \\ \qquad\qquad\qquad\qquad - \delta\, \mathrm{E}^{0}[v(X',Z')|x_i, z_j] + \delta\, \mathrm{E}^{0}[v(X',Z')|x_i, z_{j+1}], \end{cases}$$

for all $i = 1, \ldots, d_x$ and $j = 1, \ldots, d_z - 1$. We have a simpler representation of the above system of equations using equation (4.9). Define the $[d_x \cdot (d_z - 1)] \times d_s$ matrix M by

$$M \equiv I_{d_x} \otimes \begin{bmatrix} 1 & -1 & & \\ & \ddots & \ddots & \\ & & 1 & -1 \end{bmatrix}_{(d_z-1)\times d_z}.$$

Multiplying both sides of equation (4.9) with matrix M, we have

$$\begin{cases} M\phi(p) = M(\mu^{1/0} \otimes 1_{d_z}) + \delta M F^{1/0} v, \\ M\psi(p) = Mv - M(\mu^0 \otimes 1_{d_z}) - \delta M F^0 v. \end{cases}$$

Because $M(\mu^{1/0} \otimes 1_{d_z}) = M(\mu^0 \otimes 1_{d_z}) = 0_{d_s}$[8], we have the following linear system of equations,

$$Av = b, \tag{4.11}$$

where A is the $[2 \cdot d_x \cdot (d_z - 1)] \times d_s$ matrix, and b is the $[2 \cdot d_x \cdot (d_z - 1)]$-dimensional vector defined by

$$A = \begin{bmatrix} \delta M F^{1/0} \\ M(I - \delta F^0) \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} M\phi(p) \\ M\psi(p) \end{bmatrix}. \tag{4.12}$$

Step 2: Solve for the ex ante value value functions v in equation (4.11). Let A^+ be the Moore-Penrose pseudoinverse of matrix A, then the d_s-dimensional vector

$$v^+ \equiv A^+ b$$

solves equation (4.11) by the definition of pseudoinverse. If $\operatorname{rank} A = d_s - 1$, we know from lemma A.1 that the solution set for equation (4.11) is

$$\left\{ v^+ + c \cdot 1_{d_s} : c \in \mathbb{R} \right\},$$

where $c \cdot 1_{d_s} = (c, \ldots, c)^\intercal$ is a d_s-dimensional vector of constant c.

Step 3: Identify the per period utility functions $\mu^{1/0}$ and μ^0. Suppose $\operatorname{rank} A = d_s - 1$, and let $v = v^+ + c \cdot 1_{d_s}$ be an arbitrary solution of equation (4.11). Associated with the solution $v = v^+ + c \cdot 1_{d_s}$, we have the following from equation (4.9),

$$\begin{aligned} \mu^{1/0} \otimes 1_{d_z} &= \phi(p) - \delta F^{1/0}(v^+ + c \cdot 1_{d_s}) \\ &= \phi(p) - \delta F^{1/0} v^+, \end{aligned}$$

[8] Take $d_x = d_z = 2$ for example, we have $\mu^0 \otimes 1_{d_z} = (\mu^0(x_1), \mu^0(x_1), \mu^0(x_2), \mu^0(x_2))^\intercal$, henceforth $M(\mu^0 \otimes 1_{d_z}) = (\mu^0(x_1) - \mu^0(x_1), \mu^0(x_2) - \mu^0(x_2))^\intercal = 0_2$.

and

$$\begin{aligned}\mu^0 \otimes 1_{d_z} &= (v^+ + c \cdot 1_{d_s}) - \delta F^0(v^+ + c \cdot 1_{d_s}) - \psi(p) \\ &= (I - \delta F^0)v^+ - \psi(p) + (c - \delta c) \cdot 1_{d_s}.\end{aligned}$$

We have the above equations because $F^{1/0} 1_{d_s} = 0_{d_s}$ and $F^0 1_{d_s} = 1_{d_s}$. Using the matrix W of equation (4.10), we have

$$\mu^{1/0} = W(\mu^{1/0} \otimes 1_{d_z}) = W(\phi(p) - \delta F^{1/0} v^+).$$

Hence the per period utilities difference $\mu^{1/0}$ is identified. To get rid of the unknown constant c in $\mu^0 \otimes 1_{d_z}$, we use the normalization $\mu^0(x_1) = 0$ of Assumption 4.(iii). For this purpose, Define

$$L \equiv \left[\begin{array}{c|ccc} 0 & 0 & \dots & 0 \\ \hline -1 & 1 & & \\ \vdots & & \ddots & \\ -1 & & & 1 \end{array}\right]_{d_s \times d_s}. \tag{4.13}$$

We then have

$$\mu^0 = WL(\mu^0 \otimes 1_{d_z}) = WL[(I - \delta F^0)v^+ - \psi(p)].$$

Hence the per period utility function μ^0 is identified with the normalization.

Proposition 1 (Identification with the Exclusion Restriction, known discount factors and stationarity)**.** *In addition to Assumptions 1-4, suppose the Exclusion Restriction holds, the discount factors are known and that the agent's dynamic programming problem is stationary. Let the matrix A and the vector b be defined by equation* (4.12)*. If $T \geq 2$ and* $\operatorname{rank} A = d_s - 1$*, the per period utility functions $\mu^{1/0}$ and μ^0 are identified. Moreover, we have*

$$\begin{aligned}\mu^{1/0} &= W(\phi(p) - \delta F^{1/0} A^+ b), \\ \mu^0 &= WL[(I - \delta F^0)A^+ b - \psi(p)].\end{aligned}$$

We now move to the identification of general dynamic programming discrete choice models using the Exclusion Restriction. It follows from the Exclusion Restriction that equation (ID) becomes

$$\begin{cases} \phi(p_t(X_t, Z_t)) = \mu_t^{1/0}(X_t) + \delta_t \operatorname{E}_{t+1}^{1/0}[v_{t+1}(S_{t+1})|X_t, Z_t], & t = 1, \dots, T-1, \\ \psi(p_t(X_t, Z_t)) = v_t(X_t, Z_t) - \mu_t^0(X_t) - \delta_t \operatorname{E}_{t+1}^0[v_{t+1}(S_{t+1})|X_t, Z_t], & t = 2, \dots, T-1, \end{cases} \tag{4.14}$$

for all $(X_t, Z_t) \in \mathcal{S}$. There are $(2T-3) \cdot d_x + (T-1) \cdot d_s + (T-1)$ unknowns and $(2T-3) \cdot d_s$ equations. When the discount factors are known (removing $T-1$ unknowns), $d_z \geq 3$ and $T \geq 3$, we have more equations than unknowns. It should be remarked that when $T < 3$, the order condition always fails regardless of the value of d_z. When $T = 2$, we have only

$$\phi(p_1(X_1, Z_1)) = \mu_1^{1/0}(X_1) + \delta_1 \operatorname{E}_2^{1/0}[v_2(S_2)|X_1, Z_1].$$

Note that in general we do not have

$$\phi(p_2(X_2, Z_2)) = \mu_2^{1/0}(X_2) + \delta_2 \,\mathrm{E}_3^{1/0}[v_3(S_3)|X_2, Z_2],$$
$$\psi(p_2(X_2, Z_2)) = v_2(X_2, Z_2) - \mu_2^{0}(X_2) - \delta_2 \,\mathrm{E}_3^{0}[v_3(S_3)|X_2, Z_2],$$

because the state transition matrices F_3^0 and F_3^1 are unknown given data (D_1, S_1, D_2, S_2). However, if one assumes that the state transition matrices are time invariant as we did in section 3, we can use the above two equations.

We first focus on the identification with known discount factors. Let

$$\bar{v}_{t+1}(S_{t+1}) = \delta_t v_{t+1}(S_{t+1})$$

be the discounted ex ante value function. For each period $t = 2, \ldots, T-1$, we will show how to solve the unknowns $(\mu_{t-1}^{1/0}, \mu_t^{1/0}, \mu_t^0, v_t, v_{t+1})$ from the following part of equation (4.14),

$$\begin{cases} \phi(p_{t-1}(X_{t-1}, Z_{t-1})) = \mu_{t-1}^{1/0}(X_{t-1}) + \delta_{t-1} \,\mathrm{E}_t^{1/0}[v_t(S_t)|X_{t-1}, Z_{t-1}], & \forall (X_{t-1}, Z_{t-1}) \in \mathcal{S}, \\ \phi(p_t(X_t, Z_t)) = \mu_t^{1/0}(X_t) + \mathrm{E}_{t+1}^{1/0}[\bar{v}_{t+1}(S_{t+1})|X_t, Z_t], & \forall (X_t, Z_t) \in \mathcal{S}, \\ \psi(p_t(X_t, Z_t)) = v_t(X_t, Z_t) - \mu_t^0(X_t) - \mathrm{E}_{t+1}^0[\bar{v}_{t+1}(S_{t+1})|X_t, Z_t], & \forall (X_t, Z_t) \in \mathcal{S}, \end{cases} \tag{4.15}$$

or equivalently

$$\begin{cases} \phi(p_{t-1}) = \mu_{t-1}^{1/0} \otimes 1_{d_z} + \delta_{t-1} F_t^{1/0} v_t, \\ \phi(p_t) = \mu_t^{1/0} \otimes 1_{d_z} + F_{t+1}^{1/0} \bar{v}_{t+1}, \\ \psi(p_t) = v_t - \mu_t^0 - F_{t+1}^0 \bar{v}_{t+1}. \end{cases} \tag{4.16}$$

Ranging period t from 2 to $T-1$, all unknowns $\mu_1^{1/0}, \ldots, \mu_{T-1}^{1/0}, \mu_2^0, \ldots, \mu_{T-1}^0, v_2, \ldots, v_T$ will then be solved. For each t, we solve equation (4.15) by following the steps below, which are similar to the steps in subsection 3.2.

Step 1: Eliminate the unknown per period utility functions $\mu_{t-1}^{1/0}$, $\mu_t^{1/0}$ and μ_t^0 from equation (4.15) by considering the following differences,

$$\begin{cases} \phi_{t-1}(i,j) - \phi_{t-1}(i,j+1) = \delta_{t-1} \,\mathrm{E}_t^{1/0}[v_t(S_t)|x_i, z_j] - \delta_{t-1} \,\mathrm{E}_t^{1/0}[v_t(S_t)|x_i, z_{j+1}], \\ \phi_t(i,j) - \phi_t(i,j+1) = \mathrm{E}_{t+1}^{1/0}[\bar{v}_{t+1}(S_{t+1})|x_i, z_j] - \mathrm{E}_{t+1}^{1/0}[\bar{v}_{t+1}(S_{t+1})|x_i, z_{j+1}], \\ \psi_t(i,j) - \psi_t(i,j+1) = v_t(x_i, z_j) - v_t(x_i, z_{j+1}) \\ \qquad - \mathrm{E}_{t+1}^0[\bar{v}_{t+1}(S_{t+1})|x_i, z_j] + \mathrm{E}_{t+1}^0[\bar{v}_{t+1}(S_{t+1})|x_i, z_{j+1}], \end{cases} \tag{4.17}$$

for all $i = 1, \ldots, d_x$ and $j = 1, \ldots, d_z - 1$. Here,

$$\phi_t(i,j) = \phi(p_t(X_t = x_i, Z_t = z_j)) \qquad \text{and} \qquad \psi_t(i,j) = \psi(p_t(X_t = x_i, Z_t = z_j)).$$

Equation (4.17) can be organized as the following linear system of equations,

$$A_t \begin{bmatrix} v_t \\ \bar{v}_{t+1} \end{bmatrix} = b_t, \tag{4.18}$$

where A_t is a $[3 \cdot d_x \cdot (d_z - 1)] \times (2d_s)$ matrix, and b_t is a $[3 \cdot d_x \cdot (d_z - 1)]$-dimensional vector:

$$A_t \equiv \begin{bmatrix} MF_t^{1/0} & 0 \\ 0 & MF_{t+1}^{1/0} \\ M & -MF_{t+1}^{0} \end{bmatrix} \quad \text{and} \quad b_t \equiv \begin{bmatrix} \delta_{t-1}^{-1} M\phi(p_{t-1}) \\ M\phi(p_t) \\ M\psi(p_t) \end{bmatrix}. \tag{4.19}$$

Step 2: Solve v_t and $\bar{v}_{t+1}$ from equation (4.18). Let A_t^+ be the Moore-Penrose pseudoinverse of matrix A_t, then

$$\begin{bmatrix} v_t^+ \\ \bar{v}_{t+1}^+ \end{bmatrix} = A_t^+ b_t$$

solves equation (4.18) by the definition of pseudoinverse. Split the matrix A_t^+ into two parts:

$$A_t^+ = \begin{bmatrix} A_{t,u}^+ \\ A_{t,l}^+ \end{bmatrix}, \tag{4.20}$$

where $A_{t,u}^+$ and $A_{t,l}^+$ are the $d_s \times [3 \cdot d_x \cdot (d_z - 1)]$ matrices formed by the first and last d_s rows of matrix A_t^+, respectively. Then

$$\begin{bmatrix} v_t^+ \\ \bar{v}_{t+1}^+ \end{bmatrix} = \begin{bmatrix} A_{t,u}^+ b_t \\ A_{t,l}^+ b_t \end{bmatrix},$$

or more explicitly,

$$v_t^+ = A_{t,u}^+ b_t \quad \text{and} \quad \bar{v}_{t+1}^+ = A_{t,l}^+ b_t. \tag{4.21}$$

If rank $A_t = 2 \cdot (d_s - 1)$, we know from lemma A.2 that the solution set of equation (4.18) is that

$$\left\{ \begin{bmatrix} v_t^+ + c_t \times 1_{d_s} \\ \bar{v}_{t+1}^+ + c_{t+1} \times 1_{d_s} \end{bmatrix} : c_t, c_{t+1} \in \mathbb{R} \right\}. \tag{4.22}$$

Step 3: Identify the per period utility functions $\mu_{t-1}^{1/0}$, $\mu_t^{1/0}$ and μ_t^0. Suppose rank $A_t = 2 \cdot (d_s - 1)$, and let $v_t = v_t^+ + c_t \cdot 1_{d_s}$ and $\bar{v}_{t+1} = \bar{v}_{t+1}^+ + c_{t+1} \cdot 1_{d_s}$ be arbitrary solutions of equation (4.18). Then associated with $v_t = v_t^+ + c_t \cdot 1_{d_s}$ and $\bar{v}_{t+1} = \bar{v}_{t+1}^+ + c_{t+1} \cdot 1_{d_s}$, we have the following from equation (4.16),

$$\begin{aligned} \mu_t^{1/0} \otimes 1_{d_z} &= \phi(p_t) - F_{t+1}^{1/0} \bar{v}_{t+1}^+, \\ \mu_{t-1}^{1/0} \otimes 1_{d_z} &= \phi(p_{t-1}) - \delta_{t-1} F_t^{1/0} v_t^+, \\ \mu_t^0 \otimes 1_{d_z} &= v_t^+ - F_{t+1}^0 \bar{v}_{t+1}^+ - \psi(p_t) + (c_t - c_{t+1}) \cdot 1_{d_s}. \end{aligned}$$

Using the matrix W of equation (4.10), we have

$$\mu_t^{1/0} = W(\phi(p_t) - F_{t+1}^{1/0}\bar{v}_{t+1}^{+}),$$
$$\mu_{t-1}^{1/0} = W(\phi(p_{t-1}) - \delta_{t-1}F_t^{1/0}v_t^{+}).$$

Hence we claim that $\mu_t^{1/0}$ and $\mu_{t-1}^{1/0}$ are identified. With the normalization $\mu_t^{0}(x_1) = 0$ of Assumption 4.(iii), we get rid of the constant $c_t - c_{t+1}$ in $\mu_t^0 \otimes 1_{d_z}$. We then have

$$\mu_t^0 = WL(v_t^+ - F_{t+1}^0\bar{v}_{t+1}^+ - \psi(p_t)).$$

Hence the per period utility function μ_t^0 is identified with the normalization.

Up to now, we have show that for each $t = 2, \ldots, T-1$, if $\operatorname{rank} A_t = 2 \cdot (d_s - 1)$, we have unique solution of $\mu_t^{1/0}, \mu_t^0$ and $\mu_{t-1}^{1/0}$ from equation (4.14). Using the steps similar to the above, we derive the closed-form solution of $(\mu_1^{1/0}, \ldots, \mu_{T-1}^{1/0})$ and $(\mu_2^0, \ldots, \mu_{T-1}^0)$ directly from equation (4.14).

Proposition 2 (Identification with the Exclusion Restriction, known discount factors and $T \geq 3$)**.** *In addition to Assumptions 1-4, suppose the Exclusion Restriction holds, the discount factors are known and $T \geq 3$. For $t = 2, \ldots, T-1$, let the matrix A_t and the vector b_t be defined by equation (4.19). If $\operatorname{rank} A_t = 2 \cdot (d_s - 1)$, then the per period utility functions $\mu_t^{1/0}$, μ_t^0 and $\mu_{t-1}^{1/0}$ are identified. Moreover, we have*

$$\begin{bmatrix} \mu_1^{1/0} \\ \vdots \\ \mu_{T-1}^{1/0} \end{bmatrix} = (I_{T-1} \otimes W) \left(\begin{bmatrix} \phi(p_1) \\ \vdots \\ \phi(p_{T-1}) \end{bmatrix} - (\tilde{\Lambda}^{-1} \otimes I_{d_s}) F_{2:T}^{1/0} A_{1:T}^{+} b_{1:T} \right),$$
$$\begin{bmatrix} \mu_2^{0} \\ \vdots \\ \mu_{T-1}^{0} \end{bmatrix} = [I_{T-2} \otimes (WL)] \left((\Lambda^{-1} \otimes I_{d_s}) \check{F}_{3:T}^{0} A_{1:T}^{+} b_{1:T} - \begin{bmatrix} \psi(p_2) \\ \vdots \\ \psi(p_{T-1}) \end{bmatrix} \right).$$

where

$$A_{1:T} = \begin{bmatrix} (I_{T-1} \otimes M)\, F_{2:T}^{1/0} \\ (I_{T-2} \otimes M)\, \check{F}_{3:T}^{0} \end{bmatrix} \qquad \textit{and} \qquad b_{1:T} = \begin{bmatrix} (I_{T-1} \otimes M)(\tilde{\Lambda} \otimes I_{T-1}) \begin{bmatrix} \phi(p_1) \\ \vdots \\ \phi(p_{T-1}) \end{bmatrix} \\ \\ (I_{T-2} \otimes M)(\Lambda \otimes I_{T-2}) \begin{bmatrix} \psi(p_2) \\ \vdots \\ \psi(p_{T-1}) \end{bmatrix} \end{bmatrix},$$

and

$$\varLambda = \operatorname{diag}\left(\delta_1, \prod_{r=1}^{2}\delta_r, \cdots, \prod_{r=1}^{T-2}\delta_r\right) \qquad \textit{and} \qquad \tilde{\varLambda} = \operatorname{diag}\left(1, \delta_1, \prod_{r=1}^{2}\delta_r, \cdots, \prod_{r=1}^{T-2}\delta_r\right).$$

Proof. See Appendix A. □

When the panel data have the number of time periods greater than 4, we can also identify the discount factors using the strategy of subsection 3.3. Applying Proposition 2, we have two formulas of $\mu_t^{1/0}$:

$$\mu_t^{1/0} = W(\phi(p_t) - F_{t+1}^{1/0}A_{t,l}^{+}b_t)$$
$$\mu_t^{1/0} = W(\phi(p_t) - \delta_t F_{t+1}^{1/0}A_{t+1,u}^{+}b_{t+1}),$$

from which we have an equation of the discount factors δ_{t-1} and δ_t, which is hidden in b_t:

$$WF_{t+1}^{1/0}A_{t,l}^{+}b_t - \delta_t WF_{t+1}^{1/0}A_{t+1,u}^{+}b_{t+1} = 0.$$

We derive the explicit solutions of (δ_{t-1}, δ_t) below. Define

$$b_{t,u} = \operatorname{vec}\begin{bmatrix} M\phi(p_{t-1}) & 0 & 0 \end{bmatrix} \qquad \text{and} \qquad b_{t,l} = \operatorname{vec}\begin{bmatrix} 0 & M\phi(p_t) & M\psi(p_t) \end{bmatrix},$$

so that the vector b_t defined in equation (4.19) equals

$$b_t = \delta_{t-1}^{-1}b_{t,u} + b_{t,l}.$$

Let

$$H_{t,l} = WF_{t+1}^{1/0}A_{t,l}^{+} \qquad \text{and} \qquad H_{t+1,u} = WF_{t+1}^{1/0}A_{t+1,u}^{+}.$$

Then the equation

$$WF_{t+1}^{1/0}A_{t,l}^{+}b_t - \delta_t WF_{t+1}^{1/0}A_{t+1,u}^{+}b_{t+1} = 0$$

is written as follows,

$$\delta_{t-1}^{-1}H_{t,l}b_{t,u} - \delta_t H_{t+1,u}b_{t+1,l} + (H_{t,l}b_{t,l} - H_{t+1,u}b_{t+1,u}) = 0,$$

or

$$\begin{bmatrix} H_{t,l}b_{t,u} & H_{t+1,u}b_{t+1,l} \end{bmatrix}\begin{bmatrix} \delta_{t-1}^{-1} \\ -\delta_t \end{bmatrix} = (H_{t,l}b_{t,l} - H_{t+1,u}b_{t+1,u}).$$

Denote

$$\tilde{R}_t = \begin{bmatrix} H_{t,l}b_{t,u} & H_{t+1,u}b_{t+1,l} \end{bmatrix}. \tag{4.23}$$

If rank $\tilde{R}_t = 2$, we have the unique solution of $(\delta_{t-1}^{-1}, -\delta_t)^{\intercal}$:

$$\begin{bmatrix} \delta_{t-1}^{-1} \\ -\delta_t \end{bmatrix} = \tilde{R}_t^{+}(H_{t,l}b_{t,l} - H_{t+1,u}b_{t+1,u})$$

Proposition 3 (Identification of discount factors with the Exclusion Restriction and $T \geq 4$)**.** *Suppose the conditions of Proposition 2 hold with $T \geq 4$ (more than 4 consecutive periods data). If the matrices $\tilde{R}_t$, $t = 2, \ldots, T-1$, defined in equation (4.23) are of full rank, the discount factors $\delta_1, \ldots, \delta_{T-2}$ are identified. Note that the discount factor δ_{T-1} is not identified.*

In practice, the excluded state variable Z_t could be time invariant. For example, in Rust's (1987) bus engine replacement application, the excluded state variable can be a permanent route characteristic for the bus. Recently, Fang and Wang (2015) use excluded variables to identify hyperbolic discounting parameters. In their application of mammography decisions, the excluded variables include categorical variables like education and race, which do not change over time. When the excluded variable Z_t is time invariant, the conclusion of Proposition 2 and 3 hold under a different rank condition.

Proposition 4 (Identification with permanent excluded state variables)**.** *In addition to Assumptions 1-4 and the Exclusion Restriction, suppose that the excluded state variable Z_t is time invariant. For $t = 2, \ldots, T-1$, let the matrix A_t and the vector b_t be defined by equation* (4.19)*.*

(i) *If the discount factors are known, and* rank $A_t = 2d_s - d_z - 1$ *for $t = 2, \ldots, T-1$ with $T \geq 3$, the per period utility functions $\mu_1^{1/0}, \ldots, \mu_{T-1}^{1/0}$ and $\mu_2^0, \ldots, \mu_{T-1}^0$ are identified and satisfy the formulas in Proposition 2.*

(ii) *In addition to the conditions of part* (i)*, if $T \geq 4$ and the matrices $\tilde{R}_t$, $t = 2, \ldots, T-1$, defined in equation* (4.23) *are of full rank, the discount factors $\delta_1, \ldots, \delta_{T-2}$ are also identified.*

Proof. See Appendix A. □

When there are no excluded variables Z_t, an alternative way to identify per period utility functions is to assume that the per period utility functions are time invariant but the state transition matrices are time varying. Then time itself is an excluded variable. Suppose that we have at least four consecutive periods observation, that is $T \geq 4$. Assume that the per period utility functions are time invariant, $\mu_t^{1/0} = \mu^{1/0}$ and $\mu_t^0 = \mu^0$. Then for each $t \geq 3$, equation (ID') becomes the following

$$\begin{cases} \phi(p_t) = \mu^{1/0} + F_{t+1}^{1/0}\bar{v}_{t+1} \\ \phi(p_{t-1}) = \mu^{1/0} + \delta_{t-1}F_t^{1/0}v_t \\ \phi(p_{t-2}) = \mu^{1/0} + \delta_{t-2}F_{t-1}^{1/0}v_{t-1} \\ \psi(p_t) = v_t - \mu^0 - F_{t+1}^0\bar{v}_{t+1} \\ \psi(p_{t-1}) = v_{t-1} - \mu^0 - \delta_{t-1}F_t^0 v_t \end{cases} . \tag{4.24}$$

To solve the per period utility functions $\mu^{1/0}$ and μ^{0} from equation (4.24), we again first solve the ex ante value functions $v_{t-1}, v_t, \bar{v}_{t+1}$, then solve $\mu^{1/0}$ and μ^{0} given the solutions of the ex ante value functions. To solve $v_{t-1}, v_t, \bar{v}_{t+1}$, we consider the difference $\phi(p_t) - \phi(p_{t-1})$, $\phi(p_{t-1}) - \phi(p_{t-2})$ and $\psi(p_t) - \psi(p_{t-1})$. We have

$$A_t \begin{bmatrix} \bar{v}_{t+1} \\ v_t \\ v_{t-1} \end{bmatrix} = b_t,$$

where A_t is the $(3d_s) \times (3d_s)$ matrix, and b_t is the $3d_s$-dimensional vector defined by

$$A_t = \begin{bmatrix} F^{1/0}_{t+1} & -\delta_{t-1} F^{1/0}_{t} & 0 \\ 0 & \delta_{t-1} F^{1/0}_{t} & -\delta_{t-2} F^{1/0}_{t-1} \\ -F^{0}_{t+1} & I + \delta_{t-1} F^{0}_{t} & -I \end{bmatrix} \quad \text{and} \quad b_t = \begin{bmatrix} \phi(p_t) - \phi(p_{t-1}) \\ \phi(p_{t-1}) - \phi(p_{t-2}) \\ \psi(p_t) - \psi(p_{t-1}) \end{bmatrix}. \tag{4.25}$$

Proposition 5 (Identification with time-invariant per period utilities and time-varying transition matrices)**.** *In addition to Assumptions 1-4, suppose that the discount factors are known, $T \geq 4$, and that the per period utility functions are time invariant. For $t = 2, \dots, T-1$, let the matrix A_t and the vector b_t be defined by equation* (4.25)*. If $\operatorname{rank} A_t = 3d_s - 2$, the per period utility functions $\mu^{1/0}$ and μ^{0} are identified, and*

$$\begin{aligned} \mu^{1/0} &= \frac{1}{3}\left\{ \phi(p_t) + \phi(p_{t-1}) + \phi(p_{t-2}) - \left[\; F^{1/0}_{t+1} \; \middle| \; \delta_{t-1} F^{1/0}_{t} \; \middle| \; \delta_{t-2} F^{1/0}_{t-1} \; \right] A_t^{+} b_t \right\}, \\ \mu^{0} &= \frac{1}{2} L \left\{ \left[\; -F^{0}_{t+1} \; \middle| \; (I_{d_s} - \delta_{t-1} F^{0}_{t}) \; \middle| \; I_{d_s} \; \right] A_t^{+} b_t - \psi(p_t) - \psi(p_{t-1}) \right\}. \end{aligned}$$

Proof. See Appendix A. □

Remark 4 (Extension to multinomial choices). The identification arguments can be extended to multinomial choices by using the general Hotz-Miller inversion formulas (Hotz and Miller, 1993). Suppose the choice set $\{ \mathtt{0}, \mathtt{1}, \dots \mathtt{J} \}$ contains $J+1$ alternatives. By the Hotz-Miller's formula, there exists $\{ \phi^{\mathtt{j}} : \mathtt{j} = \mathtt{1}, \dots, \mathtt{J} \}$ and ψ such that

$$\begin{cases} v^{\mathtt{j}}_t(S_t) - v^{\mathtt{0}}_t(S_t) = \phi^{\mathtt{j}}(p_t(S_t)) \\ v_t(S_t) - v^{\mathtt{0}}_t(S_t) = \psi(p_t(S_t)) \end{cases},$$

where $p_t(S_t) = (\mathrm{P}(D_t = \mathtt{1}|S_t), \dots, \mathrm{P}(D_t = \mathtt{J}|S_t))^{\intercal}$. Equation (ID) becomes

$$\begin{cases} \phi^{\mathtt{1}}(p_t(S_t)) = \mu^{\mathtt{1/0}}_t(S_t) + \delta_t \, \mathrm{E}^{\mathtt{1/0}}_{t+1}[v_{t+1}(S_{t+1})|S_t], & t = 1, \dots, T-1, \forall S_t \in \mathcal{S}, \\ \qquad \vdots & \qquad \vdots \\ \phi^{\mathtt{J}}(p_t(S_t)) = \mu^{\mathtt{J/0}}_t(S_t) + \delta_t \, \mathrm{E}^{\mathtt{J/0}}_{t+1}[v_{t+1}(S_{t+1})|S_t], & t = 1, \dots, T-1, \forall S_t \in \mathcal{S}, \\ \psi(p_t(S_t)) = v_t(S_t) - \mu^{\mathtt{0}}_t(S_t) - \delta_t \, \mathrm{E}^{\mathtt{0}}_{t+1}[v_{t+1}(S_{t+1})|S_t], & t = 2, \dots, T-1, \forall S_t \in \mathcal{S}. \end{cases}$$

Each alternative j contributes $d_s\cdot(T-1)$ equations (associated with $\{\,\phi^{\mathsf{j}}(p_t(S_t)) : t=1,\dots,T-1$, for all $S_t\in\mathcal{S}\,\}$); meanwhile the alternative j brings $d_s\cdot(T-1)$ additional unknowns $\{\,\mu_t^{\mathsf{j}/\mathsf{0}} : t=1,\dots,T-1\,\}$. So the degree of underidentification does not change as we include more alternatives. However, in the presence of the Exclusion Restriction, we have

$$\begin{cases} \phi^{\mathsf{1}}(p_t(S_t)) = \mu_t^{\mathsf{1}/\mathsf{0}}(X_t) + \delta_t \operatorname{E}_{t+1}^{\mathsf{1}/\mathsf{0}}[v_{t+1}(S_{t+1})|S_t], & t=1,\dots,T-1, \forall S_t\in\mathcal{S},\\ \qquad\vdots & \vdots \\ \phi^{\mathsf{J}}(p_t(S_t)) = \mu_t^{\mathsf{J}/\mathsf{0}}(X_t) + \delta_t \operatorname{E}_{t+1}^{\mathsf{J}/\mathsf{0}}[v_{t+1}(S_{t+1})|S_t], & t=1,\dots,T-1, \forall S_t\in\mathcal{S},\\ \psi(p_t(S_t)) = v_t(S_t) - \mu_t^{\mathsf{0}}(X_t) - \delta_t \operatorname{E}_{t+1}^{\mathsf{0}}[v_{t+1}(S_{t+1})|S_t], & t=2,\dots,T-1, \forall S_t\in\mathcal{S}. \end{cases}$$

Each alternative j still contributes $d_s\cdot(T-1)$ new equations; meanwhile the alternative j brings only $d_x\cdot(T-1)$ additional unknowns $\{\,\mu_t^{\mathsf{j}/\mathsf{0}} : t=1,\dots,T-1\,\}$, because $\mu_t^{\mathsf{j}/\mathsf{0}}$ is now d_x-dimensional vector. So more alternatives provide more information about the structural parameters. The exact identification results for multinomial choices are slightly different from the above propositions, but the general idea is similar.

Remark 5 (Rank conditions). The rank conditions in the above propositions are clearly important. This remark is to show that to satisfy the rank conditions, it is necessary that the choice can change the transition matrix, and that the the excluded variable Z_t can affect the difference between the state transition probabilities under the two alternatives given the state variable X_t that affects the per period utility functions. To be specific, consider the rank condition in Proposition 1. We have

$$\begin{aligned} \operatorname{rank} A &= \operatorname{rank}\left(\begin{bmatrix} \delta M F^{\mathsf{1}/\mathsf{0}} \\ M(I-\delta F^{\mathsf{0}}) \end{bmatrix}\right) \\ &= \operatorname{rank}(M(I-\delta F^{\mathsf{0}})) + \operatorname{rank}\left[(I_{d_s} - P_{[M(I-\delta F^{\mathsf{0}})]^{\intercal}})(\delta M F^{\mathsf{1}/\mathsf{0}})^{\intercal}\right] \\ &= d_s - d_x + \operatorname{rank}\left[(I_{d_s} - P_{[M(I-\delta F^{\mathsf{0}})]^{\intercal}})(\delta M F^{\mathsf{1}/\mathsf{0}})^{\intercal}\right] \\ &= d_s - d_x + r. \end{aligned}$$

Here $P_{[M(I-\delta F^{\mathsf{0}})]^{\intercal}}$ is the projection matrix generated by matrix $[M(I-\delta F^{\mathsf{0}})]^{\intercal}$. To satisfy the rank condition ($\operatorname{rank} A = d_s - 1$), we need $r = d_x - 1$. If there are many zero rows in $MF^{\mathsf{1}/\mathsf{0}}$, r will be smaller than $d_x - 1$. Each row of $MF^{\mathsf{1}/\mathsf{0}}$ takes the form

$$(f^{\mathsf{1}/\mathsf{0}}(s_1|x_i,z_j) - f^{\mathsf{1}/\mathsf{0}}(s_1|x_i,z_{j+1}),\dots, f^{\mathsf{1}/\mathsf{0}}(s_{d_s}|x_i,z_j) - f^{\mathsf{1}/\mathsf{0}}(s_{d_s}|x_i,z_{j+1})).$$

Recall that $f^{\mathsf{1}/\mathsf{0}}(s_k|x_i,z_j) = f(s_k|x_i,z_j,\mathsf{1}) - f(s_k|x_i,z_j,\mathsf{0})$. So the row will be zero if the choice does not change transition matrix (hence $f^{\mathsf{1}/\mathsf{0}}(s|x,z)=0$), or the excluded variable Z does not affect the difference between the transition probabilities given the state variable X that affects per period utility functions (hence $f^{\mathsf{1}/\mathsf{0}}(s|x_i,z_j) - f^{\mathsf{1}/\mathsf{0}}(s|x_i,z_{j+1}) = 0$).

5 Estimation

All identification results in the previous section are constructive and follow from the linear system of equations (ID). The solution of the linear system has a closed form. Therefore, it is natural to estimate these identified structural parameters by replacing population parameters by sample estimates of the closed form solutions. The estimation proceeds in two steps. In the first step, we estimate the CCP $\{p_t(S_t) : t = 1, \ldots, T-1\}$ and the transition matrix $\{F_{t+1}^{\mathtt{d}} : t = 1, \ldots, T-1, \mathtt{d} = \mathtt{0}, \mathtt{1}\}$. Let $\hat{p}_t(S_t)$ and $\hat{F}_{t+1}^{\mathtt{d}}$ be the estimates of the CCP $p_t(S_t)$ and transition matrix $F_{t+1}^{\mathtt{d}}$ for each alternative $\mathtt{d}$ and each period t. For small state space $\mathcal{S}$, the estimator of the CCP $p_t(S_t)$ is simply the proportion of $D_t = \mathtt{1}$ in data for each S_t. When the support of S_t is large, a kernel estimator of $p_t(S_t) = \mathrm{E}(D_t|S_t)$ might be preferable. Similarly, for small state space $\mathcal{S}$, an estimator of $F_{t+1}^{\mathtt{d}}$ is simply the empirical frequency table of the transitions from S_t to S_{t+1} given $D_t = \mathtt{d}$. When the support of S_t is large, a smoothed approach may be preferable to avoid the issue of empty cells; see Aitchison and Aitken (1976). The second step is to estimate the structural parameters using the closed form solutions of the linear system under different identifying restrictions.

We focus on the case with the Exclusion Restriction and known discount factors (Proposition 2). Moreover, assume that the transition matrices are also known.

5.1 Estimation of stationary DPDC models

For a stationary DPDC model with known discount factor and state transition matrix, we need only cross-sectional data to estimate the per period utility functions $\mu^{\mathtt{1}/\mathtt{0}}$ and $\mu^{\mathtt{0}}$. Let $\{\, d_i, s_i : i = 1, \ldots, n \,\}$ be n agents' discrete choices and the corresponding states. For stationary DPDC models, it follows from Proposition 1 that

$$
\begin{aligned}
\mu^{\mathtt{1}/\mathtt{0}} &= W(\phi(p) - \delta F^{\mathtt{1}/\mathtt{0}} A^{+} b), \\
\mu^{\mathtt{0}} &= WL[(I - \delta F^{\mathtt{0}}) A^{+} b - \psi(p)].
\end{aligned}
$$

Let $\hat{p}$ be the estimator of the CCP $p = (p(s_1), \ldots, p(s_{d_s}))^{\intercal}$, and let

$$\sqrt{n}(\hat{p} - p) \to_d \mathcal{N}(0, \mathit{\Pi}).$$

We then have the estimator $\hat{\mu}^{\mathtt{1}/\mathtt{0}}$ and $\hat{\mu}^{\mathtt{0}}$ of $\mu^{\mathtt{1}/\mathtt{0}}$ and $\mu^{\mathtt{0}}$:

$$\hat{\mu}^{\mathtt{1}/\mathtt{0}} = W(\phi(\hat{p}) - \delta F^{\mathtt{1}/\mathtt{0}} A^{+} \hat{b}), \tag{5.1}$$

$$\hat{\mu}^{\mathtt{0}} = WL[(I - \delta F^{\mathtt{0}}) A^{+} \hat{b} - \psi(\hat{p})], \tag{5.2}$$

where

$$\hat{b} = \begin{bmatrix} M\phi(\hat{p}) \\ M\psi(\hat{p}) \end{bmatrix}.$$

It is easy to verify that

$$\sqrt{n}(\hat{\mu}^{1/0} - \mu^{1/0}) \to_d \mathcal{N}(0, (G^{1/0})\Pi(G^{1/0})^\intercal),$$
$$\sqrt{n}(\hat{\mu}^{0} - \mu^{0}) \to_d \mathcal{N}(0, (G^{0})\Pi(G^{0})^\intercal).$$

Here $G^{1/0}$ and G^0 are the gradient of $\mu^{1/0}$ and μ^0 with respect to p, respectively:

$$\begin{aligned} G^{1/0} &\equiv \frac{\partial \mu^{1/0}}{\partial p} \\ &= W\nabla\phi(p) - WF^{1/0}A^{+}\nabla b, \end{aligned}$$

$$\begin{aligned} G^{0} &\equiv \frac{\partial \mu^{0}}{\partial p} \\ &= WL(I - \delta F^{0})A^{+}\nabla b - WL\nabla\psi(p), \end{aligned}$$

where

$$\begin{aligned} \nabla\phi(p) &= \operatorname{diag}\left(\frac{\partial\phi(p(s_1))}{\partial p(s_1)}, \cdots, \frac{\partial\phi(p(s_{d_s}))}{\partial p(s_{d_s})}\right), \\ \nabla\psi(p) &= \operatorname{diag}\left(\frac{\partial\psi(p(s_1))}{\partial p(s_1)}, \cdots, \frac{\partial\psi(p(s_{d_s}))}{\partial p(s_{d_s})}\right), \\ \nabla b &= \begin{bmatrix} M\nabla\phi(p) \\ M\nabla\psi(p) \end{bmatrix}. \end{aligned}$$

Though parametric specification of the per period utility functions is not necessary for our estimator, it is easy to incorporate the parametric specification into our estimators. Suppose each state x_i corresponds to a vector $\tilde{x}_i$, and suppose

$$\mu^{1/0}(x_i) = \tilde{x}_i^\intercal \alpha \qquad \text{and} \qquad \mu^{0}(x_i) = \tilde{x}_i^\intercal \beta.$$

Then

$$\mu^{1/0} = \begin{bmatrix} \tilde{x}_1^\intercal \alpha \\ \vdots \\ \tilde{x}_{d_x}^\intercal \alpha \end{bmatrix} = \tilde{X}\alpha \qquad \text{and} \qquad \mu^{0} = \begin{bmatrix} \tilde{x}_1^\intercal \beta \\ \vdots \\ \tilde{x}_{d_x}^\intercal \beta \end{bmatrix} = \tilde{X}\beta.$$

In the numerical example of section 6, $\tilde{x}_i \equiv (1, x_i, x_i^2)^\intercal$ and $\mu^{1/0}(x_i) = \alpha_1 + \alpha_2 x_i + \alpha_3 x_i^2$.

Given the estimators $\hat{\mu}^{1/0}$ and $\hat{\mu}^0$ in equation (5.1) and (5.2), respectively, we can estimate α and β by

$$\hat{\alpha} = (\tilde{X}^\intercal \tilde{X})^{-1} \tilde{X}^\intercal \hat{\mu}^{1/0} \qquad \text{and} \qquad \hat{\beta} = (\tilde{X}^\intercal \tilde{X})^{-1} \tilde{X}^\intercal \hat{\mu}^{0}, \tag{5.3}$$

and

$$\sqrt{n}(\hat{\alpha}-\alpha) \to_d \mathcal{N}\Big(0, (\tilde{X}^\intercal \tilde{X})^{-1} \tilde{X}^\intercal (G^{1/0}) \Pi (G^{1/0})^\intercal \tilde{X} (\tilde{X}^\intercal \tilde{X})^{-1}\Big),$$
$$\sqrt{n}(\hat{\beta}-\beta) \to_d \mathcal{N}\Big(0, (\tilde{X}^\intercal \tilde{X})^{-1} \tilde{X}^\intercal (G^{0}) \Pi (G^{0})^\intercal \tilde{X} (\tilde{X}^\intercal \tilde{X})^{-1}\Big).$$

5.2 Estimation of nonstationary DPDC models

When the DPDC model is not stationary, we need at least two periods data when the discount factor and state transition matrices are known. Let $\{ d_{it}, s_{it} : i = 1, \ldots, n, t = 1, \ldots, T \}$ be n agents' discrete choices and the corresponding states over the T *sampling* periods. The estimation will be based on the formulas in Proposition 2. To save space, define the following new notation: for any $t < s$, let

$$\mu^{1/0}_{t:s} = \begin{bmatrix} \mu^{1/0}_t \\ \mu^{1/0}_{t+1} \\ \vdots \\ \mu^{1/0}_s \end{bmatrix}, \; \mu^{0}_{t:s} = \begin{bmatrix} \mu^{0}_t \\ \mu^{0}_{t+1} \\ \vdots \\ \mu^{0}_s \end{bmatrix}, \; p_{t:s} = \begin{bmatrix} p_t \\ p_{t+1} \\ \vdots \\ p_s \end{bmatrix}, \; \phi(p_{t:s}) = \begin{bmatrix} \phi(p_t) \\ \phi(p_{t+1}) \\ \vdots \\ \phi(p_s) \end{bmatrix}, \; \psi(p_{t:s}) = \begin{bmatrix} \psi(p_t) \\ \psi(p_{t+1}) \\ \vdots \\ \psi(p_s) \end{bmatrix}.$$

Their estimators $\hat{\mu}^{1/0}_{t:s}$, $\hat{\mu}^{0}_{t:s}$, $\hat{p}_{t:s}$, $\phi(\hat{p}_{t:s})$ and $\psi(\hat{p}_{t:s})$ are defined similarly.

We have the following estimator the per period utility functions:

$$\hat{\mu}^{1/0}_{1:(T-1)} = (I_{T-1} \otimes W) \left(\phi(\hat{p}_{1:(T-1)} - (\tilde{\Lambda}^{-1} \otimes I_{d_s}) F^{1/0}_{2:T} A^{+}_{1:T} \hat{b}_{1:T} \right),$$
$$\hat{\mu}^{0}_{2:(T-1)} = [I_{T-2} \otimes (WL)] \left((\Lambda^{-1} \otimes I_{d_s}) \check{F}^{0}_{3:T} A^{+}_{1:T} \hat{b}_{1:T} - \psi(\hat{p}_{2:(T-1)}) \right).$$

where

$$\hat{b}_{1:T} = \begin{bmatrix} (I_{T-1} \otimes M)(I_{T-1} \otimes \tilde{\Lambda}) \phi(\hat{p}_{1:(T-1)}) \\ (I_{T-2} \otimes M)(I_{T-2} \otimes \Lambda) \psi(\hat{p}_{2:(T-1)}) \end{bmatrix}.$$

Given

$$\sqrt{n} \left(\hat{p}_{1:(T-1)} - p_{1:(T-1)} \right) \to_d \mathcal{N}(0, \Pi).$$

we have

$$\sqrt{n} \left(\hat{\mu}^{1/0}_{1:(T-1)} - \mu^{1/0}_{1:(T-1)} \right) \to_d \mathcal{N}(0, (G^{1/0}_{1:T}) \Pi (G^{1/0}_{1:T})^\intercal),$$
$$\sqrt{n} \left(\hat{\mu}^{0}_{2:(T-1)} - \mu_{2:(T-1)} \right) \to_d \mathcal{N}(0, (G^{0}_{1:T}) \Pi (G^{0}_{1:T})^\intercal).$$

Here

$$\begin{aligned}G^{1/0}_{1:T} &\equiv \frac{\partial \mu^{1/0}_{1:(T-1)}}{\partial p_{1:(T-1)}} \\ &= (I_{T-1}\otimes W)\left(\operatorname{diag}\left(\nabla\phi(p_1),\ldots,\nabla\phi(p_{T-1})\right) - (\tilde{\Lambda}^{-1}\otimes I_{d_s})F^{1/0}_{2:T}A^{+}_{1:T}\nabla b_{1:T}\right),\end{aligned}$$

$$\begin{aligned}G^{0}_{1:T} &\equiv \frac{\partial \mu^{0}_{2:(T-1)}}{\partial p_{1:(T-1)}} \\ &= [I_{T-2}\otimes (WL)]\left((\Lambda^{-1}\otimes I_{d_s})\check{F}^{0}_{3:T}A^{+}_{1:T}\nabla b_{1:T} - \begin{bmatrix}0 & \nabla\psi(p_2) & & \\ \vdots & & \ddots & \\ 0 & & & \nabla\psi(p_{T-1})\end{bmatrix}\right),\end{aligned}$$

$$\nabla b_{1:T} = \begin{bmatrix}(I_{T-1}\otimes M)(\tilde{\Lambda}\otimes I_{T-1})\operatorname{diag}\left(\nabla\phi(p_1),\ldots,\nabla\phi(p_{T-1})\right) \\ \\ (I_{T-2}\otimes M)(\Lambda\otimes I_{T-2})\begin{bmatrix}0 & \nabla\psi(p_2) & & \\ \vdots & & \ddots & \\ 0 & & & \nabla\psi(p_{T-1})\end{bmatrix}\end{bmatrix}.$$

When there is parametric specification of the per period utility functions, we estimate the unknown parameters involved in the specification by the minimum distance principle. In particular, suppose each state x_i corresponds to a vector $\tilde{x}_i$, and suppose

$$\mu^{1/0}_t(x_i) = \tilde{x}_i^{\intercal}\alpha_t \qquad \text{and} \qquad \mu^{0}_t(x_i) = \tilde{x}_i^{\intercal}\beta_t,$$

and then

$$\mu^{1/0}_t = \begin{bmatrix}\tilde{x}_1^{\intercal}\alpha \\ \vdots \\ \tilde{x}_{d_x}^{\intercal}\alpha\end{bmatrix} = \tilde{X}\alpha_t \qquad \text{and} \qquad \mu^{0}_t = \begin{bmatrix}\tilde{x}_1^{\intercal}\beta \\ \vdots \\ \tilde{x}_{d_x}^{\intercal}\beta\end{bmatrix} = \tilde{X}\beta_t.$$

We then have following estimators of $\alpha^{\intercal}_{1:(T-1)} \equiv (\alpha_1^{\intercal},\ldots,\alpha_{T-1}^{\intercal})$ and $\beta^{\intercal}_{2:(T-1)} \equiv (\beta_2^{\intercal},\ldots,\beta_{T-1}^{\intercal})$:

$$\begin{aligned}\hat{\alpha}_{1:(T-1)} &= I_{T-1}\otimes (\tilde{X}^{\intercal}\tilde{X})^{-1}\tilde{X}^{\intercal}\hat{\mu}^{1/0}_{1:(T-1)}, \\ \hat{\beta}_{2:T} &= I_{T-2}\otimes (\tilde{X}^{\intercal}\tilde{X})^{-1}\tilde{X}^{\intercal}\hat{\mu}^{0}_{2:(T-1)},\end{aligned}$$

whose asymptotic variances are

$$\begin{aligned}\operatorname{Var}(\hat{\alpha}_{1:(T-1)}) &= \left[I_{T-1}\otimes (\tilde{X}^{\intercal}\tilde{X})^{-1}\tilde{X}^{\intercal}\right] G^{1/0}_{1:T}\Pi(G^{1/0}_{1:T})^{\intercal}\left[I_{T-1}\otimes \tilde{X}(\tilde{X}^{\intercal}\tilde{X})^{-1}\right], \\ \operatorname{Var}(\hat{\beta}_{2:(T-1)}) &= \left[I_{T-2}\otimes (\tilde{X}^{\intercal}\tilde{X})^{-1}\tilde{X}^{\intercal}\right] G^{0}_{1:T}\Pi(G^{0}_{1:T})^{\intercal}\left[I_{T-2}\otimes \tilde{X}(\tilde{X}^{\intercal}\tilde{X})^{-1}\right].\end{aligned}$$

6 Numerical studies

In the numerical experiments below, we consider a stationary dynamic programming discrete choice model with a single X_t and a single excluded variable Z_t, and let $S_t = (X_t, Z_t)$.

The support $\mathcal{X} \equiv \{ x_1, \dots x_{d_x} \}$ of X_t are the d_x cutting points that split the interval $[0, 2]$ into $d_x - 1$ equally spaced subintervals. The support of Z_t is $\mathcal{Z} \equiv \{ 1, \dots, d_z \}$. Let the state space $\mathcal{S} \equiv \mathcal{X} \times \mathcal{Z}$ and $d_s = d_x \times d_z$. The observable states $S_t = (X_t, Z_t)$ follows a homogenous controlled first-order Markov chain. For $\mathtt{d} \in \{ \mathtt{0}, \mathtt{1} \}$, let $F^{\mathtt{d}}$ be the time invariant $d_s \times d_s$ transition matrix describing the transition probability law from S_t to S_{t+1} given the discrete choice $D_t = \mathtt{d}$. The transition matrix $F^{\mathtt{d}}$ is randomly generated subjecting to the sparsity restriction that there are at most m_s number of states that can be reached in the next period. In the experiments below, we let $d_x = 40$, $d_z = 3$ and $m_s = 5$, and let the discount factor $\delta = 0.8$. The per period utility functions are

$$\mu^{\mathtt{1}}(X) = 1 + X - \frac{X^2}{2} \qquad \text{and} \qquad \mu^{\mathtt{0}}(X) = X.$$

The unobserved utilities shocks $\varepsilon^{\mathtt{0}}$ and $\varepsilon^{\mathtt{1}}$ are independent and follow the type 1 extreme value distribution. In the estimation below, we assume both the state transition matrices and the discount factor are known.

We are going to compare our closed-form estimator with the three well known parametric estimators in the literature, including the nested fixed point (NFXP) algorithm (Rust, 1987), the pseudo-maximum likelihood (PML) estimator and the nested pseudo-likelihood (NPL) algorithm (Aguirregabiria and Mira, 2002). To implement their methods, we assume the parametric specification of the per period utility functions

$$\mu_t^{\mathtt{1/0}}(X_t) = \alpha_1 + \alpha_2 X_t + \alpha_3 X_t^2 \qquad \text{and} \qquad \mu_t^{\mathtt{0}}(X_t) = \beta_1 + \beta_2 X_t + \beta_3 X_t^2,$$

with $\beta_1 = 0$. So the true value is $(\alpha_1, \alpha_2, \alpha_3) = (1, 0, -1/2)$ and $(\beta_1, \beta_2, \beta_3) = (0, 1, 0)$. We describe the implementation of these three estimators without using normalization assumption below. Our parametric estimation will use the formula (5.3).

- *Nested fixed point (NFXP) algorithm*

Denote $\theta = (\alpha, \beta)$. For a stationary DPDC model, we have

$$\begin{cases} \phi(p) = \mu^{\mathtt{1/0}}(\theta) + \delta F^{\mathtt{1/0}} v \\ \psi(p) = v - \mu^{\mathtt{0}}(\theta) - \delta F^{\mathtt{0}} v \end{cases}.$$

So

$$v = (I - \delta F^{\mathtt{0}})^{-1} (\psi(p) + \mu^{\mathtt{0}}(\theta)), \tag{6.1}$$

and

$$p = \phi^{-1} \left[\mu^{\mathtt{1/0}}(\theta) + \delta F^{\mathtt{1/0}} v \right]. \tag{6.2}$$

We then have

$$\begin{aligned} v &= (I - \delta F^0)^{-1}(\psi(p) + \mu^0(\theta)) \\ &= (I - \delta F^0)^{-1}(\psi\left(\phi^{-1}\left[\mu^{1/0}(\theta) + \delta F^{1/0} v\right]\right) + \mu^0(\theta)) \\ &\equiv \Gamma(v; \theta). \end{aligned}$$

The NFXP algorithm proceeds in the following steps:

1. For each θ, we find the ex ante value function $v^*(\theta)$ that solves $v = \Gamma(v; \theta)$ by iteration. Let $v^{(s)}(\theta)$ be the current guess of the solution. We update $v^{(s)}(\theta)$ by letting $v^{(s+1)} = \Gamma(v^{(s)}; \theta)$ until convergence.

2. Given the solution $v^*(\theta)$, we have the CCP $p(\theta)$ as a function of θ:

$$p(\theta) = \phi^{-1}\left[\mu^{1/0}(\theta) + \delta F^{1/0} v^*(\theta)\right],$$

   and the log likelihood function $l(\theta)$ can be formed from $p(\theta)$.

3. The NFXP is the maximizer of the log likelihood function $l(\theta)$.

- *Pseudo-maximum likelihood (PML) estimator*

The PML estimator is based on the following equation of CCP,

$$p(\theta) = \phi^{-1}\left[\mu^{1/0}(\theta) + \delta F^{1/0}(I - \delta F^0)^{-1}(\psi(p) + \mu^0(\theta))\right],$$

which follows from substituting the value function v in equation (6.2) with its representation of equation (6.1). This estimator proceeds in two steps. In the first step, we estimate CCP p nonparametrically. Let $\hat{p}(x, z)$ be the kernel estimator of $\mathrm{P}(D = 1|x, z)$ based on the data $\{ d_i, x_i, z_i : i = 1, \ldots, n \}$. In the second step, we define

$$\tilde{p}(\theta; \hat{p}) = \phi^{-1}\left[\mu^{1/0}(\theta) + \delta F^{1/0}(I - \delta F^0)^{-1}(\psi(\hat{p}) + \mu^0(\theta))\right], \tag{6.3}$$

and a pseudo log likelihood $\tilde{l}(\theta; \hat{p})$ can be defined correspondingly. The PML estimator $\tilde{\theta}$ is the maximizer of the pseudo log likelihood function $\tilde{l}(\theta; \hat{p})$.

- *Nested pseudo-likelihood (NPL) algorithm*

Let $\hat{p}$ be the CCP estimate used in the PML estimation, and let $\tilde{\theta}$ be the PML estimator. The NPL algorithm then updates the CCP estimates by replacing the existing CCP estimate $\hat{p}$ with $\tilde{p}(\tilde{\theta}; \hat{p})$ as defined in equation (6.3). In particular, NPL algorithm proceeds in the following steps:

1. Given the current CCP estimates $\hat{p}^{(s)}$, obtain the PML estimator $\tilde{\theta}^{(s)}$.

Table 1: Estimation of Stationary DPDC Models

		$\alpha_1 = 1$	$\alpha_2 = 0$	$\alpha_3 = -0.5$	$\beta_1 = 0$	$\beta_2 = 1$	$\beta_3 = 0$	Time[1]
NFXP	Bias	0.008	0.003	-0.006	-	-0.017	0.014	
	Var.	0.044	0.218	0.051	-	0.704	0.152	*240.118*
	MSE	*0.044*	*0.218*	*0.051*	-	*0.704*	*0152*	
PML	Bias	0.008	-0.003	-0.003	-	0.158	-0.070	
	Var.	0.045	0.231	0.054	-	0.651	0.146	*9.825*
	MSE	*0.045*	*0.231*	*0.054*	-	*0.676*	*0.151*	
NPL	Bias	0.004	0.003	-0.004	-	0.013	-0.003	
	Var.	0.045	0.231	0.054	-	0.809	0.186	*170.360*
	MSE	*0.045*	*0.231*	*0.054*	-	*0.809*	*0.186*	
Closed-Form[2]	Bias	0.103	-0.269	0.128	-0.048	0.172	-0.070	
	Var.	0.032	0.122	0.029	0.147	0.299	0.059	*0.282*
	MSE	*0.043*	*0.194*	*0.045*	0.150	*0.328*	*0.064*	

Note: The results are based on 10 pairs of state transition matrices and $1,000$ replications for each pair. The cross-sectional sample size is $1,000$, and there is one period observation.

[1] The computation time is measured in seconds based on the average of the replications.

[2] We first estimate the per period utility functions nonparametrically using the formulas in equation (5.1) and (5.2). Then estimate the parameters α and β by the formulas of equation (5.3).

2. Let
$$\hat{p}^{(s+1)} = \phi^{-1}\left[\mu^{1/0}(\tilde{\theta}^{(s)}) + \delta F^{1/0}(I - \delta F^0)^{-1}(\psi(\hat{p}) + \mu^0(\tilde{\theta}^{(s)}))\right].$$

3. Let $\hat{p}^{(s)} = \hat{p}^{(s+1)}$ and go back step 1 until convergence ($\|\tilde{\theta}^{(s+1)} - \tilde{\theta}^{(s)}\| < \varepsilon$).

Table 1 reports the parametric estimation performance of the NFXP, PML, NPL and our minimum distance estimator based on the nonparametric closed form estimator. Our estimator outperforms the other three estimators in terms of mean squared error (MSE) and computation time. Our estimator is 34, 600 and 850 times faster than the PML, NPL and NFXP algorithms, respectively. In addition to computation time, our estimator has smaller variance. In the experiments, we also found that NFXP, PML and NPL are numerically unstable due to the existence of multiple local maximum in the maximization of log likelihood function. Our estimator does not suffer from this issue because there is no numerical optimization involved at all. Figure 6.1 shows the 95% confidence interval of the per period utility functions based on our nonparametric closed form estimator, NFXP, PML and NPL. The estimates of the utility functions from NFXP, PML and NPL are based on the estimates of the parametric utility functions. Since our closed-form estimator does not have the information about the parametric form of the utility functions, its confidence interval is slightly wider than those of the parametric estimators, but the difference is marginal.

We now consider the counterfactual intervention in the dynamic discrete choice model. In particular, we want to know how large would be the bias from using normalization, which is wrong in this example since $\mu^0(x) = x$. Let $\tilde{F}^0$ and $\tilde{F}^1$ be the counterfactual state transition matrices, and let $\tilde{p}$ be the counterfactual CCP of the stationary DPDC model with every-

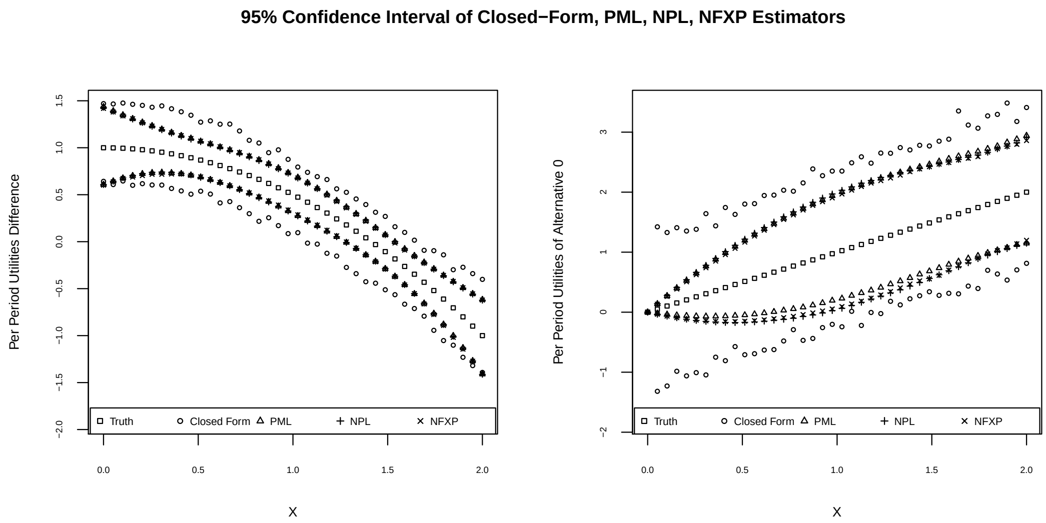

Figure 6.1: Estimation of Per Period Utilities in Stationary DPDC Models

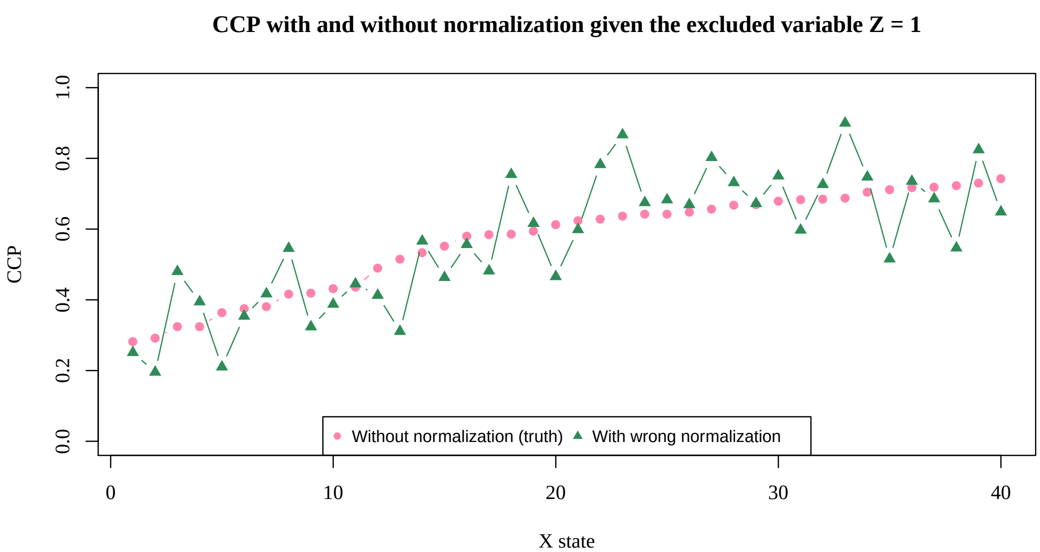

Figure 6.2: Counterfactual CCP with and without Normalization

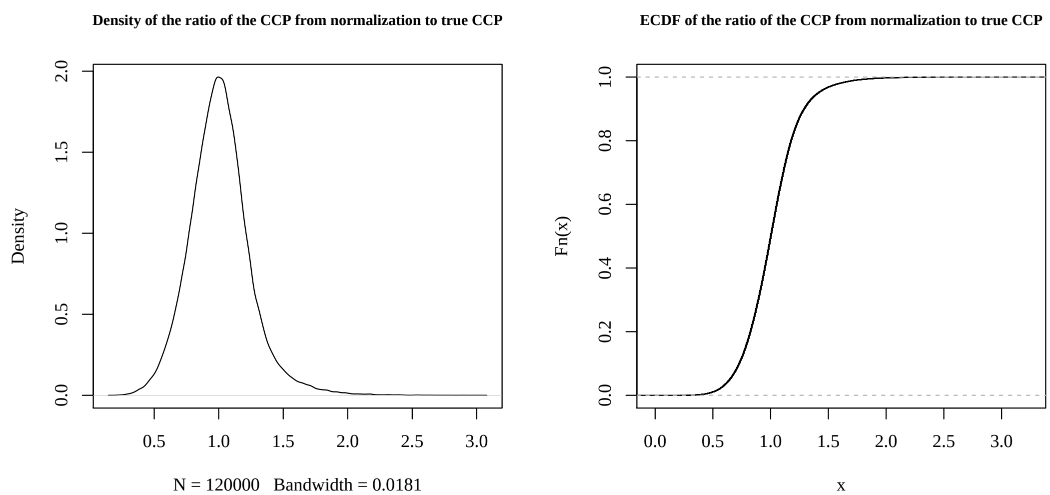

Figure 6.3: Density and Distribution of the Ratio of True Counterfactual CCP and Counterfactual CCP under Normalization

thing unchanged excepting for that the state transition matrices are $\tilde{F}^0$ and $\tilde{F}^1$. Let $\tilde{p}$ be the counterfactual CCP under the true per period utility functions, and let $\tilde{p}_{\text{nm}}$ be counterfactual CCP under the normalization assumption. Figure 6.2 shows the bias between the true counterfactual CCP $\tilde{p}$ and and the counterfactual CCP under normalization assumption $\tilde{p}_{\text{nm}}$. By trying $1,000$ pairs of state transition matrices (F^0, F^1) and $(\tilde{F}^0, \tilde{F}^1)$, we report the empirical density and CDF of the ratio $\tilde{p}(s_i)/\tilde{p}_{\text{nm}}(s_i)$ in figure 6.3. Based on these two figures, we can conclude that bias in counterfactual CCP from using normalization is noticeably large. More importantly, the counterfactual CCP from normalization could give the wrong direction of how the state affects the choice probabilities, which could lead to completely different conclusions about policy effects.

7 Concluding Remarks

The identification and estimation of DPDC models are considered to be complicated and numerically difficult. This paper shows that the identification of DPDC model is indeed equivalent to the identification of a linear GMM system. So the identification and estimation of DPDC models become easy to address. We show how to identify DPDC models under a variety of restrictions. In particular, we show how to identify the DPDC model without normalizing the period utility function of any alternative. This case is particularly important because we show that normalization of period utility functions can usually bias the counterfactual policy predictions. Due to the equivalence to a linear GMM system, we show how to estimate DPDC models using linear estimation approach without using any terminal conditions or assuming the dynamic programming problem is stationary. The implementation of our estimator does not involve any numerical optimization or iteration.

There are two practically important extensions of this paper. First, one can extend the paper by incorporating the unobservable heterogeneity. Several papers, including Kasahara and Shimotsu (2009) and Hu and Shum (2012), have studied the identification of the CCP, when there are unobservable heterogeneity, such as discrete types in Kasahara and Shimotsu (2009). Since our identification of per period utility functions depends on the state transition matrix and the CCP only, one can then identify the type-specific per period utility functions by using the identified type specific CCP (and state transition distributions).

Second, most paper in the literature of DPDC study the identification under the assumption that the distribution of utility shocks are known. Depending on the sensitivity of the parameter estimates on the specification of the error distribution, allowing the error distribution to be unknown could be practically important. Suppose $\mathcal{G}$ is a set of possible distributions of the utility shocks. Each error distribution $G \in \mathcal{G}$ defines a pair of functionals ϕ and ψ being used in our identification arguments. In addition, each pair of functions ϕ and ψ will give rise to a set of formulas of the identified structural parameters according to Proposition 2 or other propositions. Therefore, we can explicitly characterize the identified set of the per period utility functions.

APPENDIX

A Proofs

Lemma A.1. *Let A be an $m \times n$ real matrix with $m \geq n-1$. Suppose each row of A sums to be zero and* $\operatorname{rank} A = n-1$*. Suppose linear equation $Ax = b$ has solutions. Then the solution set is $\{A^+ b + c \times 1_n : \forall c \in \mathbb{R}\}$, where A^+ is the Moore-Penrose pseudoinverse of A, and 1_n is a n-dimensional vector of ones.*

Proof. We know that the solution set of equation $Ax = b$ is $\{A^+ b + (I - A^+ A)a : \forall a \in \mathbb{R}^n\}$. It suffices to show that $(I - A^+ A)$ is an $n \times n$ matrix, whose elements are identical.

Let $A = U \Sigma V^\intercal$ be an singular value decomposition (SVD) of matrix A. We know that $A^+ = V \Sigma^+ U^\intercal$, where Σ^+ is the pseudoinverse of Σ. Because U and V are both orthogonal matrices, we have $A^+ A = V \Sigma^+ \Sigma V^\intercal$ as an eigenvalue decomposition (EVD). When $\operatorname{rank} A = n-1$, we have

$$\Sigma^+ \Sigma = \left[\begin{array}{c|c} I_{n-1} & \\ \hline & 0 \end{array}\right],$$

where I_{n-1} is $(n-1) \times (n-1)$ identity matrix. So the columns of V are eigenvectors of $A^+ A$ corresponding to the eigenvalues 1 and 0. Because the sum of columns of A is zero, 1_n is an eigenvector of $A^+ A$ corresponding to eigenvalue zero, and $n^{-1/2} \times 1_n$ is one column of V. Removing the column $n^{-1/2} \times 1_n$ from matrix V, we obtain an $n \times (n-1)$ matrix $\tilde{V}$ and $A^+ A = V \Sigma^+ \Sigma V^\intercal = \tilde{V} \tilde{V}^\intercal$.

As V is an orthogonal matrix, we have

$$\begin{aligned} I = VV^\intercal &= \begin{bmatrix} \tilde{V} & n^{-1/2} \times 1_n \end{bmatrix} \begin{bmatrix} \tilde{V}^\intercal \\ n^{-1/2} \times 1_n^\intercal \end{bmatrix} \\ &= \tilde{V} \tilde{V}^\intercal + n^{-1} 1_n {1_n}^\intercal \\ &= A^+ A + n^{-1} 1_{n \times n}. \end{aligned}$$

Here $1_{n \times n}$ is a $n \times n$ matrix whose elements are all 1. So we have $I - A^+ A = n^{-1} 1_{n \times n}$, and the lemma will follow. □

Lemma A.2. *Let A_1 and A_2 both be $m \times n$ real matrices with $m \geq 2(n-1)$. Define a block matrix $A \equiv \left[\begin{array}{c|c} A_1 & A_2 \end{array}\right]$. For each $i = 1, 2$, suppose each row of A_i sums to be zero, and* $\operatorname{rank} A = 2n-2$*. Suppose linear equation*

$$A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = b$$

has solutions. Let

$$\begin{bmatrix} x_{1,+} \\ x_{2,+} \end{bmatrix} \equiv A^{+}b.$$

Then the solution set of the equation is

$$\left\{ \begin{bmatrix} x_{1,+} + c_1 1_n \\ x_{2,+} + c_2 1_n \end{bmatrix} : c_1, c_2 \in \mathbb{R} \right\}.$$

Proof. The proof is similar to the proof of lemma A.1. The solution set of equation $Ax = b$ is $\{A^{+}b + (I - A^{+}A)a : \forall a \in \mathbb{R}^n\}$. Let $A = U\Sigma V^{\intercal}$ be an SVD of matrix A. We have $A^{+}A = V\Sigma^{+}\Sigma V^{\intercal}$ as an EVD. Because $\operatorname{rank} A = 2n-2$ and the row sums of each A_i $(i = 1, 2)$ are zero, we have

$$\Sigma^{+}\Sigma = \left[\begin{array}{c|c} I_{2n-2} & \\ \hline & 0_{2\times 2} \end{array}\right].$$

So V has two columns $w_1^{\intercal} = n^{-1/2}(1_n^{\intercal}, 0_n^{\intercal})$ and $w_2^{\intercal} = n^{-1/2}(0_n^{\intercal}, 1_n^{\intercal})$, because they are two orthonormal eigenvectors corresponding to eigenvalue 0. Removing w_1 and w_2 from the columns of matrix V, we obtain an $2n\times(2n-2)$ matrix $\tilde{V}$ whose columns are eigenvectors corresponding to the $2n - 2$ nonzero eigenvalues. We then have $A^{+}A = V\Sigma^{+}\Sigma V^{\intercal} = \tilde{V}\tilde{V}^{\intercal}$.

As V is an orthogonal matrix, we have

$$\begin{aligned} I = VV^{\intercal} &= \begin{bmatrix} \tilde{V} & w_1 & w_2 \end{bmatrix} \begin{bmatrix} \tilde{V}^{\intercal} \\ w_1^{\intercal} \\ w_2^{\intercal} \end{bmatrix} \\ &= \tilde{V}\tilde{V}^{\intercal} + w_1 {w_1}^{\intercal} + w_2 w_2^{\intercal} \\ &= A^{+}A + \left[\begin{array}{c|c} 1_{n\times n} & \\ \hline & 1_{n\times n} \end{array}\right]. \end{aligned}$$

The rest of the proof follows immediately. $\square$

Proof of Proposition 2. Equation (4.14) is equivalent to the following,

$$
\begin{aligned}
\mu_1^{1/0} \otimes 1_{d_z} + F_2^{1/0}(\delta_1 \cdot v_2) &= \phi(p_1) \\
\left(\prod_{r=1}^{1} \delta_r\right) \cdot \mu_2^{1/0} \otimes 1_{d_z} + F_3^{1/0}\left[\left(\prod_{r=1}^{2} \delta_r\right) \cdot v_3\right] &= \left(\prod_{r=1}^{1} \delta_r\right) \cdot \phi(p_2) \\
&\vdots \\
\left(\prod_{r=1}^{T-2} \delta_r\right) \cdot \mu_{T-1}^{1/0} \otimes 1_{d_z} + F_T^{1/0}\left[\left(\prod_{r=1}^{T-1} \delta_r\right) \cdot v_T\right] &= \left(\prod_{r=1}^{T-2} \delta_r\right) \cdot \phi(p_{T-1}) \\
(\delta_1 \cdot v_2) - \delta_1 \cdot \mu_2^{0} \otimes 1_{d_z} - F_3^{0}\left(\prod_{r=1}^{2} \delta_r\right) \cdot v_3 &= \delta_1 \cdot \psi(p_2) \\
\left(\prod_{r=1}^{2} \delta_r\right) \cdot v_3 - \left(\prod_{r=1}^{2} \delta_r\right) \cdot \mu_3^{0} \otimes 1_{d_z} - F_4^{0}\left(\prod_{r=1}^{3} \delta_r\right) \cdot v_4 &= \left(\prod_{r=1}^{2} \delta_r\right) \cdot \psi(p_3) \\
&\vdots \\
\left(\prod_{r=1}^{T-2} \delta_r\right) \cdot v_T - \left(\prod_{r=1}^{T-2} \delta_r\right) \cdot \mu_{T-1}^{0} \otimes 1_{d_z} - F_T^{0}\left(\prod_{r=1}^{T-1} \delta_r\right) \cdot v_T &= \left(\prod_{r=1}^{T-2} \delta_r\right) \cdot \psi(p_{T-1}).
\end{aligned}
\tag{A.1}
$$

Multiplying both sides of equation (A.1) with the M matrix, we have

$$
A_{1:T} \begin{bmatrix} \delta_1 \cdot v_2 \\ (\prod_{r=1}^{2} \delta_r) \cdot v_3 \\ \vdots \\ (\prod_{r=1}^{T-1} \delta_r) \cdot v_T \end{bmatrix} = b_{1:T}, \tag{A.2}
$$

where

$$
A_{1:T} = \begin{bmatrix}
MF_2^{1/0} & & & & \\
& MF_3^{1/0} & & & \\
& & MF_4^{1/0} & & \\
& & & \ddots & \\
& & & & MF_T^{1/0} \\
M & -MF_3^{0} & & & \\
& M & -MF_4^{0} & & \\
& & \ddots & \ddots & \\
& & & M & -MF_T^{0}
\end{bmatrix}
= \begin{bmatrix} (I_{T-1} \otimes M)\, F_{2:T}^{1/0} \\ (I_{T-2} \otimes M)\, \check{F}_{3:T}^{0} \end{bmatrix}
$$

and

$$b_{1:T} = \begin{bmatrix} M\phi(p_1) \\ (\prod_{r=1}^{1} \delta_r) \cdot M\phi(p_2) \\ \vdots \\ (\prod_{r=1}^{T-2} \delta_r) \cdot M\phi(p_{T-1}) \\ \delta_1 \cdot M\psi(p_2) \\ (\prod_{r=1}^{2} \delta_r) \cdot M\psi(p_3) \\ \vdots \\ (\prod_{r=1}^{T-2} \delta_r) \cdot M\psi(p_{T-1}) \end{bmatrix} = \begin{bmatrix} (I_{T-1} \otimes M)(\tilde{\Lambda} \otimes I_{T-1}) \begin{bmatrix} \phi(p_1) \\ \vdots \\ \phi(p_{T-1}) \end{bmatrix} \\ (I_{T-2} \otimes M)(\Lambda \otimes I_{T-2}) \begin{bmatrix} \psi(p_2) \\ \vdots \\ \psi(p_{T-1}) \end{bmatrix} \end{bmatrix}.$$

Note that $A_{1:T}^{+} b_{1:T}$ is one solution for $(\delta_1 \cdot v_2, (\prod_{r=1}^{2} \delta_r) \cdot v_3 \ldots, (\prod_{r=1}^{T-1} \delta_r) \cdot v_T)^\intercal$ of equation (A.2).

It follows from equation (A.1) that

$$\begin{bmatrix} \mu_1^{1/0} \otimes 1_{d_z} \\ (\prod_{r=1}^{1} \delta_r) \cdot \mu_2^{1/0} \otimes 1_{d_z} \\ \vdots \\ (\prod_{r=1}^{T-2} \delta_r) \cdot \mu_{T-1}^{1/0} \otimes 1_{d_z} \end{bmatrix} = (\tilde{\Lambda} \otimes I_{d_s}) \begin{bmatrix} \phi(p_1) \\ \phi(p_2) \\ \vdots \\ \phi(p_{T-1}) \end{bmatrix} - F_{2:T}^{1/0} \begin{bmatrix} \delta_1 \cdot v_2 \\ (\prod_{r=1}^{2} \delta_r) \cdot v_3 \\ \vdots \\ (\prod_{r=1}^{T-1} \delta_r) \cdot v_T \end{bmatrix}, \tag{A.3}$$

and

$$\begin{bmatrix} \delta_1 \cdot \mu_2^{0} \otimes 1_{d_z} \\ (\prod_{r=1}^{2} \delta_r) \cdot \mu_3^{0} \otimes 1_{d_z} \\ \vdots \\ (\prod_{r=1}^{T-2} \delta_r) \cdot \mu_{T-1}^{0} \otimes 1_{d_z} \end{bmatrix} = \check{F}_{3:T}^{0} \begin{bmatrix} \delta_1 \cdot v_2 \\ (\prod_{r=1}^{2} \delta_r) \cdot v_3 \\ \vdots \\ (\prod_{r=1}^{T-1} \delta_r) \cdot v_T \end{bmatrix} - (\Lambda \otimes I_{d_s}) \begin{bmatrix} \psi(p_2) \\ \psi(p_3) \\ \vdots \\ \psi(p_{T-1}) \end{bmatrix}. \tag{A.4}$$

Substituting $(\delta_1 \cdot v_2, (\prod_{r=1}^{2} \delta_r) \cdot v_3 \ldots, (\prod_{r=1}^{T-1} \delta_r) \cdot v_T)^\intercal$ in equation (A.3) with $A_{1:T}^{+} b_{1:T}$, and multiplying both sides of equation (A.3) with $\tilde{\Lambda}^{-1} \otimes I_{d_s}$, we have

$$\begin{bmatrix} \mu_1^{1/0} \otimes 1_{d_z} \\ \mu_2^{1/0} \otimes 1_{d_z} \\ \vdots \\ \mu_{T-1}^{1/0} \otimes 1_{d_z} \end{bmatrix} = \begin{bmatrix} \phi(p_1) \\ \phi(p_2) \\ \vdots \\ \phi(p_{T-1}) \end{bmatrix} - (\tilde{\Lambda}^{-1} \otimes I_{d_s}) F_{2:T}^{1/0} A_{1:T}^{+} b_{1:T}.$$

Similar operations for equation (A.4) gives

$$\begin{bmatrix} \mu_2^{0} \otimes 1_{d_z} \\ \mu_3^{0} \otimes 1_{d_z} \\ \vdots \\ \mu_{T-1}^{0} \otimes 1_{d_z} \end{bmatrix} = (\Lambda^{-1} \otimes I_{d_s}) \check{F}_{3:T}^{0} A_{1:T}^{+} b_{1:T} - \begin{bmatrix} \psi(p_2) \\ \psi(p_3) \\ \vdots \\ \psi(p_{T-1}) \end{bmatrix}.$$

We then have the closed-form solution:

$$\begin{bmatrix} \mu_1^{1/0} \\ \vdots \\ \mu_{T-1}^{1/0} \end{bmatrix} = (I_{T-1} \otimes W) \left(\begin{bmatrix} \phi(p_1) \\ \vdots \\ \phi(p_{T-1}) \end{bmatrix} - (\tilde{\Lambda}^{-1} \otimes I_{d_s}) F_{2:T}^{1/0} A_{1:T}^{+} b_{1:T} \right),$$

$$\begin{bmatrix} \mu_2^{0} \\ \vdots \\ \mu_{T-1}^{0} \end{bmatrix} = [I_{T-2} \otimes (WL)] \left((\Lambda^{-1} \otimes I_{d_s}) \check{F}_{3:T}^{0} A_{1:T}^{+} b_{1:T} - \begin{bmatrix} \psi(p_2) \\ \vdots \\ \psi(p_{T-1}) \end{bmatrix} \right).$$

Here we used $(I_{T-2} \otimes W)(I_{T-2} \otimes L) = I_{T-2} \otimes (WL)$. □

Proof of Proposition 4. The key observation is that when Z_t is time invariant, the state transition matrix $F_t^{\mathtt{d}}$ is a d_x-by-d_x block matrix,

$$F_t^{\mathtt{d}} = \begin{bmatrix} D_t(x_1, x_1) & D_t(x_1, x_2) & \dots & D_t(x_1, x_{d_x}) \\ D_t(x_2, x_1) & D_t(x_2, x_2) & \dots & D_t(x_2, x_{d_x}) \\ \vdots & \vdots & \vdots & \vdots \\ D_t(x_{d_x}, x_1) & D_t(x_{d_x}, x_2) & \dots & D_t(x_{d_x}, x_{d_x}) \end{bmatrix},$$

of which each element $D_t(x_i, x_j)$ is a d_z-by-d_z diagonal matrix, for each period t and each choice $\mathtt{d}$. The diagonal matrix $D_t(x_i, x_j)$ has the following form,

$$D_t(x_i, x_j) = \begin{bmatrix} f_t(x_j, z_1 | x_i, z_1, Y_{t-1} = \mathtt{d}) & & \\ & \ddots & \\ & & f_t(x_j, z_{d_z} | x_i, z_{d_z}, Y_{t-1} = \mathtt{d}) \end{bmatrix},$$

because $f_t(x_j, z_k | x_i, z_l, Y_{t-1} = \mathtt{d}) = 0$ whenever $z_k \neq z_l$. Let e_i be an d_z-dimensional vector whose elements are all zero excepting for the i-th element being 1. One can verify that

$$\tilde{e}_1 \equiv 1_{2d_x} \otimes e_1, \dots, \tilde{e}_{d_z} \equiv 1_{2d_x} \otimes e_{d_z}, \tilde{e}_{d_z+1} \equiv (0_{d_x}^{\intercal}, 1_{d_x}^{\intercal})^{\intercal},$$

belong to the null space of A_t, and are linearly independent. Hence, if rank $A_t = 2d_s - d_z - 1$, we have $\mathcal{N}(A_t) = \operatorname{span}(\tilde{e}_1, \dots, \tilde{e}_{d_z+1})$. Then the solution set for equation (4.18) is

$$\left\{ \begin{bmatrix} v_{t,+} \\ \bar{v}_{t+1,+} \end{bmatrix} + \lambda_1 \tilde{e}_1 + \dots + \lambda_{d_z+1} \tilde{e}_{d_z+1} : (\lambda_1, \dots, \lambda_{d_z+1}) \in \mathbb{R}^{d_z+1} \right\}. \tag{A.5}$$

Let

$$\tilde{e}_i = \begin{bmatrix} \tilde{e}_{i,h} \\ \hline \tilde{e}_{i,l} \end{bmatrix}.$$

Then $\mu_{t-1}^{1/0}$ and $\mu_t^{1/0}$ are identified because $F_t^{1/0} \tilde{e}_{i,h} = F_{t+1}^{1/0} \tilde{e}_{i,l} = 0_{d_s}$ for each $i = 1, \dots, d_z + 1$.

The period utility function μ_t^0 is identified up to an additive constant because for any $\left(v_t', \underset{\sim}{\bar{v}}_{t+1}'\right)$ belonging to the solution set (A.5), we have

$$\begin{aligned} v_t' - F_{t+1}^0 \underset{\sim}{\bar{v}}_{t+1}' &= \left(v_{t,+} - F_{t+1}^0 \underset{\sim}{\bar{v}}_{t+1,+}\right) + 1_{d_x} \otimes (\lambda_{d_z+1} \times 1_{d_z}) \\ &= \left(v_{t,+} - F_{t+1}^0 \underset{\sim}{\bar{v}}_{t+1,+}\right) + \lambda_{d_z+1} \times 1_{d_s}. \end{aligned}$$

So the conclusion of Proposition 2 can be established for the permanent excluded variable case. The same proof of Proposition 3 can be used to identify the discount factors in the permanent excluded variable situation. □

Proof of Proposition 5. We have a system

$$A_t \begin{bmatrix} \underset{\sim}{\bar{v}}_{t+1} \\ v_t \\ v_{t-1} \end{bmatrix} = b_t, \tag{A.6}$$

where A_t and b_t are as defined in equation (4.25).

Note that the two vectors

$$\begin{aligned} q_1^\intercal &= (1_{d_s}^\intercal, 0_{d_s}^\intercal, -1_{d_s}^\intercal), \\ q_2^\intercal &= (0_{d_s}^\intercal, 1_{d_s}^\intercal, (1+\delta_{t-1}) \times 1_{d_s}^\intercal), \end{aligned}$$

are linearly independent, and $A_t q_1 = A_t q_2 = 0_{3d_s}$. So q_1 and q_2 are contained in the null space of matrix A_t, denoted by $\mathcal{N}(A_t)$. By the assumption that $\operatorname{rank} A_t = 3d_s - 2$, we then have $\mathcal{N}(A_t) = \operatorname{span}(b_1, b_2)$. So the solution set of equation (A.6) is

$$\left\{A_t^+ b_t + \lambda_1 q_1 + \lambda_2 q_2 : \lambda_1, \lambda_2 \in \mathbb{R}\right\}.$$

We then have

$$\mu^{1/0} = \frac{1}{3} \cdot \left\{\phi(p_t) + \phi(p_{t-1}) + \phi(p_{t-2}) - \left[\begin{array}{c|c|c} F_{t+1}^{1/0} & \delta_{t-1} F_t^{1/0} & \delta_{t-2} F_{t-1}^{1/0} \end{array}\right] A_t^+ b_t\right\},$$

is unique. Here

$$\left[\begin{array}{c|c|c} F_{t+1}^{1/0} & \delta_{t-1} F_t^{1/0} & \delta_{t-2} F_{t-1}^{1/0} \end{array}\right]$$

is 1×3 block matrix. And the solution set of μ^0 is

$$\{\mu_+^0 + c \cdot 1_{d_s} : c \in \mathbb{R}\},$$

where

$$\mu_+^0 = \frac{1}{2} \cdot \left\{\left[\begin{array}{c|c|c} -F_{t+1}^0 & (I_{d_s} - \delta_{t-1} F_t^0) & I_{d_s} \end{array}\right] A_t^+ b_t - \psi(p_t) - \psi(p_{t-1})\right\}.$$

Applying the normalization, μ^0 is identified with $\mu^0 = L\mu_+^0$. □

B Counterfactual Policy Predictions under Normalization of Period Utility Functions

For simplicity, we focus on the case where state transition matrices and discount factors are known and time invariant. Let F^0 and F^1 be the state transition matrices that generate the observed data. And let δ be discount factor. Consider a counterfactual experiment that changes state transition matrices but do *not* change per period utility functions. Let F_c^0 and F_c^1 the state transition matrices under counterfactual experiment. We are interested in predicting the counterfactual CCP.

B.1 Consequence for stationary DPDC models

Suppose the agent's dynamic programming problem is stationary. One way to identify the stationary DPDC model is to assume that μ^0 and the discount factor δ are known. Let $\mu^0 = (\mu^0(s_1), \ldots, \mu^0(s_{d_s}))^\intercal$ and $\tilde{\mu}^0 = (\tilde{\mu}^0(s_1), \ldots, \tilde{\mu}^0(s_{d_s}))^\intercal$ be two normalized period utility functions. Let F_c^0 and F_c^1 be counterfactual state transition matrices. Given $(\mu^0, \delta, F_c^0, F_c^1)$, we will have one counterfactual CCP $p_c(S)$. Similarly, the set $(\tilde{\mu}^0, \delta, F_c^0, F_c^1)$ also defines a counterfactual CCP $\tilde{p}_c(S)$. Write $p_c = (p_c(s_1), \ldots p_c(s_{d_s}))^\intercal$ and $\tilde{p}_c = (\tilde{p}_c(s_1), \ldots \tilde{p}_c(s_{d_s}))^\intercal$. The following proposition answers the question when will $p_c = \tilde{p}_c$, so the normalization of per period utility function μ^0 will be innocuous for predicting counterfactual policy effects.

Proposition B.1. *Define $B \equiv \delta F^{1/0}(I - \delta F^0)^{-1}$ and $B_c \equiv \delta F_c^{1/0}(I - \delta F_c^0)^{-1}$. One necessary condition for $p_c = \tilde{p}_c$ is that $(\mu^0 - \tilde{\mu}^0) \in \mathcal{N}(B - B_c)$, where $\mathcal{N}(B - B_c)$ is the null space of matrix $B - B_c$. One sufficient condition for $p_c = \tilde{p}_c$ is that $\mu^0 - \tilde{\mu}^0$ equals to a vector whose entries are identical, and this condition would also be necessary if* $\operatorname{rank}(B - B_c) = d_s - 1$.

Proof. By the definition of the ASVF $v^0 \equiv (v^0(s_1), \ldots, v^0(s_{d_s}))^\intercal$ and $v^1 \equiv (v^1(s_1), \ldots, v^1(s_{d_s}))^\intercal$, we have

$$v^1 - v^0 = \mu^{1/0} + \delta F^{1/0} v.$$

Also, it follows from equation (4.6),

$$v = \mu^0 + \delta F^0 v + \psi(p).$$

So we have

$$\begin{aligned} v^1 - v^0 &= \mu^{1/0} + \delta F^{1/0}(I - \delta F^0)^{-1}(\mu^0 + \psi(p)) \\ &= \mu^{1/0} + B(\mu^0 + \psi(p), \end{aligned}$$

with

$$B = \delta F^{1/0}(I - \delta F^0)^{-1}.$$

Similarly, the difference between the counterfactual ASVF of two alternatives, $v_c^1 - v_c^0$, is

$$\begin{aligned} v_c^1 - v_c^0 &= \mu^{1/0} + \delta F_c^{1/0}(I - \delta F_c^0)^{-1}(\mu^0 + \psi(p_c)) \\ &= \mu^{1/0} + B_c(\mu^0 + \psi(p_c)), \end{aligned}$$

with

$$B_c = \delta F_c^{1/0}(I - \delta F_c^0)^{-1}.$$

We know that $v^1 - v^0 = \phi(p)$ and $v_c^1 - v_c^0 = \phi(p_c)$. So we conclude

$$\phi(p) - \phi(p_c) = (B - B_c)\mu^0 + B\psi(p) - B_c\psi(p_c).$$

We have similar conclusion for using the per period utility functions $\tilde{\mu}^0$:

$$\phi(p) - \phi(\tilde{p}_c) = (B - B_c)\,\tilde{\mu}^0 + B\psi(p) - B_c\psi(\tilde{p}_c).$$

Hence, we have

$$\begin{aligned} \phi(\tilde{p}_c) - \phi(p_c) &= (\phi(p) - \phi(p_c)) - (\phi(p) - \phi(\tilde{p}_c)) \\ &= (B - B_c)(\mu^0 - \tilde{\mu}^0) - B_c\psi(p_c) + B_c\psi(\tilde{p}_c). \end{aligned}$$

In other words,

$$(\phi(\tilde{p}_c) - B_c\psi(\tilde{p}_c)) - (\phi(p_c) - B_c\psi(p_c)) = (B - B_c)(\mu^0 - \tilde{\mu}^0).$$

Define a mapping $g : \mathbb{R}^{d_s} \mapsto \mathbb{R}^{d_s}$ such that $g(p) \equiv \phi(p) - B_c\psi(p)$ for any $p \in \mathbb{R}^{d_s}$. We then have

$$g(\tilde{p}_c) - g(p_c) = (B - B_c)(\mu^0 - \tilde{\mu}^0). \tag{B.1}$$

By the mean value theorem for vector valued mappings, we have

$$\left(\int_0^1 \nabla g(p_c + \tau(\tilde{p}_c - p_c))\,\mathrm{d}\tau\right)(\tilde{p}_c - p_c) = (B - B_c)\,(\mu^0 - \tilde{\mu}^0).$$

Suppose $\tilde{p}_c = p_c$. We must have $(B - B_c)\,(\mu^0 - \tilde{\mu}^0) = 0$, that is $\mu^0 - \tilde{\mu}^0$ belongs to the null space of $B - B_c$. When $\operatorname{rank}(B - B_c) = d_s - 1$, the null space of $B - B_c$ contains only the vectors, whose elements are identical. This proves the necessary part of the proposition.

For any $\tilde{\mu}^0 = \mu^0 + a$, where a is a vector whose elements are all a, we have $\tilde{v} = v + (1-\delta)^{-1}a$ and $\tilde{v}_c = v_c + (1-\delta)^{-1}a$. Because $F^{1/0}v = F^{1/0}\tilde{v}$, we have $\mu^{1/0} = \tilde{\mu}^{1/0}$. Then we have $v_c^1 - v_c^0 = \tilde{v}_c^1 - \tilde{v}_c^0$ for $F^{1/0}v_c = F^{1/0}\tilde{v}_c$, which implies that $\tilde{p}_c = p_c$. This shows the sufficiency part. □

B.2 Consequence for DPDC models with finite horizon

Suppose the agent's dynamic programming problem has finite horizon, and the last sampling period T is the decision horizon T_*. Let $\mu_t^{\mathtt{0}}$ and $\tilde{\mu}_t^{\mathtt{0}}$ be two assumed per period utility of alternative 0. Let $p_{t,c}$ and $\tilde{p}_{t,c}$ be the counterfactual CCP vectors associated with the assumed per period utility functions $\mu_t^{\mathtt{0}}$ and $\tilde{\mu}_t^{\mathtt{0}}$. The following proposition answers the question when will $p_{T-1,c} = \tilde{p}_{T-1,c}$ ($p_{T,c} = \tilde{p}_{T,c}$ is always true). Of course, the proposition can be extended to cover the other periods at the expense of more complicated notation.

Proposition B.2. *Define $B \equiv \delta F^{\mathtt{1/0}}$ and $B_c \equiv \delta F_c^{\mathtt{1/0}}$. One necessary condition for $p_{T-1,c} = \tilde{p}_{T-1,c}$ is that $(\mu_T^{\mathtt{0}} - \tilde{\mu}_T^{\mathtt{0}}) \in \mathcal{N}(B - B_c)$, where $\mathcal{N}(B - B_c)$ is the null space of matrix $B - B_c$. One sufficient condition for $p_{T-1,c} = \tilde{p}_{T-1,c}$ is that $(\mu_T^{\mathtt{0}} - \tilde{\mu}_T^{\mathtt{0}})$ equals to a vector of which all entries are identical, and this condition would also be necessary if* $\operatorname{rank}(B - B_c) = d_s - 1$.

Proof. We first have $\phi(p_t) = \mu_T^{\mathtt{1/0}} = \tilde{\mu}_T^{\mathtt{1/0}}$, $v_T = \mu_T^{\mathtt{0}} + \psi(p_T)$ and $\tilde{v}_T = \tilde{\mu}_T^{\mathtt{0}} + \psi(p_T)$ for the last period T. Next, it follows from $v_{T-1}^{\mathtt{1}} - v_{T-1}^{\mathtt{0}} = \tilde{v}_{T-1}^{\mathtt{1}} - \tilde{v}_{T-1}^{\mathtt{0}}$ that

$$\mu_{T-1}^{\mathtt{1/0}} + \delta F^{\mathtt{1/0}} v_T = \tilde{\mu}_{T-1}^{\mathtt{1/0}} + \delta F^{\mathtt{1/0}} \tilde{v}_T,$$

which implies that

$$\mu_{T-1}^{\mathtt{1/0}} - \tilde{\mu}_{T-1}^{\mathtt{1/0}} = \delta F^{\mathtt{1/0}} \left(\tilde{\mu}_T^{\mathtt{0}} - \mu_T^{\mathtt{0}}\right).$$

Now consider the counterfactual experiment. We have $p_{T,c} = \tilde{p}_{T,c} = p_T$ because the counterfactual experiment does not change per period utilities. So $v_{T,c} = v_T$ and $\tilde{v}_{T,c} = \tilde{v}_T$. For period $T-1$, however, we have

$$\phi\left(p_{T-1,c}\right) = v_{T-1,c}^{\mathtt{1}} - v_{T-1,c}^{\mathtt{0}} = \mu_{T-1}^{\mathtt{1/0}} + \delta F_c^{\mathtt{1/0}} v_T$$

and

$$\phi\left(\tilde{p}_{T-1,c}\right) = \tilde{v}_{T-1,c}^{\mathtt{1}} - \tilde{v}_{T-1,c}^{\mathtt{0}} = \tilde{\mu}_{T-1}^{\mathtt{1/0}} + \delta F_c^{\mathtt{1/0}} \tilde{v}_T.$$

Then

$$\begin{aligned}
\phi\left(p_{T-1,c}\right) - \phi\left(\tilde{p}_{T-1,c}\right) &= \left(\mu_{T-1}^{\mathtt{1/0}} - \tilde{\mu}_{T-1}^{\mathtt{1/0}}\right) + \delta F_c^{\mathtt{1/0}} \left(v_T - \tilde{v}_T\right) \\
&= \delta F^{\mathtt{1/0}} \left(\tilde{\mu}_T^{\mathtt{0}} - \mu_T^{\mathtt{0}}\right) - \delta F_c^{\mathtt{1/0}} \left(\tilde{\mu}_T^{\mathtt{0}} - \mu_T^{\mathtt{0}}\right) \\
&= \delta \left(F^{\mathtt{1/0}} - F_c^{\mathtt{1/0}}\right) \left(\tilde{\mu}_T^{\mathtt{0}} - \mu_T^{\mathtt{0}}\right).
\end{aligned}$$

The above display is similar to equation (B.1). So we can apply the arguments in the proof of Proposition B.1 to prove the present proposition. □

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