Quasilinear approximations

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Abstract

This paper demonstrates that the canonical stochastic infinite horizon framework from both the macro and finance literature, with sufficiently patient consumers, is indistinguishable in ordinal terms from a simple quasilinear economy. In particular, with a discount factor approaching one, the framework converges to a quasilinear limit in its preferences, allocation, prices, and ordinal welfare. Consequently, income effects associated with economic policies vanish, and equivalent and compensating variations coincide and become approximately additive for a set of temporary policies. This ordinal convergence holds for a large class of potentially heterogenous preferences that are in Gorman polar form. The paper thus formalizes the Marshallian conjecture, whereby quasilinear approximation is justified in those settings in which agents consume many commodities (here, referring to consumption in many periods), and unifies the general and partial equilibrium schools of thought that were previously seen as distinct. We also provide a consumption-based foundation for normative analysis based on social surplus.

Key words:

JEL classification numbers: D43, D53, G11, G12, L13

1 Introduction

One of the most prevalent premises in economics is the ongoing notion that consumers behave rationally. The types of rational preferences studied in the literature, however, vary considerably across many fields. The macro, financial and labor literature typically adopt consumption-based preferences within a general equilibrium framework, while in Industrial Organization, Auction Theory, Public Finance and other fields, the partial equilibrium approach using quasilinear preferences is much more popular.

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These two approaches, partial and general equilibrium, each have their own advantages and disadvantages. The general equilibrium model uses natural preferences, which are defined over primitive consumption. On the other hand, however, the framework is highly complex and income effects may result in ill-behaved consumer demands and multiple equilibria. Most importantly, the framework does not define a preference-based welfare index that can consistently quantify the effects of various economic policies. In particular, the two classic ordinal indices, equivalent and compensating variations, are not additive on a set of policies.[1] As a result, normative predictions depend on the assumed *status quo* policy, and the order in which such policies are implemented. Typically the two indices also give divergent predictions. These problems, however, vanish in the partial equilibrium setting with quasilinear preferences. Here, the income effects are null, the law of demand always holds, and the equilibrium is unique. Moreover, each policy is associated with a surplus, a real number such that for any pair of policies, equivalent variation is given as the difference between the corresponding values. Consequently, the welfare index is additive. Nonetheless, the quasilinear framework assumes the existence of a special commodity, interpreted as money, that enters utility linearly. This formulation is often informally interpreted as a shorthand for the opportunity cost of money that can be utilized beyond the specifically considered markets.

In this paper we ask whether these two approaches can be reconciled; in particular, we present conditions under which a quasilinear model is a good approximation of a general equilibrium framework with consumption-based preferences. The main result of this paper can be summarized as follows. Consider a stationary, infinite horizon, complete market economy with I consumers with (potentially heterogeneous) preferences as represented by von Neumann-Morgenstern utilities

$$E\sum_{t=1}^{\infty}\beta^t u^i(x_t) \text{ for } i=1,...,I.$$

Suppose the policies are temporary, i.e., they affect fundamentals, such as consumers endowments in finite time $T<\infty$. We then show that under natural assumptions on the preferences, when consumers become patient, this framework becomes indistinguishable from its quasilinear counterpart in ordinal terms. More precisely, when the discount factor approaches one, preferences over consumption in the periods $t\leq T$ and savings (which we call *reduced-form preferences*), equilibrium allocation and prices during these periods, as well as the welfare effects of economic policies as measured in ordinal terms (i.e., in terms of equivalent variation), all converge to those predicted by a quasilinear economy with preferences

[1] We say that a welfare index is additive if, for any three policies p, p', and p'', the sum of welfare effects of p' relative to p and p'' relative to p' coincides with the welfare effect of policy p'' relative to p.

obtained from

$$\lambda^{i*}x_0^i + E\sum_{t=1}^{T}\beta^t u^i(x_t) \text{ for } i = 1, ..., I.$$

where scalars $\{\lambda^{i*}\}_{i=1}^{I}$ can be uniquely derived from the primitives of the infinite horizon economy.

The significance of our result is the following. The canonical framework from the macro and financial literature—an infinite horizon economy with heterogenous consumers—for a discount factor close to one acquires all the desirable ordinal properties of the quasilinear framework. In particular, income effects vanish and welfare indices become additive. More generally, we identify conditions under which a quasilinear model constitutes a good approximation of the consumption-based framework, thereby bridging the two seemingly unrelated approaches that are at the heart of economic analyses. Finally, we also offer a consumption-based foundation for the normative analysis based on social surplus.

In the considered problem, as the discount factor approaches one, cardinal utilities become unbounded, and hence they do not converge. The underlying consumers' reduced-form preferences, however, continuously transform into the well-behaved quasilinear limits. One of the conceptual contributions of this paper, therefore, is the formalization of such a continuity of a parametric family of preferences, which we call *ordinal joint continuity*. We also demonstrate that this property is satisfied in our applications and derive its implications for both choice and preference-based welfare.

An important comparative statics result in mathematical economics, the maximum theorem[2] (Claude Berge, 1959) shows that in a maximization problem, whenever utility function is jointly continuous (in alternatives and parameters) and a budget correspondence is continuous, then the value function of the program is continuous as well. The usefulness of this result in normative analyses is partially limited by the fact that it characterizes a cardinal object that does not have any welfare interpretation in contemporary economics. Thus, the main technical contribution this paper advances is a modern, ordinal, variant of the maximum theorem that formulates assumptions exclusively in terms of underlying preferences and, further, demonstrates the continuity of equivalent variation that is also measurable with respect to preferences. Since equivalent variation is formally defined as a minimal distance between certain contour sets, we call the result an *ordinal minimum theorem*. We also demonstrate by means of example that the assumptions of the Berge theorem, such as representation by a jointly continuous utility or continuity of budget correspondence, are insufficient for our result to hold true.

We are not the first to seek a foundation for quasilinear preferences. Already in *Principles of Economics*[3] Alfred Marshall argued that *"... expenditure on any one thing, as, for*

[2]See [Berge, 1963], pp. 115-117, and [Ok, 2007], p. 306 for the formal exposition of the theorem.
[3]See [Marshall, 1895] p. 12.

instance, tea, is only a small part of his whole..." He further pointed out that, with many markets, policies that affect only one can have only negligible impact on the marginal utility of money, since the resulting monetary outflows from the market are spread over many other commodities. This conjecture was then formalized by [Vives, 1987] who considered a single agent consuming many commodities with income proportional to the number of commodities. Vives showed that, as the number of commodities grows to infinity, the limit of marginal utility of money, as given by the Lagrangian multiplier in the optimization program, becomes invariant to changes in income and, thus, income effect for each commodity vanishes. Therefore, in each market a consumer increasingly behaves as if he or she has maximized a quasilinear utility.

Vives' formalization of Marshallian conjecture offers useful insights regarding the marginal utility of money, as well as the choice of an individual good in each market. Its limitation stems from the so-called fallacy of composition.[4] Even if a distortion associated with income effects disappears market-by-market, the overall welfare effects of policies may remain substantial even at the limit, since the number of markets goes to infinity. Also, the experiment involves adding new commodities, by which it changes the consumption space for a consumer, making policies not comparable. For these two reasons the result is insufficient to conclude that there is convergence of a consumption-based framework to a quasilinear limit in terms of preferences, choice or welfare predictions.

The rest of this paper is organized as follows. In Section 2 we introduce key ideas within a simple example. In Section 3, within an abstract problem, we develop mathematical tools for ordinal continuity, including an ordinal minimum theorem. We then use these tools in Section 4 to demonstrate quasilinear approximation for a problem of a trader in financial markets and then in Section 5 for a complete market setting with many such traders. These applications are central in the macroeconomics and financial literature. The mechanism identified in this paper, however, operates more broadly in markets with frictions. In Section 6, we discuss the applecations of our results to these more challenging settings.

2 A Motivating Example

We first introduce the key ideas in a simple example, wherein we derive our formulas in closed forms. Consider a consumer within a deterministic infinite-horizon setting, whose preferences over consumption flows $x^i = \{x^i\}_{t=1}^{\infty}$ admit utility representation $U^i\left(x^i, \beta\right) = \sum_{t=1}^{\infty} \beta^t \ln(x_t^i)$. Suppose that, among all the available policies, there are two that affect fundamentals in period $t = 1$. Under the *status quo* (or factual) policy, the price of consumption and the endowment are $\zeta_1 = 1$, and $e_1^i = 1$ respectively. The counterfactual policy alters these values

[4] For the detailed discussion see [Mas-Colell et al., 1995], p. 89.

to $\zeta_1' = 1/2$ and $e_1^{i\prime} = 2$. In the subsequent periods, the price of consumption in $t \geq 2$ is β^{t-2} and endowment is 2 for both policies. We denote factual policy by $p = (\zeta_1, e_1^i)$ and the counterfactual policy by $p' = (\zeta_1', e_1^{i\prime})$.

The fundamental assumption in modern economics is that satisfaction from the consumption of goods cannot be measured effectively in cardinal units. How then can one quantify the welfare effect brought about by the counterfactual scenario within the ordinal framework of a consumer choice? Fix a consumption flow $d = \{d_t\}_{t=1}^{\infty}$ that defines a welfare unit, *a welfare numeraire*. A variant of the preferences-based index introduced by [Hicks, 1939], an equivalent variation, $EV^i_{p,p',d}$, gives a sufficient transfer of d, that makes the factual policy equally as attractive as the counterfactual one. Thus defined index is expressed in real terms, and similarly to optimal choice, that index is invariant to either utility or price normalizations.

An alternative index, *compensating variation* $CV^i_{p,p',d}$, informs how much of flow d a consumer is willing to sacrifice in order not to return to the factual policy once the counterfactual policy is implemented. In terms of equivalent variation, the index is then given by $CV^i_{p,p',d} = -EV^i_{p',p,d}$. The two indices offer a simple test of welfare additivity. This property implies that a round trip from p to p' and back yields a zero welfare change, $EV^i_{p,p',d} = -EV^i_{p',p,d}$. Consequently, the equivalent and compensating variation are equal to each other.

In the example, optimal choice and equivalent variation can be derived in a simpler reduced-form setting with a two-dimensional commodity space. For this purpose, we break down the original problem into the choice of consumption x_1^i and savings, $x_0^i \equiv \zeta_1(e_1^i - x_1^i)$, and the optimal consumption flow in periods $t \geq 2$, conditional on x_0^i. For the latter problem, the value function

$$v^i(x_0^i, \beta) \equiv \max_{\{x_t^i\}_{t\geq 2}} \sum_{t\geq 2} \beta^t \ln(x_t^i) : \sum_{t\geq 2} \beta^{t-2}(x_t^i - e_t^i) \leq x_0^i, \tag{1}$$

is well-defined whenever the borrowing constraint given by $x_0^i > \underline{x}_0^i \equiv -\sum_{t\geq 2} \beta^{t-2} e_t^i$ is satisfied. Note that since policies do not affect prices and endowments after period one, the value function (1) is the same for both policies.

The reduced-form preferences over tuples (x_0^i, x_1^i), as represented by function $\tilde{U}^i(x^i, \beta) \equiv v^i(x_0^i, \beta) + \beta \ln(x_1^i)$, are sufficient for period one consumption and welfare in the infinite horizon example. More precisely, equivalent variation in the reduced-form problem, measured in terms of money x_0, coincides with the analogous index in the infinite horizon problem, as expressed in the price numeraire, in the example given by consumption in period 2.

Utility function can be derived in a closed form. It takes a Cobb-Douglass form on a shifted domain

$$\tilde{U}^i(x^i, \beta) \equiv \alpha \ln(x_0^i - \underline{x}_0^i) + \beta \ln(x_1^i) + \gamma, \tag{2}$$

where the respective coefficients are given by $\alpha = \beta^2/(1-\beta)$ and $\gamma = \beta^2 \ln(1-\beta)/(1-\beta)$.

The borrowing bound is $\underline{x}_0^i = -2/(1-\beta)$. The associated indifference curves, as depicted in Figure 1, are proportional expansions along tangency rays emanating from origin $(\underline{x}_0^i, 0)$. Figure 2 shows budget sets, optimal choices that correspond to the assumed policies, income and welfare effect derived for $\beta = 0.5$. It is clear that for the homothetic preferences, the income effect is non-zero and equivalent and compensating variations offer different values. Therefore preference-based welfare is not additive, despite an infinite number of commodities, (i.e., consumption in an infinite number of periods). This extends to an arbitrary value of a discount factor. Table 1 summarizes the optimal choices for the two policies, income effects and welfare indices, for the different values of β.

Table 1.

	$\beta = 0.5$	$\beta = 0.7$	$\beta = 0.9$	$\beta = 0.99$	Q
x_p^i	$(-1.5, 2.5)$	$(-1.3, 2.3)$	$(-1.1, 2.1)$	$(-1.01, 2.01)$	$(-1, 2)$
$x_{p'}^i$	$(-1.5, 5)$	$(-1.3, 4.6)$	$(-1.1, 4.2)$	$(-1.01, 4.02)$	$(-1, 4)$
IE	1.46	0.86	0.28	0.02	0
EV^i	2.07	1.77	1.5	1.4	1.39
CV^i	1.46	1.44	1.4	1.39	1.39
%	29%	18%	6%	0.7%	0

Note: The first two rows offer optimal choices under two policies, the third row gives the income effect. The next two rows are welfare indices and the bottom row indicates the gap between the two indices in percentage terms, $\% \equiv (EV^i - CV^i)/EV^i$.

Next, consider a thought experiment in which a consumer becomes infinitely patient. When the discount factor approaches one, coefficient γ in function (2) goes off to infinity, and the utility attained under each policy becomes unbounded. Therefore, the problem does not permit meaningful comparisons in terms of limit cardinal utilities. This issue, however, does not preclude normative analyses in terms of a preference-based index of equivalent variation, as preferences converge to well-behaved limits. To see this, observe that borrowing bound $\underline{x}_0^i$ increases in absolute terms to infinity. As a result, the origin of the domain shifts to the left, and on the relevant part of the domain (i.e., the part near the budget lines where are optimal choices) tangency rays effectively become flat. The indifference curves become parallel displacements of each other along a horizontal axis, as in a quasilinear problem. Indeed, Figure 1 illustrates how the homothetic indifference curves change continuously in β, and with a discount factor sufficiently high, on a fixed compact set these curves can be made arbitrarily close to the counterparts derived from utility

$$\tilde{U}^i(x^i, 1) \equiv \frac{1}{2}x_0^i + \ln(x_1^i). \tag{3}$$

Indeed, in our figure the indifference curves for $\beta = 0.99$ are essentially indistinguishable from the analogous curves derived from function (3). In this sense, quasilinear preferences

(3) constitute a good approximation of the reduced-form preferences with a discount factor close to one.

The alignment of preferences in turn gives rise to a convergence of optimal choices and equivalent variation. The corresponding values for the quasilinear preferences are reported here in the last column of Table 1. As we show in Figure 3, in the infinite horizon problem with a patient consumer, (i.e., for $\beta = 0.99$) income effects are negligible, and the alternative welfare indices give almost identical predictions.

More generally, in the example with a sufficiently high discount factor, equivalent variation is approximately additive on an arbitrary set of temporary policies. For any pair $p = \{e_1^i, \zeta_1\}$ and $p' = \{e_1^{i\prime}, \zeta_1'\}$, and any non-negative welfare numeraire $d = \{d_t\}_{t=2}^{\infty} \neq 0$ that has finite limit market value $\lim_{\beta \to 1} \sum_{t=2}^{\infty} \beta^{t-2} d_t < \infty$, the limit of equivalent variation is given by

$$\lim_{\beta \to 1} EV_{p,p',d}^i(\beta) = S_d^i(p') - S_d^i(p), \tag{4}$$

where function $S_d^i(\cdot)$ is given by the surplus function obtained from the quasilinear limit (3), and then normalized by the value of numeraire, $S_d^i(p) = [e_1^i \zeta_1 - 2\ln(\zeta_1)] / \sum_{t=2}^{\infty} d_t$. Note that the numerator and the denominator of the surplus function are homogenous of degree one with respect to prices and their ratio is not affected by price normalizations;[5] the limit surplus in the infinite horizon setting is then expressed in real terms.

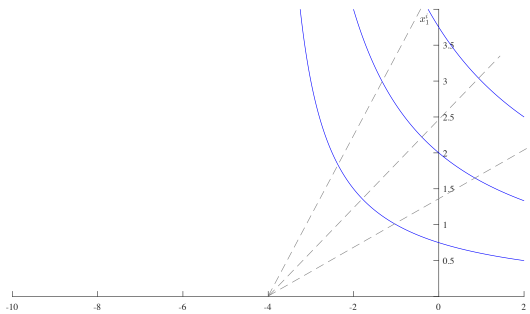
To summarize, in the infinite horizon example with $\beta < 1$, income effects are strictly positive, and equivalent variation is not additive. However, with the discount factor approaching one, the problem converges to the quasilinear one in terms of reduced-from preferences. This property in turn implies a convergence of consumption choice and equivalent variation. Consequently, income effects vanish, and welfare indices have a common additive limit, such that (1) for any pair of policies, the limit is finite (2) typically, it is non-zero, and (3) it can take an arbitrary value, depending on the policy pairs. Therefore, unless one can measure the cardinal utility of a consumer, the infinite horizon problem with a patient consumer is effectively indistinguishable from the quasilinear problem in terms of observables and normative predictions.

The economic intuition behind the quasilinear approximation is closely related to the Marshallian conjecture discussed in the introduction. With an infinite number of consumption goods, assumed policies affect only one market for consumption in period one. Even though the considered policies induce different savings x_0^i, the latter is spread over consumption during many other periods and, as a result, their effect on consumption in each

[5]Suppose in the example for policy p the prices are normalized by factor $c > 0$. The limit quasilinear preferences for the reduced-form problem, represented by $\tilde{U}^i(x^i, 1) = \frac{1}{2c} x_0^i + \ln(x_1^i)$, define surplus $e_1^i \zeta_1 c - 2c \ln(\zeta_1)$ while the limit market value of flow d is $c \sum_{t=2}^{\infty} d_t$. The ratio of the two is given by the surplus from the example, with $c = 1$.

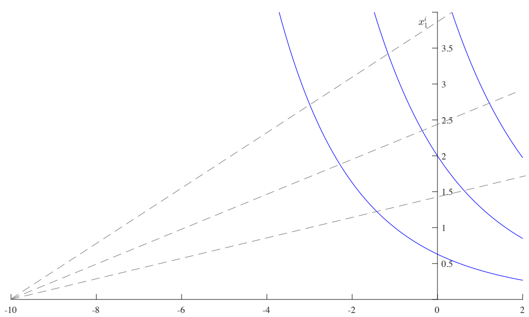
Figure 1. Convergence of the reduced-form preferences

A)



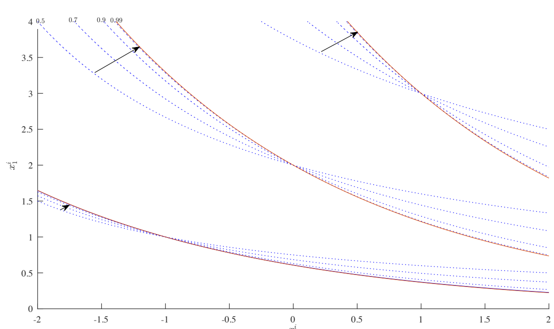
Note: The figure depicts the indifference curves for $\beta = 0.5$, passing thorough points $(-1, 1), (0, 2)$ and $(1, 3)$. The associated homothetic curves are proportional expansions along rays with origin $(-4, 0)$.

B)



Note: The figure shows the indifference curves for $\beta = 0.8$, passing thorough points $(-1, 1), (0, 2)$ and $(1, 3)$. The associated homothetic curves are proportional expansions along rays with origin $(-10, 0)$.

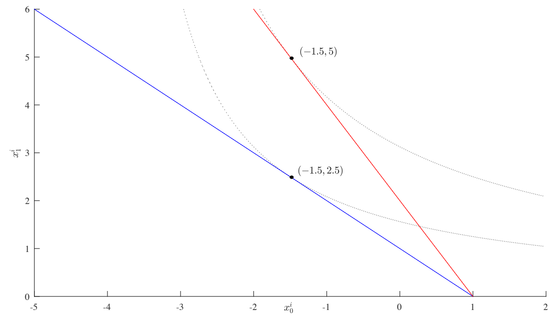
C)



Note: The figure demonstrates the evolution of the respective indifference curves in the reduced-form problem and convergence to quasilinear limit on the part of the domain that is relevant for the consumer's choice and welfare. The dotted (blue) curves represent the indifference maps for the values of a discount factor of 0.5, 0.7, 0.9 and 0.99. The solid (red) indifference curves are derived from the quasilinear utility function (3).

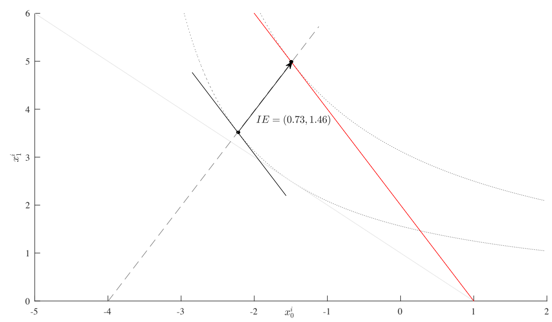
Figure 2. Choice, income effects and welfare ($\beta = 0.5$)

A)



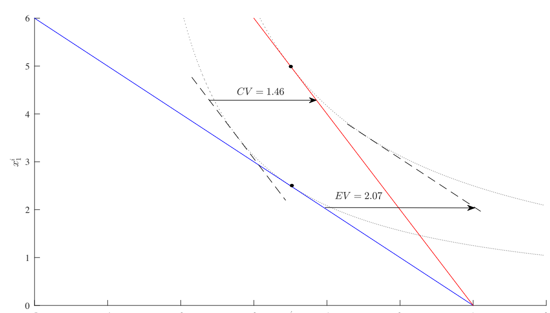
Note: Figure depicts the factual budget set (blue line) and the counterfactual one (red line) along with the optimal choices and the corresponding indifference curves.

B)



Note: Figure shows income effect, geometrically represented by the change in consumption in counterfactual scenario (red line) relative to the auxiliary scenario that preserves factual level of utility at counterfactual prices (black line). Increase in income results in a parallel shift of the auxiliary budget line, and optimal choice shifting along one of the tangency rays (dashed line), with strictly upward-slopping rays. Thus, for both goods income effects are strictly positive.

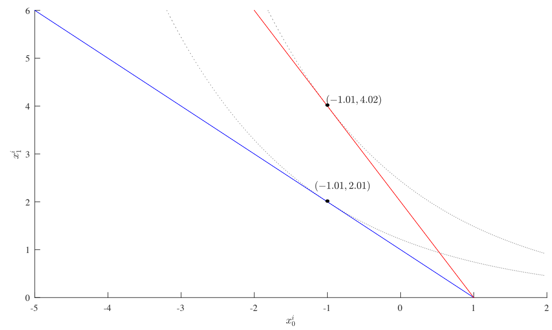
C)



Note: Equivalent variation is represented by the horizontal shift of the factual budget set sufficient to attain the counterfactual indifference curve. Compensating variation is a leftward shift of the counterfactual budget line that makes it tangent to the factual indifference curve. For the homothetic preferences, these two distances do not coincide.

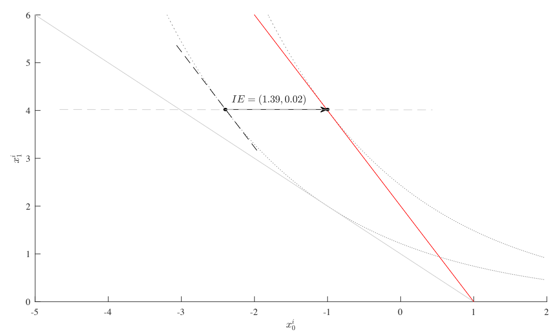
Figure 3. Choice, income effects and welfare ($\beta = 0.99$)

A)



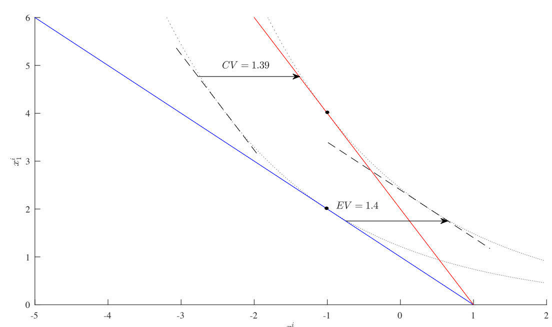
Note: Figure depicts the factual budget set (blue line) and the counterfactual one (red line) along with the optimal choices and the corresponding indifference curves.

B)



Note: Figure shows income effect, geometrically represented by the change in consumption in counterfactual scenario (red line) relative to the auxiliary scenario that preserves factual level of utility at counterfactual prices (black line). Increase in income results in a parallel shift of the auxiliary budget line, and optimal choice shifting along one of the tangency rays (dashed line). With approximately flat rays, for consumption in period $t = 2$ income effect is effectively zero.

C)



Note: Equivalent variation is represented by the horizontal shift of the factual budget set sufficient to attain the counterfactual indifference curve. Compensating variation is a leftward shift of the counterfactual budget line, that makes it tangent to the factual indifference curve. For nearly quasilinear preferences on the relevant part of the domain, these two distances are almost identical.

individual period $t > 1$, and hence on the marginal utility of money, is negligible. This mechanism gives rise to ordinal convergence. The example qualifies this conjecture by demonstrating that a large number of commodities *per se* is not sufficient for the quasilinear approximation, as the latter does not hold for moderate values of β. Instead, one needs a large number of non-negligible commodities in the consumer's budget. In our example, this requirement is satisfied for a discount factor that is close to one.

3 Ordinal Joint Continuity

In this section, we develop a mathematical toolbox related to the ordinal comparative statics that relies on the notion of continuity of a family of preferences.

3.1 A Decision-maker Problem

Consider an abstract problem of decision-maker i, characterized by a parametric family of preferences $\{\succeq_\theta^i\}_{\theta\in\Theta}$ over the set of alternatives $X^i \subset \mathbb{R}^N$ where N can be finite or infinite and $\Theta \subset \mathbb{R}^M$ is a parametric space. Each policy $p \in \mathbb{P}$ determines a subset of available alternatives that can potentially depend on θ. Policy p is represented by a budget correspondence denoted by $B_p^i : \Theta \to \mathbb{R}^N$. For each p, we can define choice correspondence in a standard way,[6] namely, as

$$x_p^i(\theta) \equiv \left\{x^i \in B_p^i(\theta) \cap X^i | x^i \succeq_\theta^i y^i \text{ for all } y \in B_p^i(\theta) \cap X^i\right\},$$

and the maximal upper contour set attained under policy p is

$$\bar{\Psi}_p^i(\theta) \equiv \left\{y^i \in X^i | y^i \succeq_\theta^i x^i \text{ for some } x^i \in x_p^i(\theta)\right\}.$$

Fix welfare numeraire $d \in \mathbb{R}_+^N$, such that $d \neq 0$. Recall that equivalent variation is a minimal (possibly negative) transfer of numeraire, which makes the set of alternatives under factual policy $B_p^i(\theta)$ equally attractive as the counterfactual set $B_{p'}^i(\theta)$. In an abstract setting, this definition can be formalized as follows. For any θ and $p, p' \in \mathbb{P}$, equivalent variation is given by the solution to the following program

$$EV_{p,p',d}^i(\theta) \equiv \min_{z^i \in X^i, \tau\in\mathbb{R}} \tau, \tag{5}$$

[6] Note that a budget set is defined on a set that is larger than the set of alternatives. For the optimal choice, our formulation is equivalent to the standard definition. However, this formulation allows for avoidance of the problems of corner solutions for the welfare index and, hence, it is more suitable for the considered problem.

subject to

$$z^i \in \bar{\Psi}^i_{p'}(\theta) \text{ and } z^i \in B^i_p(\theta) + \tau d.$$

By $z^i_{p,p',d}(\cdot)$ we denote correspondence that for each $\theta \in \Theta$ gives all the alternatives $z^i \in X^i$ for which tuple (z^i, τ) solves the program (5). Geometrically, equivalent variation is a minimal (signed) distance between the factual budget set $B^i_p(\theta)$ and the counterfactual upper contour set $\bar{\Psi}^i_{p'}(\theta)$, as measured along direction d.

Definition (5) reformulates the Hicksian notion in real terms and highlights the fact that the index is measurable with respect to preferences and sets of available alternatives. As such, it is not affected by any normalization of utility or prices. For d that coincides with the price numeraire (for the factual policy), our notion is equivalent to the standard money metric welfare. The index also coincides with the *consumption equivalent*, [Lucas, 1987], a preference-based measure of welfare commonly used in the macro literature that assumes d to be an aggregate consumption realized under factual policy.

3.2 Joint Continuity of a Family of Preferences

In the next two sections, we assume that the set of alternatives X^i and the parametric space Θ are compact and that N and M are finite. In the example found in Section 2, we have demonstrated that the reduced-form preferences transform continuously with a discount factor into the quasilinear limits. How can one formalize this ordinal continuity in an abstract problem by a decision-maker? Intuitively, a family of preferences is continuous in θ whenever the associated upper counter sets do not vary too much with the small perturbations of a parameter. More precisely, for any convergent sequence of that parameter, contour sets do not implode or explode in the limit. This property is captured by the continuity of a weakly-better-than-x^i correspondence, $\Psi^i(x^i, \theta) \equiv \{y^i \in X^i | y^i \succeq^i_\theta x^i\}$ that for any pair (x^i, θ) gives a collection of all alternatives that are at least as good as x^i with respect to preferences $\succeq^i_\theta$.

Definition 1. *Family of preferences $\{\succeq^i_\theta\}_{\theta\in\Theta}$ is jointly continuous on $X^i \times \Theta$ whenever associated correspondence Ψ^i is upper and lower hemicontinuous.*

Before we derive the implications of the notion of ordinal continuity for the comparative statics of choice and welfare, we find it insightful to relate our concept to the continuity of utility representation. Suppose a family of preferences can be represented by a function, $U^i : X^i \times \Theta \to \mathbb{R}$, that is jointly continuous in x^i and θ. It is then straightforward to show that the associated correspondence Ψ^i is upper hemicontinuous (see proof of Lemma 1, Step 1). Still, a family of preferences may fail to be jointly continuous according to our definition, due to a lack of lower hemicontinuity of this correspondence. To see this, consider a set of alternatives given by a box $X^i = \{x^i \in \mathbb{R}^2_+ | x^i \leq (4, 4)\}$ and parametric space

$\Theta = [0, 1]$. For a jointly continuous utility function

$$U^i(x^i, \theta) = (1 - \theta) \times \min(x_0^i, x_1^i) \tag{6}$$

at $x^i = (2, 2)$, and $\theta \in [0, 1)$ the upper contour set is $\Psi^i(x^i, \theta) = \{x^i \in X^i | x^i \geq (2, 2)\}$. At $\theta = 1$, this set discontinuously expands to the entire box X^i and correspondence Ψ^i is not lower hemicontinuous.

Note that the joint continuity of family $\{\succeq_\theta^i\}_{\theta \in \Theta}$ implies a continuity of preferences $\succeq_\theta^i$ in x^i, for each fixed value $\theta \in \Theta$. It then follows from Debreu's theorem that any individual member of a family necessarily admits a utility representation that is continuous in x^i. Whether the entire family admits a representation that is *jointly* continuous, however, remains an open question.

3.3 The Ordinal Minimum Theorem

A celebrated result in mathematical economics, the maximum theorem [Berge, 1963] shows that whenever utility function is jointly continuous and budget correspondence is continuous, the value function of the program itself is continuous as well and choice is upper hemicontinuous. In this section we offer a more modern, ordinal, variant of the theorem that demonstrates continuity of an equivalent variation. To this end, we make the following assumption regarding preferences and budget sets:

Assumption 1. *Family $\{\succeq_\theta^i\}_{\theta \in \Theta}$ is jointly continuous, and correspondence $B_p^i(\cdot) \cap X^i$ is continuous and non-empty for all p.*

Further, our theorem requires a well-behaved boundary of a budget set. For this purpose, we assume that budget correspondence is derived from a budget constraint, $B_p^i(\theta) \equiv \left\{x^i \in \mathbb{R}^N | b_p^i(x^i, \theta) \leq 0\right\}$ where function $b_p^i : \mathbb{R}^N \times \mathbb{R}^M \to \mathbb{R}$ is differentiable with the derivatives that satisfy the following assumption:

Assumption 2. *There exist strictly positive scalars $\underline{b}, \bar{b} \in \mathbb{R}_{++}$ such that $\bar{b} \geq \partial b_p^i / \partial x_n^i \geq \underline{b}$ for all n and $(x^i, \theta) \in X^i \times \Theta$.*

Under this assumption, function $b_p^i(\cdot, \cdot)$ is jointly continuous, strictly increasing, and has uniformly bounded slopes. We now state the ordinal minimum theorem.

Theorem 1. *(Ordinal Minimum Theorem): Suppose Assumptions 1-2 hold. Equivalent variation $EV_{p,p',d}^i(\cdot)$ is a well-defined continuous function on Θ, while correspondences $z_{p,p',d}^i(\cdot)$ and $x_{p'}^i(\cdot)$ are non-empty and upper hemicontinuous.*

At a high level of abstraction, the continuity of equivalent variation can be easily understood in terms of the geometry of contour sets, $\bar{\Psi}_{p'}^i(\theta)$ and of the function $b_p^i(\cdot, \theta)$. Given continuous weakly-better-than-x^i correspondence, for parameter values of θ close to $\bar{\theta}$, the sets $\bar{\Psi}^i_p(\theta)$ cannot be too different in terms of shape from their limit at $\bar{\theta}$. Analogously, budget sets are very similar to the limit set. It then follows that for $\theta \simeq \bar{\theta}$, the minimal distance between these two sets, measured along direction d, should not be too different from the same statistic evaluated at $\bar{\theta}$.

We next give examples to demonstrate that the assumptions of the maximum theorem are insufficient for the continuity of equivalent variation and that our assumptions are tight. In all the examples we assume $X^i = \{x^i \in \mathbb{R}^2_+ | x^i \leq (4,4)\}$ and $\Theta = [0,1]$ and we depict them in Figure 4. We first show that our result does not hold when the assumption of joint continuity of preferences is replaced by the representation by a jointly continuous utility. To accomplish this we consider a family of preferences represented by (6). In the previous section, we have demonstrated that this jointly continuous utility function defines correspondence Ψ^i that fails to be lower hemicontinuous. Consider constant budget correspondences, $B^i_p(\theta) = \{x^i \in \mathbb{R}^2 | x^i_0 + x^i_1 \leq 0\}$ and $B^i_{p'}(\theta) = \{x^i \in \mathbb{R}^2 | x^i_0 + x^i_1 - 4 \leq 0\}$ that satisfy Assumptions 1-2. Suppose welfare numeraire is given by commodity x^i_0, so that the equivalent variation is measured along horizontal axis. For all values $\theta \in [0,1)$, counterfactual choice is $(2,2)$; the upper contour set is $\bar{\Psi}^i_{p'}(\theta) = \{x^i \in X^i | x^i \geq (2,2)\}$, and the minimal horizontal distance between this set and factual budget set $B^i_p(\theta)$ is $EV^i_{p,p',d}(\theta) = 4$. For $\bar{\theta} = 1$ the upper contour set is given by box $\bar{\Psi}^i_{p'}(\bar{\theta}) = X^i$, and the distance discontinuously drops to $EV^i_{p,p',d}(\bar{\theta}) = 0$ (see Figure 4.A).

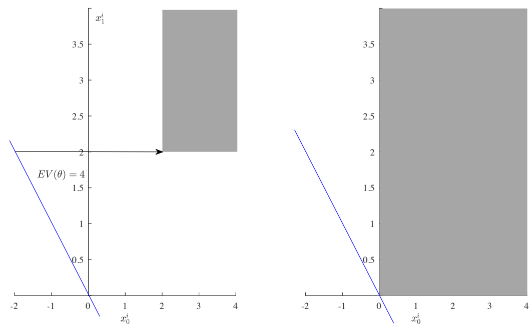
Continuity of the budget correspondence assumed by the maximum theorem is also by itself insufficient for the continuity of the welfare index, even with jointly continuous preferences. To see this aspect, consider a continuous factual budget correspondence $B^i_p(\theta) = \{x^i \in X^i | x^i_0 = 1 \text{ or } x^i_1 = 2\theta\}$ that defines cross-shaped budget sets. Let the counterfactual correspondence be given by a singleton $B^i_{p'}(\theta) = \{2,2\}$ and the family of the preferences be represented by utility function $U^i(x^i,\theta) = \min(x^i_0, x^i_1)$ for which the counterfactual upper contour set $\bar{\Psi}^i_{p'}(\theta) = \{x^i \in X^i | x^i \geq (2,2)\}$ is independent from the parameter value. As it is clear from Figure 4.B, for any $\theta \in [0,1)$ the minimal signed horizontal distance between the upper contour set and the factual budget set is $EV^i_{p,p',d}(\theta) = 1$ and the limit equivalent variation at $\bar{\theta} = 1$ discontinuously decreases to $EV^i_{p,p',d}(\bar{\theta}) = -2$. Assumption 2 eliminates such downward discontinuities of the welfare index.

Finally, equivalent variation may fail to exist in settings with jointly continuous preferences and smooth $b^i_p(\cdot,\cdot)$, for which derivatives are not uniformly bounded. To see this, in the previous example replace the factual budget set with the set derived from the budget constraint function

$$b^i_p(x^i,\theta) = \left\{ \begin{array}{c} -(x^i_0 - 3)(x^i_1 - 2) + 1 \text{ if } x^i < (3,2) \\ 1 \text{ otherwise} \end{array} \right\}.$$

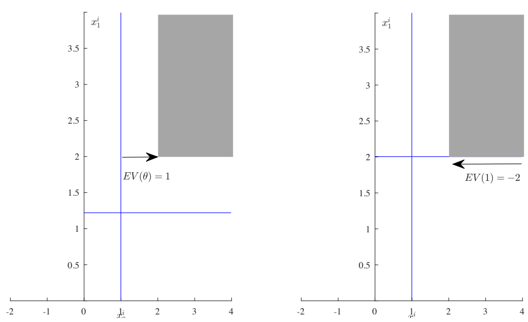
Figure 4. Failures of the Ordinal Minimum Theorem

A)



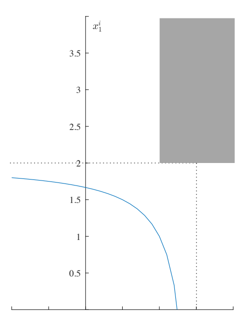
Note: The figure demonstrates the failure of the ordinal minimum theorem due to a lack of lower hemicontinuity of correspondence Ψ^i. The left panel shows the counterfactual upper contour set $\bar{\Psi}^i_{p'}(\theta)$ (shaded area) and the factual budget set $B^i_p(\theta)$ (the area south-west of the solid blue line) for $\theta \in [0, 1)$. The minimal horizontal distance between the sets is 4. The panel on the right shows the analogous sets for $\bar{\theta} = 1$. Due to the discontinuous "explosion" of the upper contour set, the distance between sets drops to zero.

B)



Note: The figure demonstrates the necessity of a smooth downward-slopping boundary of budget sets. The cross-shaped budget set consists of the horizontal and vertical line. For each $\theta \in [0, 1)$ the horizontal part of the budget set is below set $\bar{\Psi}^i_{p'}(\theta)$ and thus the minimal horizontal distance between the two sets is determined by the vertical part and is equal to one (left panel). In the limit $\bar{\theta} = 1$, the horizontal part of the budget set becomes relevant, and the signed minimal horizontal distance is -2.

C)



Note: The figure demonstrates the non-existence of the equivalent variation due to the unbounded derivatives of $b^i_p(\cdot, \theta)$. As it is clear from the picture, no horizontal shift of a budget set allows for the attainment of a point in $\bar{\Psi}^i_{p'}(\theta)$ and equivalent variation is not well-defined.

As we show in Figure 4.C, even though the budget set has a smooth boundary, with counterfactual upper contour set $\bar{\Psi}^i_{p'}(\theta) = \{x^i \in X^i | x^i \geq (2,2)\}$ an equivalent variation is not well-defined for any θ.

As a by-product, our proof also clarifies the implicit ordinal assumptions of the maximum theorem that underlie the upper hemicontinuity of the choice correspondence. In the Appendix, we show this result under the assumption that weakly-better-than-x^i correspondence is upper but not necessarily lower hemicontinuous (see proof of Lemma 4, Step 2).[7] As we argued in the previous section, this property is implied by jointly continuous utility representation, as assumed by the Berge theorem.

3.4 Local Extension of the Ordinal Minimum Theorem

The ordinal minimum theorem is not directly applicable to problems such as the one from Section 2. In the considered problem, consumption space is unbounded and it depends on a discount factor (implicitly, through a bound on borrowing $\underline{x}^i_0$). We next offer a simple, yet powerful, local extension of the theorem to settings with non-compact sets of alternatives. For this purpose we restrict our attention to problems with strictly convex preferences and convex-valued budget correspondences.

Fix $p, p' \in \mathbb{P}$ and some value of a parameter $\theta^* \in \Theta$, and let Θ^* denote some compact neighborhood of θ^*. Consider a family $\{\succeq^i_\theta\}_{\theta \in \Theta^*}$ where for each $\theta \in \Theta^*$ strictly convex preferences $\succeq^i_\theta$ are defined over the arbitrary domain $X^i(\theta) \subseteq \mathbb{R}^N$ that can potentially depend on θ. Let functions $b^i_p(\cdot, \theta)$ be quasi-convex in x^i for all $p \in \mathbb{P}$ and $\theta \in \Theta^*$. Suppose for θ^* that the equivalent variation is well-defined and attained on z^{i*}, and the optimal choice under p' is x^{i*}. Let $X^{i*} \equiv \left\{x^i \in \mathbb{R}^N | \underline{x} \leq x^i \leq \overline{x}\right\}$ for some $\underline{x}, \overline{x} \in \mathbb{R}^N$ be a box, such that choice x^{i*} and z^{i*} are in its interior and $X^{i*} \subset X^i(\theta)$ for all $\theta \in \Theta^*$.

Assumption 3. *Restriction $\Psi^i : X^{i*} \times \Theta^* \to X^{i*}$ is continuous.*

The next proposition extends the theorem further as follows:

Proposition 1. *Suppose that Assumptions 2-3 hold. There exists a neighborhood of θ^*, denoted by Θ^{**}, such that the equivalent variation $EV^i_{p,p',d}(\cdot)$, and correspondences $z^i_{p,p',d}(\cdot)$ and $x^i_{p'}(\cdot)$ are continuous functions on Θ^{**}.*

Suppose one can demonstrate the existence of an equivalent variation and an optimal choice for a particular value of a parameter (in our applications for a discount factor equal to one) and also show that, in a neighborhood of this parameter, the family of preferences is jointly continuous, locally, on a compact box around the choice and the equivalent variation.

[7] Strengthening this assumption to the continuity of a weakly-better-than-x^i correspondence does not suffice for the continuity of choice. This can be seen in the standard example of perfect substitutes.

The proposition shows that the choice and the welfare index are uniquely defined and they are continuous in some neighborhood.

The argument supporting the proposition consists of two steps. Lemma 10 verifies that the problem restricted to the box X^{i*} and parameter neighborhood Θ^* satisfies the assumptions of Theorem 1; hence, in the restricted problem the choice and welfare index exist and are continuous. Lemma 11 then shows that, for the parameter values sufficiently close to θ^*, the solutions for the restricted problem uniquely solve the unrestricted problems as well.

The uniqueness of the solution to program (5) can be established using an argument similar to the one for the uniqueness of choice. With convex budget sets and strictly convex preferences, any convex combination of two distinct solutions to Program (5) necessarily satisfies the optimization constraints, with one of the constraints not binding (i.e., z^i is in the interior of $\bar{\Psi}^i_{p'}(\theta)$). Therefore, one can reduce the value of the program without violating the constraint, resulting in a contradiction.

Until now, we have developed our theory exclusively in ordinal terms. We conclude this section with a simple test for the continuity of correspondence Ψ^i for a strictly monotonic family of preferences that uses a utility representation.

Lemma 1. *Suppose a parametric family of preferences defined over $X^{i*} \times \Theta^*$ admits utility representation $U^i : X^{i*} \times \Theta^* \to \mathbb{R}$ that is jointly continuous, and that for each $\theta \in \Theta^*$ the preferences are strictly monotone. Correspondence $\Psi^i : X^{i*} \times \Theta^* \to X^{i*}$ is thereby continuous.*

Thus, one can verify Assumption 3 by constructing a utility representation that is jointly continuous in (x^i, θ) on its domain. The lemma makes clear that, for a family of preferences that is represented by a jointly continuous utility function, such as e.g., (6), correspondence Ψ^i may fail to be lower hemicontinuous due to thick indifference sets that potentially appear in a limit. Such sets are ruled out by the assumption of a strict monotonicity of preferences.

4 A Trader in Financial Markets

Equipped with the necessary tools, we can now formalize the observations from our example. We first consider a problem of a single trader who is choosing a portfolio of securities in complete financial markets. In the subsequent section, we look at markets with many such traders.

4.1 A Trader Problem

Consider the problem of a trader in an infinite-horizon complete market setting. The trader's preferences are defined over random consumption flows $x^i = \{x^i_t\}_{t=1}^{\infty}$ that satisfy $x^i_t > \underline{x}^i$

for all $t \geq 1$ where $\underline{x}^i \in \mathbb{R}$ is some exogenous lower bound on consumption. The family of preferences, parametrized by a discount factor $\beta \in (0,1)$, is represented by the von Neumann-Morgenstern utility function $U^i(x^i, \beta) = E\sum_{t=1}^{\infty} \beta^t u^i(x_t^i)$, that satisfies standard assumptions; function $u^i : (\underline{x}^i, \infty) \to \mathbb{R}$ is twice continuously differentiable, strictly increasing, strictly concave and satisfying the Inada conditions, i.e., $\lim_{x_t^i \to \underline{x}^i} u^{i\prime}(x_t^i) = \infty$ and $\lim_{x^i \to \infty} u^{i\prime}(x_t^i) = 0$. To preserve the Euclidian structure of the problem with the standard metric topology, we make a technical assumption that, in each period t, collection of potential date-events (t, s) is finite, i.e., $s \in 1, 2, \ldots, S$.[8] The consumption space thus consists of a set of all integrable processes that give finite utility:

$$X^i(\beta) = \{\{x_t^i\}_{t=1}^{\infty} | x_t^i > \underline{x}^i \text{ for all } t \text{ and } U^i(x^i, \beta) \in \mathbb{R}\}.$$

It is well known that, with no arbitrage condition and market completeness, the prices of assets define a unique stochastic process $\zeta = \{\zeta_t\}_t$ that is sufficient for the value of any consumption flow; market value of flow $x^i \in X^i$ is given by $E\sum_{t=1}^{\infty} \beta^t \zeta_t x_t^i$. Realizations of the process ζ in respective date-events are given by Arrow prices, appropriately normalized by a discount factor and probability.[9] Therefore, without any loss of generality we can consider a trader to be choosing consumption flow, given the implicit prices of consumption ζ. In the financial application, the trader's budget set is fully determined by prices and endowment so that each policy is sufficiently summarized by a tuple $p = \{\zeta, e^i\}$. For any $p \in \mathbb{P}$, budget correspondence $B_p^i(\beta)$ is derived from the standard budget constraint $b_p^i(x^i, \beta) \equiv E\sum_{t=1}^{\infty} \beta^t \zeta_t (x_t^i - e_t^i) \leq 0$.

In the example we have considered here, policies affect the fundamentals in period one. In the general problem, policies can have effects on the arbitrary *finite* number of periods $T < \infty$, after which both prices and endowments are stationary and independent of any policy. Formally,

Assumption 4. *For each $\beta \in (0,1)$ asset prices admit a normalization, for which corresponding $\{\zeta_t\}_{t=T+1}^{\infty}$, as well as endowments $\{e_t^i\}_{t=T+1}^{\infty}$, are Markov chains that coincide for all p and β.*

We restrict our attention to Markov chains defined by an irreducible aperiodic symmetric stochastic matrix, so that they will have a unique stationary distribution.

Note that Assumption 4 is satisfied in the example from Section 2. Prices of consumption normalized by factor β^2 are given by $\zeta_t = 1$ for all $t \geq 2$; thus $\{\zeta_t\}_{t\geq 2}^{\infty}$ is a stationary Markov process independent from a discount factor or policy. In the subsequent section below we

[8] By this assumption, an economy truncated to any finite number of periods has a finite number of date events $n = (t, s)$ and consumption space has a structure of $\mathbb{R}^N$ with $N < \infty$; hence we can apply the ordinal minimum theorem.

[9] The Arrow price for a date-event (t, s) is given by $\zeta_{t,s}\pi_{t,s}\beta^t$.

show that this property arises endogenously in a competitive equilibrium for a large class of preferences.

In periods $t \leq T$, prices and endowments are affected by policies in an arbitrary way, i.e., they take the values $\zeta_{t,s} > 0$ and $e^i_{t,s} > \underline{x}^i$ for all t, s that can vary in p. The trader's choice defined in this section can thus be interpreted as a stationary problem perturbed by policies within a specific finite time horizon. Note, however, that even though policies affect fundamentals only temporarily, their effects on optimal consumption typically persist throughout the entire life span of a trader.

Let $\tilde{X}^i(\beta)$ be a collection of all consumption flows in $t \leq T$ and savings, $(x^i_0, \{x^i_t\}^T_{t=1})$ that satisfy borrowing constraint and $v^i(x^i_0, \beta)$ be the value function of the maximization program after period T. Following the steps in the example for each value of a discount factor, we can derive reduced-form preferences, $\succeq^i_\beta$ over $\tilde{X}^i(\beta)$ from representation $\tilde{U}^i(x^i, \beta) \equiv v^i(x^i_0, \beta) + E\sum^T_{t=1}\beta^t u^i(x^i_t)$. These preferences, along with the budget correspondence obtained from $\tilde{b}^i_p(x^i, \beta) \equiv x^i_0 + E\sum^T_{t=1}\beta^t\zeta_t(x^i_t - e^i_t)$ are sufficient for optimal consumption and welfare in the infinite horizon problem (Lemmas 13-15). In the reduced-form model, welfare $\tilde{EV}^i_{p,p',\tilde{d}}$ is measured in terms of money x^i_0.

4.2 Quasilinear Approximations (Trader)

The limits of reduced-form preferences, with the discount factor approaching one, can be derived as follows. Let λ^{i*} be a scalar that solves equality

$$E(\tilde{\zeta}u^{i\prime-1}(\tilde{\zeta}\lambda^i)) = E(\tilde{\zeta}\tilde{e}^i) \tag{7}$$

where $u^{i\prime-1}(\cdot)$ is the inverse of marginal utility and $\tilde{\zeta}$, and $\tilde{e}^i$ are stationary prices and endowment (for a given period) for the assumed Markov chain. This equation has a unique and strictly positive solution[10] that corresponds to the marginal utility of money, for which the optimal choice in a quasilinear model satisfies the budget constraint in a steady state, period-by-period. For $\beta = 1$, limit preferences $\succeq^i_1$ are represented by the quasilinear utility function

$$\tilde{U}^i\left(x^i, 1\right) = \lambda^{i*}x^i_0 + E\sum^T_{t=1}u^i(x^i_t). \tag{8}$$

In the deterministic example from Section 2, the stationary price is $\tilde{\zeta} = 1$, endowment is $\tilde{e}^i = 2$ and equation (7) reduces to $1/\lambda^{i*} = 2$, thereby giving rise to the utility function (3).

The next (well-known) lemma characterizes the choice and the equivalent variation in the quasilinear problem.

[10]The left-hand side of equation (7) is a continuous function strictly decreasing in λ^i with range $(E(\tilde{\zeta}\underline{x}^i), \infty)$. The right-hand side, a real number, is in this range by assumption that $\tilde{e}^i > \underline{x}^i$. It follows that a solution exists and is unique.

Lemma 2. *Fix $p, p' \in \mathbb{P}$. Equivalent variation is well-defined and attained on z^{i*}, as well as a choice for policy p', which is uniquely defined and given by x^{i*}. Moreover, there exists function $\tilde{S}^i : \mathbb{P} \to \mathbb{R}$ such that the equivalent variation satisfies $\tilde{EV}^i_{p,p',\tilde{d}} = \tilde{S}^i(p') - \tilde{S}^i(p)$.*

The limit preferences are well-behaved: The equivalent variation is additive, and the income effects for consumption in all date-events are zero.

Fix the arbitrary compact box $X^{i*} \equiv \{(x^i_0, \{x^i_t\}^T_{t=1}) | \underline{x} \leq x^i \leq \overline{x}\}$, such that x^{i*} and z^{i*} from Lemma 2 are in its interior. In the example, we show that, for certain values of β, some alternatives in box X^{i*} may fail to satisfy the borrowing constraint and, hence, they might not be ranked by the reduced-form preferences. The next proposition shows that with the discount factors being sufficiently high, the constraint is not binding and reduced-form preferences are well-defined on the entire set X^{i*}. They are also jointly continuous.

Proposition 2. *There exists threshold $\beta^{i*} \in (0, 1)$, such that weakly-better-than-x^i correspondence $\Psi^i : X^{i*} \times [\beta^{i*}, 1] \to X^{i*}$ derived from function $\tilde{U}^i(\cdot, \cdot)$ is continuous.*

The proposition formalizes the idea that, on the relevant part of a domain, given by box X^{i*}, reduced-form preferences transform continuously in a discount factor and converge to the quasilinear limits at $\beta = 1$.

The argument supporting the result can be easily understood within the example in Section 2. The borrowing constraint satisfies $\lim_{\beta\to1} \underline{x}^i_0(\beta) = \lim_{\beta\to1} -2/(1-\beta) = -\infty$, and, therefore, there exists threshold $\beta^{i*} < 1$ for which $\underline{x}^i_0(\beta) < \underline{x}_0$ for all $\beta \in [\beta^{i*}, 1]$ where $\underline{x}_0$ is the zero component of vector $\underline{x}$ that defines the box. Consequently $X^{i*} \subset X^i(\beta)$. Consider function $\tilde{V}^i : X^{i*} \times [\beta^{i*}, 1] \to \mathbb{R}$ to be defined as follows. For $\beta \in [\beta^{i*}, 1)$ the function is a monotonic transformation of $\tilde{U}^i(x^i, \beta)$,

$$\tilde{V}^i(x^i, \beta) \equiv \tilde{U}^i(x^i, \beta) - \frac{\beta^2}{1-\beta} \ln(2), \tag{9}$$

while for $\beta = 1$ it is $\tilde{V}^i(x^i, 1) \equiv \tilde{U}^i(x^i, 1) = \frac{1}{2}x^i_0 + \ln(x^i_1)$. It can be shown that function $\tilde{V}^i(\cdot, \cdot)$, which represents preferences Ψ^i, is jointly continuous on $X^{i*} \times [\beta^{i*}, 1]$.[11] Since associated preferences are strictly monotonic, by Lemma 1, the restriction Ψ^i is continuous. For general preferences the argument relies on similar logic. Construction of the jointly continuous representation is more involved and is delegated to the Appendix.

[11] For any sequence $x^{i,h}, \beta^h \to x^i, 1$, the limit of the function coincides with the value of the function at the limit. For $\beta < 1$ the utility function can be derived in closed form

$$\tilde{V}^i(x^i, \beta) = \frac{\beta^2}{1-\beta} \ln\left(\frac{1-\beta}{2}x^i_0 + 1\right) + \beta \ln(x^i_1).$$

For any fixed x^i and $\beta^h \to 1$ one can derive the limit by applying L'Hospital's rule. In the proof of Lemma 2 we show that this limit obtains for an arbitrary convergent sequence $x^{i,h}, \beta^h, \to x^i, 1$.

Finally, we employ the local variant of the ordinal minimum theorem to characterize choice and welfare. Consider a stationary infinite horizon problem parametrized by a discount factor. The next proposition characterizes the consumption within the first T periods.

Proposition 3. *For any policy $p \in \mathbb{P}$, there exists $\beta^{i**} \in (0,1)$, such that for all $\beta \in [\beta^{i**}, 1)$ optimal consumption under policy p is well-defined. Its truncation to $t \leq T$ periods along with savings is continuous in a discount factor and has finite limit x^{i*} as defined in Lemma 2.*

An immediate corollary of this proposition is that income effects in the reduced-form problem are negligible for sufficiently patient consumers. An analogous result holds for an equivalent variation as well. Let $d = \{d_t\}_{t=T+1}^{\infty} \neq 0$ be an arbitrary stochastic process with a finite limit value $E\sum_{t=T+1}^{\infty} \zeta_t d_t < \infty$ and let $S_d^i(p) \equiv \tilde{S}^i(p)/E\sum_{t=T+1}^{\infty} \zeta_t d_t$.

Theorem 2. *For any pair $p, p' \in \mathbb{P}$, there exists $\beta^{i***} \in (0,1)$, such that for all the values $\beta \in [\beta^{i***}, 1)$ equivalent variation is well-defined. It is continuous in β and it has a finite additive limit,*

$$\lim_{\beta \to 1} EV_{p,p',d}^i(\beta) = S_d^i(p') - S_d^i(p).$$

These results jointly demonstrate that, with a sufficiently high discount factor, the infinite horizon problem offers predictions regarding choice and welfare that are very close to the ones in the quasilinear model. As a result, the equivalent variation is nearly additive with the approximation error decreasing in β. The proofs for these results proceed as follows. We first construct reduced-form preferences and derive the quasilinear limit according to (8). By Lemma 2, the choice and equivalent variation for the limit preferences are well-defined. We can thus invoke Proposition 1 to establish continuity of choice and an equivalent variation for β close, but strictly smaller than one, as well as convergence to the quasilinear limits. Finally, we can use Lemmas 13-15, to demonstrate that the values derived in the reduced-form model coincide with the counterparts in the infinite horizon problem.

The limit surplus $S_d^i(\cdot)$ in the infinite horizon setting is not well-defined for all the numeraire flows d. The choice of numeraire is restricted in two ways. First, to eliminate the differential effects of the policies on market value of numeraire, the viable flows d have to give zero consumption in all date-events for the initial T periods. Also, the limit value of the numeraire has to be bounded, for otherwise, the numeraire would be infinitely more preferred relative to the effects of policies, and the equivalent variation measured in terms of d would vanish for all policy pairs. The second requirement rules out, for example, stationary welfare numeraire flows.

5 The Financial Markets

In economics, we are often interested in making normative predictions for an entire economy or a market. Can the quasilinear approximation for an individual consumer be extended to settings with many such consumers? The answer is 'yes' but under more stringent assumptions on preferences. The approximation does not hold with fully general preferences for the following reasons. For a single consumer, we could impose assumptions directly on prices and endowments. In the equilibrium framework, the first of these elements is determined endogenously through market clearing conditions, and our additional assumptions ensure that the required properties hold in equilibrium.

5.1 Example

Consider a deterministic infinite horizon economy with two consumers with endowments $\{e_t^i\}_{t=1}^{\infty}$ for $i = 1, 2$. Suppose two policies affect these consumers' endowments in period one, after which the endowments are constant and independent of policies. Normative predictions in an economy with von Neumann-Morgenstern preferences are associated with the following complications. First, a policy may give rise to multiple equilibria, in which case it is not clear which prices should be used to define the welfare index. Moreover, even if the equilibria are unique, the price effects of policies may persist throughout an infinite number of periods. Consequently, even though policies affect the exogenous fundamentals in period one, they are not temporary, and the economy does not admit a reduced-form representation.

These problems are not present for the homothetic preferences assumed in Section 2. Here, the competitive price of consumption in $t = 1, 2, ...,$ in terms of period-two consumption, is unique and given by $\beta^{t-2}e_2/e_t$ where $e_t \equiv \sum_{i=1,2} e_t^i$ is the aggregate endowment.[12] With constant endowments after $t \geq 2$, corresponding prices are as the ones assumed in the single agent example. We can thus construct a two-period economy with reduced-form preferences (2) that are sufficient for competitive equilibrium and welfare in the infinite horizon economy.

For the sake of concreteness, suppose that factual policy gives $e_1^i = 2$, while the counterfactual one is $e_1^{i\prime} = 4$ for both $i = 1, 2$ and in the remaining periods endowments are $e_t^i = 2$ for both policies. The equilibrium prices of consumption are $1/\beta$ and $1/2\beta$ for the factual

[12] To see this fact, note that a competitive equilibrium $(\{\bar{x}^i\}_i, \bar{\zeta})$ is efficient, and thus, there exist non-negative weights $\{\omega^i\}_{i=1,2} \neq 0$ such as the allocation solves $U(e) \equiv \max_{\{x^i\}_i} \sum_{i=1,2} \omega^i U^i$ subject to $\sum_{i=1,2} x^i \leq e$. Moreover, for any t, t', equilibrium relative prices coincide with the marginal rate of substitution derived from $U(e)$ i.e., $\zeta_t/\zeta_{t'} = MRS_{t,t'}^U$. Straightforward calculations reveal that $U(e) = \sum_{t=1}^{\infty} \beta^t \ln e_t + const$ and the weights enter the function only though a constant. It then follows that $MRS_{t,t'}^U$ and the relative prices are independent of weights and, hence, they must coincide across equilibria. It follows that the equilibrium is unique up to a price normalization.

and the counterfactual policy, respectively. Social welfare, measured as a sum of equivalent (compensating) variations for the two consumers, evaluated at equilibrium prices, is given in Table 2.

Table 2.

	$\beta = 0.5$	$\beta = 0.7$	$\beta = 0.9$	$\beta = 0.99$	Q
EV	6.63	4.40	3.19	2.81	2.77
CV	4.69	3.58	2.98	2.79	2.77
%	29%	19%	6.7%	0.6%	0

Note: The first two rows offer social welfare indices. The bottom row reports the gap between the indexes in percentage terms.

Similarly to a single trader, aggregate indices give divergent predictions, and equivalent variation is not additive. With patient consumers, however, the reduced form economy approaches the quasilinear limit with preferences (6) for $i = 1, 2$, and so does the equilibrium consumption and aggregate welfare indices, as reported in the last column of Table 2. More generally, for any pair of policies that potentially affect consumers asymmetrically, giving rise to factual aggregate endowment e_1, and the counterfactual one, e_1', the limit aggregate equivalent variation can be derived as the difference in surpluses, given by $S_d(e_1) = 4\ln(e_1)/\sum_{t=2}^{\infty} d_t$.

5.2 A Gorman Economy

We now consider a financial economy with $i = 1, 2, ..., I$ consumers with heterogenous von Neumann-Morgenstern preferences that satisfy the assumptions laid out in Section 4. However, traders share a common discount factor β. Budget correspondences are determined by policies that determine the profiles of endowments. The other determinant, prices, now emerges in a competitive equilibrium. In the reminder of this paper the policies are given by profiles $p = \{e^i\}_{i=1}^{I}$. As in the case of a single trader, policies are temporary; for some $T < \infty$ endowments $\{e_t^i\}_{t=T+1}^{\infty}$ are finite stationary Markov chains that are unaffected by the policies $p \in \mathbb{P}$ that satisfy assumptions from the previous section. For $t \leq T$, the endowments can take the arbitrary values $e_t^i > \underline{x}^i$ that can vary in policies.

Social welfare is measured as an *aggregate equivalent variation*, defined as $EV_{p,p',d}(\beta) \equiv \sum_i EV_{p,p^i,d}^i(\beta)$. Intuitively, welfare effects are quantified as a minimal transfer of numeraire d such that when appropriately redistributed among consumers, at prices observed in a factual equilibrium, all consumers are at least as well off under a factual policy as they are in the counterfactual equilibrium. Social welfare is measurable with respect to the profile of ordinal preferences, and it does not require any comparability of utilities among consumers. Moreover, the index is Paretian; whenever a counterfactual policy improves the preferences of all the consumers, the welfare effect is necessarily positive. $EV_{p,p',d}$ is independent of any normalization of equilibrium prices, that can vary across policies and discount factors. Finally, the index is affected by the choice of the welfare numeraire d only up to the scaling factor.

In order to assure well-behaved equilibrium prices, we make a further assumption regarding the profile of the traders' preferences that generalizes our assumption of homotheticity from the example.

Assumption 5. *For any $\beta \in (0,1)$, preferences $\{\succeq^i_\beta\}_{i=1}^I$ are in Gorman*[13] *polar form.*

Our assumption admittedly has strong implications for the demands: Wealth expansion paths for all consumers are parallel lines. Still, Gorman preferences include most of the specifications used in applied work in the macroeconomic and financial literature.[14] An important family that satisfies the assumption is defined by the instantaneous utility function that exhibits hyperbolic absolute risk aversion (HARA)

$$u^i(x^i_t) = \frac{1-\sigma}{\sigma}\left(\frac{a^i x^i_t}{1-\sigma} + \eta^i\right)^\sigma, \tag{10}$$

wherein values $a^i > 0, \eta^i$ can be heterogeneous and $\sigma < 1$ is common to all consumers.[15] By setting $a^i = 1-\sigma$ and choosing the appropriate bound η^i, the family specializes to the power utility function with arbitrary borrowing constraints or standard CRRA or CES preferences if we further assume $\eta^i = 0$. Logarithmic representation occurs for $a^i = 1$ and $\sigma \to 0$, while CARA for $\eta^i = 1$ as a limit $\sigma \to -\infty$.

The next proposition characterizes the competitive equilibrium in a financial economy with Gorman preferences and stationary endowments after period T.

Proposition 4. *Suppose Assumption 5 holds. Each policy $p \in \mathbb{P}$ defines a unique competitive equilibrium, up to price normalization, with prices satisfying Assumption 4.*

From the proposition it then follows that for each pair p, p' the two ingredients of equivalent variation, $B^i_p(\cdot)$ and $\bar{\Psi}^i_{p'}$ are unambiguously defined.

[13]Gorman preferences admit an indirect utility representation of the form $v^i\left(\zeta, w^i\right) = a^i(\zeta) + b(\zeta)w^i$. This assumption assures that consumers demands aggregate: aggregate demand does not depend on wealth distribution. Formally, let $x^i\left(\zeta, w^i\right)$ be the individual demand of consumer i given prices ζ and wealth w^i. For Gorman preferences holds $\sum_{i=1}^I x^i\left(\zeta, w^i\right) = \sum_{i=1}^I x^i\left(\zeta, w^i + \tau^i\right)$ for arbitrary profile of zero-sum transfers $\left\{\tau^i\right\}_{i=1}^I$.

[14]For the centrality of the Gorman assumption in normative analysis see e.g., [McFadden, 2004].

[15]Observe that HARA preferences are strictly increasing and strictly convex, and they define a lower bound on consumption $x^i_t > \underline{x}^i \equiv -\left(1-\sigma\right)\eta^i/a^i$ for all t for which they satisfy Inada conditions. Thus, the preferences satisfy the assumptions from Section 4.

5.3 Quasilinear Approximations (Economy)

With prices satisfying Assumption 4, we can define a reduced-form economy with $i = 1, ..., I$ traders that is sufficient for the allocation and prices in all date-events in $t \leq T$ and social welfare. This economy converges in terms of individual preferences to quasilinear limits (8), where scalars λ^{i*} are determined by (7) with respect to endogenous prices. The limit quasilinear economy for each policy defines a unique competitive equilibrium, and gives rise to the social surplus function that is measurable with respect to aggregate endowment.

Lemma 3. *For any policy $p \in \mathbb{P}$, the quasilinear economy defines a unique competitive equilibrium $\{x^{i*}\}_{i=1}^{I}, \zeta^*$. There exists function $\tilde{S}(e)$, such that any $p, p' \in \mathbb{P}$, the aggregate equivalent variation with respect to x_0, is well-defined and given by $\tilde{EV}_{p,p',\tilde{d}}(1) = \tilde{S}(e') - \tilde{S}(e)$.*

For the infinite horizon economy the quasilinear approximation results are as follows:

Proposition 5. *For any policy $p \in \mathbb{P}$, there exists threshold $\beta^{**} \in (0, 1)$, such that for all $\beta \in [\beta^{**}, 1)$ competitive equilibrium exists, and its truncation to periods $t \leq T$ along with savings is continuous in β on this interval, with limit, $\{x^{i*}\}_{i=1}^{I}, \zeta^*$ defined in Lemma 3.*

Testable implications of the competitive framework with patient consumers are arbitrarily close to the predictions of the quasilinear economy. The infinite horizon economy is also approximately equivalent in terms of normative indices. Let $S_d(e) \equiv \tilde{S}(e)/E \sum_{t=T+1}^{\infty} \zeta_t d_t$ be the aggregate surplus as measured in terms of flow $d = \{d_t\}_{t=T+1}^{\infty}$ that has a finite limit value, $E \sum_{t=T+1}^{\infty} \zeta_t d_t < \infty$. The following theorem is then proposed:

Theorem 3. *For any pair $p, p' \in \mathbb{P}$, there exists $\beta^{***} \in (0, 1)$, such that the aggregate equivalent variation $EV_{p,p',d}(\beta)$ is well-defined for all $\beta \in [\beta^{***}, 1)$. Moreover, the welfare index is continuous in β and has a finite additive limit,*

$$\lim_{\beta \to 1} EV_{p,p',d}(\beta) = S_d(e') - S_d(e).$$

With an appropriate choice of welfare numeraire, an infinite horizon Gorman economy defines a social surplus function that approximates well welfare effects. The function is given by $S(e) \equiv E \sum_{t=1}^{T} s(e_t)$ where $s(\bar{y}) \equiv \max_{\{y^i\}_i} \sum_i u^i(y) / \lambda^{i*} : \sum_i y^i \leq \bar{y}$ and hence is measurable with respect to aggregate endowment. Note that the instantaneous surplus function $s(\cdot)$ is smooth, strictly increasing, and strictly concave and hence it inherits all the properties of well behaved utility function. It follows that the function can be interpreted as utility function of an "approximate" normative representative consumer.

6 Discussion

In this paper we provide formal arguments for the convergence of a canonical infinite horizon framework with heterogenous preferences to quasilinear limits. We now discuss some of the extensions of our quasilinear approximation results.

6.1 Assumptions

We made several technical assumptions that simplify our notation, but are not critical for our results. For example, we have considered Markov chains that are defined by a stochastic matrix that is symmetric. This assumption assures that a matrix is diagonalizable with all real eigenvalues. This assumption is quite restrictive in terms of admissible stochastic processes. At some notational cost, our arguments can be straightforwardly extended to non-symmetric irreducible aperiodic matrices as well.[16]

We also assumed that instantaneous utility functions are time independent. Our results carry over to settings in which instantaneous utility functions $u_t^i(\cdot)$ are time dependent in periods $t \leq T$. Given this generalization, the infinite horizon economy converges to the quasilinear economy with preferences

$$U^{i*} = \lambda^{i*} x_0^i + E \sum_{t=1}^{T} u_t^i(x_t^i) \tag{11}$$

for each i. Convergence of the reduced form preferences does require, however, a time independence after horizon T, as this assumption allows for the elimination of instances in which the marginal utility of money oscillates without reaching a limit as the discount factor approaches one.

We further conjecture that our results hold in some form for economies outside of the Gorman class. Demonstrating this fact formally, however, requires some selection criteria in the case of multiple equilibria. Moreover, temporary policies may differentially affect prices within an infinite number of periods. Consequently, the value functions $v^i(\cdot)$ depend on the assumed policies, and the economy does not admit a reduced-form representation. Thus, our formal arguments do not directly extend to these settings.

[16]For such matrices, from the Perron-Frobenius theorem it follows that there exists an eigenvector with positive components corresponding to the eigenvalue equal to one. This eigenvector defines a unique stationary distribution. All other potentially complex-valued eigenvalues have a modulus less than one. Therefore, for the Jordan form of the transition matrix, all entries vanish except for the diagonal entry corresponding to the largest eigenvalue and the limit preferences are determined solely by the stationary distribution. This extension mimics the steps of the argument for the existence and uniqueness of a stationary distribution of the Markov processes.

6.2 Markets with Frictions

The approximation results do not rely on the details of assumed market structure or a solution concept to make predictions. In [Makarski and Weretka, 2018], we consider a variant of the infinite horizon economy with a single (riskless) asset. The framework is essentially a variant of an Aiyagari economy, a workhorse model with heterogenous consumers. Using the mathematical results developed in this paper, we show that an incomplete market economy admits a reduced-form representation that in ordinal terms also converges to a well behaved limit. The approximate social surplus, however, is not measurable with respect to the aggregate endowment. We calibrate the Aiyagari extension to determine the errors of quasilinear approximation and the speed of convergence for the standard policy experiments that are considered in the macroeconomic literature.

The quasilinear approximation also holds in noncompetitive settings. This finding can be seen by reinterpreting the example from Section 5.1 as a static economy with I consumers and an infinite number of monopolistic firms $t = 1, 2, ...$, each producing one perfectly divisible commodity. In this application coefficient β^t in the utility function $\sum_{t=1}^{\infty} \beta^t \ln(x_t^i)$ measures the importance of commodity t in the basket of commodities. In the Cournot-Nash equilibrium, each firm takes the quantities produced by other firms as given, and hence a firm affects only its own market. One can then show that, as all commodities become equally important (i.e., $\beta^t \to 1$), in the Nash equilibrium each individual firm faces a residual demand that increasingly resembles the demand obtained from the quasilinear preferences. This of course generalizes to settings in which each produces choses an arbitrary finite number of goods $T < \infty$.

6.3 Identification of Surplus from Market Level Data

In the companion paper [Weretka, 2018], we show that within a dynamic quasilinear economy the social surplus function can be non-parametrically recovered from the prices of a relatively small portfolio of aggregate securities, realized under a single policy (technically from a single observation of the equilibrium correspondence). This identification result for the social surplus is feasible even with time-dependent preferences of the form (11). Since the infinite horizon framework considered in this paper is indistinguishable in terms of prices, and social welfare, from its quasilinear limit, this identification straightforwardly carries over to the consumption-based infinite horizon setting.

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Appendices

A Ordinal joint continuity

A.1 The Ordinal Minimum Theorem

Proof of Theorem 1:

The proof of the theorem proceeds as follows. In Lemma 4 we demonstrate existence and continuity of choice correspondence. Lemma 5 shows that equivalent variation is well-defined for any parameter value and policy pair, and that it is attained on a compact set. Lemma 6 gives an auxiliary result that we use in Lemma 7 to demonstrate continuity of equivalent variation on a parametric space. Finally Lemma 8 concludes by showing continuity of $z^i_{p,p',d}(\cdot)$. Throughout the proof we fix p, p', d and use compact notation $EV^i(\cdot) = EV^i_{p,p',d}(\cdot)$ and $z^i(\cdot) = z^i_{p,p',d}(\cdot)$.

Lemma 4. *For any $p \in \mathbb{P}$ correspondence $x^i_p(\cdot)$ is non-empty and upper hemicontinuous.*

Proof of Lemma 4:

Step 1. Non-emptiness of $x^i_p(\cdot)$: Fix $\theta \in \Theta$. For any $x^i \in X^i$ sets $\left\{y^i \in X^i | y \succeq^i_\theta x^i\right\}$ and $\left\{y^i \in X^i | x^i \succeq^i_\theta y\right\}$ are closed for otherwise there would exist convergent sequences in X^i, $x^{i,h}, y^{i,h} \to \bar{y}^i, \bar{x}^i$ such that $x^{i,h} \succeq^i_\theta y^{i,h}$ but $\bar{y}^i \succ^i_\theta \bar{x}^i$, contradicting upper hemicontinuity of Ψ^i (that is equivalent to the property of a closed graph, given compact X^i). Thus preferences $\succeq^i_\theta$ are continuous for all $\theta \in \Theta$. By the Debreu representation theorem preferences $\succeq^i_\theta$ admit a continuous (in x^i but not necessarily jointly continuous in (x^i, θ)) utility representation. Set $B^i_p(\theta)$ is a preimage of a closed interval by a continuous function $b^i_p(\cdot, \theta)$, and hence it is closed. Thus $B^i_p(\theta) \cap X^i$ is compact. By Assumption 1 it is also non-empty. It then follows from the extreme value theorem that optimal choice is attained on $B^i_p(\theta) \cap X^i$ and choice correspondence $x^i_p(\cdot)$ is non-empty.

Step 2. Upper hemicontinuity of $x^i_p(\cdot)$: Consider convergent sequence $x^{i,h}, \theta^h \to \bar{x}^i, \bar{\theta}$ for which $x^{i,h} \in x^i_p(\theta^h)$. Since $B^i_p(\theta) \cap X^i$ is upper hemicontinuous, it has a closed graph and therefore $\bar{x}^i \in B^i_p(\bar{\theta}) \cap X^i$. Consider any alternative $\bar{y}^i \in B^i_p\left(\bar{\theta}\right) \cap X^i$ and arbitrary sequence $y^{i,h}, \theta^h \to \bar{y}^i, \bar{\theta}$. Since by Assumption 1 correspondence $B^i_p(\cdot) \cap X^i$ is lower hemicontinuous, there exists convergent subsequence $\left(y^{i,k}, \theta^k\right) \to (\bar{y}^i, \bar{\theta})$ such that $y^{i,k} \in B^i_p\left(\theta^k\right) \cap X^i$. By optimality $x^{i,k} \in x^i_p(\theta^k)$ and $y^{i,k} \in B^i_p\left(\theta^k\right) \cap X^i$ therefore one has $x^{i,k} \in \Psi^i\left(y^{i,k}, \theta^k\right)$. Since Ψ^i is upper hemicontinuous with compact range, it has closed graph and $\bar{x}^i \in \Psi^i\left(\bar{y}^i, \bar{\theta}\right)$. This in turn implies $\bar{x}^i \succeq^i_{\bar{\theta}} \bar{y}^i$. This is true for all $\bar{y}^i \in B^i_p\left(\bar{\theta}\right) \cap X^i$ and $\bar{x}^i \in x^i_p\left(\bar{\theta}\right)$. □

In the next lemma we show that equivalent variation is well-defined.

Lemma 5. *Equivalent variation exists and is uniformly bounded.*

Proof of Lemma 5:

Define closed interval $Q \equiv [\tau^-, \tau^+] \subset \mathbb{R}$ where the corresponding bounds are defined as $\tau^+ \equiv \max_{x^i, \theta \in X^i \times \Theta} b^i_p(x^i, \theta)/(\underline{b} \sum_{n=1}^N d_n)$ and $\tau^- \equiv \min_{x^i, \theta \in X^i \times \Theta} b^i_p(x^i, \theta)/(\bar{b} \sum_{n=1}^N d_n)$ and $\bar{b}, \underline{b} > 0$ are scalars from Assumption 2. By extreme value theorem bounds are attained on a compact set $X^i \times \Theta$ and hence they are finite. In addition since $\tau^- \leq \tau^+$ interval Q is non-empty. In Step 1 we show that Program 5, augmented by additional constraint $\tau \in Q$ has a solution. In Step 2 we demonstrate that the constraint is not binding, and so the solution to the restricted program defines equivalent variation.

Step 1. Let

$$A \equiv \left\{\left(z^i, \tau\right) \in X^i \times Q | z^i \in B^i_p(\bar{\theta}) + \tau d\right\},$$

and

$$B \equiv \{(z^i, \tau) \in X^i \times Q | z^i \in \bar{\Psi}^i_{p'}(\bar{\theta})\}.$$

Program 5 augmented by constraint $\tau \in Q$ can be reformulated as $\min \tau$ subject to $(z^i, \tau) \in A \cap B \subset \mathbb{R}^{N+1}$. Set A is closed (by continuity of b_p^i and closeness of $X^i \times Q$) and bounded (by boundedness of $X^i \times Q$). Similarly, set B is closed (by closeness of X^i, Q and continuity of preferences) and bounded (by boundedness of $X^i \times Q$). It follows that $A \cap B \subset \mathbb{R}^{N+1}$ is compact. Pick arbitrary $z^i \in x_{p'}^i(\bar{\theta})$, that by Lemma 4 is well defined. Note that $z^i \in \bar{\Psi}_{p'}^i(\bar{\theta})$ and hence $z^i \in X^i$. For any $\tau \in \mathbb{R}$

$$b_p^i\left(z^i, \bar{\theta}\right) - b_p^i\left(z^i - \tau d, \bar{\theta}\right) \geq (\underline{b} \sum_{n=1}^N d_n)\tau = \max_{x^i, \theta \in X^i \times \Theta} b_p^i(x^i, \theta) \frac{\tau}{\tau^+} \geq b_p^i\left(z^i, \bar{\theta}\right) \frac{\tau}{\tau^+}, \tag{12}$$

where the first inequality holds by mean value theorem and $d_n \geq 0$ for all n and the equality by the definition of τ^+. For $\tau = \tau^+$ one has $b_p^i\left(z^i - \tau^+ d, \bar{\theta}\right) \leq 0$ and hence $z^i \in B_p^i(\bar{\theta}) + \tau^+ d$ and $\left(z^i, \tau^+\right) \in A \cap B$. It follows that set $A \cap B$ is non-empty. By the extreme value theorem, a solution to the program exists and is attained on set $A \cap B$. Denote it by $\bar{z}, \bar{\tau}$.

Step 2. In this step we argue that constraint $\tau \in Q$ introduced in Step 1 is not binding. First note that any $z^i, \tau \in X^i \times \mathbb{R}$ for which $\tau > \tau^+$ necessarily gives strictly higher value of Program 5 than solution $\bar{z}, \bar{\tau}$, and hence additional constraint $\tau \leq \tau^+$ is not binding. Next consider any $z^i, \tau \in X^i \times \mathbb{R}$ such that $\tau < \tau^-$. By the analogous arguments as before

$$b_p^i\left(z^i, \bar{\theta}\right) - b_p^i\left(z^i - \tau d, \bar{\theta}\right) \leq (\bar{b} \sum_{n=1}^N d_n)\tau = \min_{x^i, \theta \in X^i \times \Theta} b_p^i(x^i, \theta) \frac{\tau}{\tau^-} \leq b_p^i\left(z^i, \bar{\theta}\right) \frac{\tau}{\tau^-},$$

which at $\tau = \tau^-$ reduces to $0 \leq b_p^i\left(z^i - \tau^- d, \bar{\theta}\right)$. Since b_p^i is strictly increasing in each component, $d \in \mathbb{R}_+^N$ and $d \neq 0$, the derivative of the composite function $b_p^i(z^i - d\tau, \bar{\theta})$ with respect to τ is $-\nabla b_p^i \cdot d < 0$, and function is strictly decreasing in τ. Therefore one has $0 < b_p^i\left(z^i - \tau d, \bar{\theta}\right)$ and consequently $z^i \notin B_p^i(\bar{\theta}) + \tau d$. It follows that pair z^i, τ violates one of the constraints in Program 5, i.e., restriction $\tau \geq \tau^-$ is implied by other constraints of Program 5. The two observations imply that condition $\tau \in Q$ is not binding and $\tau < \tau^-$ is a solution to unrestricted Program 5. Since minimum takes at most one value, equivalent variation $EV^i(\bar{\theta})$ is a well-defined function on Θ and correspondence $z^i(\cdot)$ is non-empty. □

Before we prove continuity of equivalent variation we give an auxiliary result.

Lemma 6. *Fix $\bar{\theta} \in \Theta$ and let $\bar{z}^i, \bar{\tau}$ be a solution to Program 5. There exist open neighborhoods of $(\bar{z}^i, \bar{\theta})$ and $\bar{\tau}$, denoted by $V_{\bar{z}^i, \bar{\theta}}$ and $V_{\bar{\tau}}$, respectively, and a continuous bijection $\tilde{\tau} : V_{\bar{z}^i, \bar{\theta}} \to V_{\bar{\tau}}$, such that $EV^i(\bar{\theta}) = \tilde{\tau}(\bar{z}^i, \bar{\theta})$ and $(z^i - \tilde{\tau}(z^i, \theta) d) \in B_p^i(\theta)$ for all $(z^i, \theta) \in V_{\bar{z}^i, \bar{\theta}}$.*

Proof of Lemma 6:

Observe that alternative $\bar{z}^i - d\bar{\tau}$ is necessarily a boundary point of a budget set, i.e., $b_p^i(\bar{z}^i - d\bar{\tau}, \bar{\theta}) = 0$ for otherwise there would exist $\varepsilon \geq 0$ such that $y^i \equiv \bar{z}^i - (\bar{\tau} - \varepsilon) d \in B_p^i(\theta)$ contradicting the fact that $\bar{z}^i, \bar{\tau}$ solves the minimization problem. The derivative of the composite function $b_p^i(\bar{z}^i - d\tau, \bar{\theta})$ with respect to τ satisfies $-\nabla b_p^i \cdot d < 0$, and hence it is non-zero, by the implicit function theorem there exists a differentiable bijection $\tilde{\tau}(\cdot,\cdot)$ from some neighborhood of $(\bar{z}^i,\bar{\theta})$, denoted by $V_{\bar{z}^i,\bar{\theta}} \subset \mathbb{R}^N \times \mathbb{R}^M$, to neighborhood of $\bar{\tau}$, given by $V_{\bar{\tau}} \subset \mathbb{R}$, such that $b_p^i(z^i - \tilde{\tau}(z^i,\theta)d,\theta) = 0$ for all $(z^i,\theta) \in V_{\bar{z}^i,\bar{\theta}}$. It follows that for all such values one has $z^i - \tilde{\tau}(z^i,\theta)d \in B_p^i(\theta)$. By construction equivalent variation is given by $EV^i(\bar{\theta}) \equiv \bar{\tau} = \tilde{\tau}(\bar{z}^i,\bar{\theta})$. □

We next prove continuity of equivalent variation.

Lemma 7. *Equivalent variation is continuous.*

Proof of Lemma 7:

Suppose for some $\bar{\theta} \in \Theta$ equivalent variation is not continuous. This implies that there exists $\varepsilon > 0$ such that for any $\delta > 0$ one can find θ satisfying $\left\|\theta - \bar{\theta}\right\| < \delta$ for which either $EV^i(\theta) \geq EV\left(\bar{\theta}\right) + \varepsilon$ or $EV^i(\theta) \leq EV\left(\bar{\theta}\right) - \varepsilon$. This in turn implies that either there exists sequence $\theta^h \to \bar{\theta}$ in Θ for which $EV^i\left(\theta^h\right) \geq EV^i(\bar{\theta}) + \varepsilon$ or $EV^i\left(\theta^h\right) \leq EV^i(\bar{\theta}) - \varepsilon$ for all $h = 1, 2, ...$ We argue that this is an impossibility.

Upper bound. Suppose there exists sequence $\theta^h \to \bar{\theta}$ and $\varepsilon > 0$ such that $EV^i\left(\theta^h\right) \geq EV^i\left(\bar{\theta}\right) + \varepsilon$ for all h. Pick arbitrary $x^{i,h} \in x_{p'}^i\left(\theta^h\right)$. Note that by Lemma 4 set $x_{p'}^i\left(\theta^h\right)$ is non-empty so such $x^{i,h}$ exists. For any θ^h the maximal upper contour set attainable under p' can be written as $\bar{\Psi}_{p'}^i\left(\theta^h\right) = \Psi^i\left(x^{i,h},\theta^h\right)$. Since $x^{i,h} \in X^i$, and X^i is compact, there exists convergent subsequence $\theta^k, x^{i,k} \to \bar{\theta}, \bar{x}^i$. By Lemma 4 correspondence $x_{p'}^i(\cdot)$ is upper hemicontinuous and thus by compactness of X^i it has a closed graph. This implies that $\bar{x}^i \in x_{p'}^i\left(\bar{\theta}\right)$. The maximal upper contour set at $\bar{\theta}$ is given by $\bar{\Psi}_{p'}^i\left(\bar{\theta}\right) = \Psi^i\left(\bar{x}^i,\bar{\theta}\right)$.

Consider convergent sequence $\theta^k, x^{i,k} \to \bar{\theta}, \bar{x}^i$, defined in the previous paragraph. Let $\bar{z}^i, \bar{\tau} \in X^i \times Q$ be a solution to the program that defines equivalent variation at $\bar{\theta}$. By Lemma 5 such solutions exist. By definition of equivalent variation $\bar{z}^i = \Psi^i\left(\bar{x}^i,\bar{\theta}\right)$ and $\bar{z}^i - d\bar{\tau} \in B_p^i\left(\bar{\theta}\right)$. Since by Assumption 1 correspondence Ψ^i is lower hemicontinuous, there exists subsequence $x^{i,l}, \theta^l \to \bar{x}^i, \bar{\theta}$ and $z^{i,l} \to \bar{z}^i$ such that $z^{i,l} \in \Psi^i\left(x^{i,l},\theta^l\right) = \bar{\Psi}_{p'}^i\left(\theta^l\right)$ for all l. Truncate this subsequence to elements $z^{i,l}, \theta^l$ that are in the neighborhood $V_{\bar{z}^i,\bar{\theta}}$ defined in Lemma 6. By Lemma 6 one has $z^{i,l} - \tilde{\tau}(z^{i,l},\theta^l)d \in B_p^i\left(\theta^l\right)$ for all l. For any l equivalent variation cannot be larger than $\tilde{\tau}(z^{i,l},\theta^l)$ since pair $(z^{i,l},\tilde{\tau}(z^{i,l},\theta^l))$ satisfies constraints in Program 5 and $EV^i(\theta^l) \leq \tilde{\tau}(z^{i,l},\theta^l)$. By construction $z^{i,l}, \theta^l \to \bar{z}^i, \bar{\theta}$ and $\tilde{\tau}(\cdot,\cdot)$ is continuous, hence $\lim_{l\to\infty} EV^i\left(\theta^l\right) \leq \tilde{\tau}(\bar{z}^i,\bar{\theta}) = EV^i\left(\bar{\theta}\right)$. This shows that equivalent variation evaluated at $\bar{\theta}$ is an upper bound for the limit of subsequence $EV^i\left(\theta^l\right)$. This contradicts the proposition that $EV^i\left(\theta^h\right) \geq EV^i\left(\bar{\theta}\right) + \varepsilon$ for all elements of the sequence θ^h for some fixed $\varepsilon > 0$.

Lower bound: Suppose there exists sequence $\theta^h \to \bar{\theta}$ and $\varepsilon > 0$ such that $EV^i\left(\theta^h\right) \leq EV^i\left(\bar{\theta}\right) - \varepsilon$ for all h. For each h pick arbitrary $x^{i,h} \in x_{p'}^i\left(\theta^h\right)$ and $z^{i,h}, \tau^h \in X^i \times Q$ that solves Program 5. By Lemmas 4 and 5 such sequences exist. Since $x^{i,h}, z^{i,h} \in X^i$ and $\tau^h \in Q$ for all h, and sets are compact, sequence $x^{i,h}, z^{i,h}, \tau^h$ has convergent subsequence $x^{i,k}, z^{i,k}, \tau^k \to \bar{x}^i, \bar{z}^i, \bar{\tau}$ where $\bar{x}^i, \bar{z}^i \in X^i$

and $\bar{\tau} \in Q$. By definition of equivalent variation, for every k one has $z^{i,k} \in \bar{\Psi}^i_{p'}\left(\theta^k\right) = \Psi^i\left(x^{i,k}, \theta^k\right)$ and $z^{i,k} \in B^i_p\left(\theta^k\right) + \tau^k d$. By Assumption 1 correspondences are upper hemicontinuous and hence have closed graphs, $\bar{z}^i \in \Psi^i\left(\bar{x}^i, \bar{\theta}\right) = \bar{\Psi}^i_{p'}\left(\bar{\theta}\right)$ and $\bar{z}^i \in B^i_p\left(\bar{\theta}\right) + \bar{\tau} d$. Thus, pair $\bar{z}^i, \bar{\tau}$ satisfies constraints in Program 5 at $\bar{\theta}$. Hence

$$\lim_{k\to\infty} EV^i\left(\theta^k\right) = \lim_{k\to\infty} \tau^k \equiv \bar{\tau} \geq EV^i\left(\bar{\theta}\right).$$

Therefore, equivalent variation evaluated at $\bar{\theta}$ is the lower bound for the limit of the subsequence $EV^i\left(\theta^k\right)$. This in turn contradicts proposition $EV^i\left(\theta^h\right) \leq EV^i\left(\bar{\theta}\right) - \varepsilon$ for all n given fixed $\varepsilon > 0$. □

By Lemma 5 correspondence $z^i(\cdot)$ is non-empty. We conclude the proof by demonstrating upper hemicontinuity of $z^i(\cdot)$.

Lemma 8. *Correspondence $z^i(\cdot)$ is upper hemicontinuous.*

Proof of Lemma 8:

Since the range of correspondence is compact, it suffices to demonstrate that the latter has a closed graph. Consider arbitrary convergent sequence $z^{i,h}, \theta^h \to \bar{z}^i, \bar{\theta}$ for which $z^{i,h} \in z^i(\theta^h)$. By continuity of equivalent variation (Lemma 7), $\tau^h \equiv EV^i\left(\theta^h\right) \to \bar{\tau} \equiv EV^i\left(\bar{\theta}\right)$. Choice correspondence $x^i_{p'}\left(\cdot\right)$ is non-empty (Lemma 4). Pick arbitrary $x^{i,h} \in x^i_{p'}\left(\theta^h\right)$. By compactness of X^i, there exists convergent subsequence $x^{i,k} \to \bar{x} \in X^i$. $x^i_{p'}\left(\cdot\right)$ is upper hemicontinuous and hence it has closed graph, it follows that $\bar{x}^i \in x^i_{p'}\left(\bar{\theta}\right)$ and $\bar{\Psi}^i_{p'}\left(\bar{\theta}\right) = \Psi^i\left(\bar{x}^i, \bar{\theta}\right)$. Elements $z^{i,k}, \tau^k$ solve Program (5) for each k and thus they satisfy constraints: $z^{i,k} \in \bar{\Psi}^i_{p'}\left(\theta^k\right) = \Psi^i\left(x^{i,k}, \theta^k\right)$ and $z^{i,k} \in B^i_p\left(\theta^k\right) + d\tau^k$. By Assumption 1 correspondence Ψ^i has closed graphs, and hence $\bar{z}^i \in \Psi^i\left(\bar{x}^i, \bar{\theta}\right) = \bar{\Psi}^i_{p'}\left(\bar{\theta}\right)$. The latter condition in turn implies that $b^i_p\left(z^{i,k} - \tau^k d, \theta\right) \leq 0$ which, by joint continuity of b^i_p is preserved in the limit $b^i_p\left(\bar{z}^i - \bar{\tau} d, \theta\right) \leq 0$ and hence $\bar{z}^i \in B^i_p\left(\bar{\theta}\right) + d\bar{\tau}$. It follows that $\bar{z}^i, \bar{\tau}$ satisfies constraints and attains $\bar{\tau} = EV^i\left(\bar{\theta}\right)$ so $\bar{z}^i \in z^i(\bar{\theta})$. □

A.2 Local Extension

Proof of Proposition 1:

The proof of proposition proceeds as follows. Lemma 10 demonstrates that the problem restricted to set of alternatives X^{i*} and a compact neighborhood Θ^* satisfies the assumptions of Theorem 1 and, hence, choice and equivalent variation are well-defined and continuous. Lemma 11 then shows that for some (potentially smaller) neighborhood of θ^* choice and equivalent variations from the restricted problem give corresponding values in the unrestricted problem and that they are unique. We start with an auxiliary result that we use in the subsequent lemmas.

Lemma 9. *Sets $x^i_{p'}(\theta^*)$ and $z^i(\theta^*)$ are singletons i.e., $x^i_{p'}(\theta^*) = \{x^{i*}\}$ and $z^i(\theta^*) = \{z^{i*}\}$.*

Proof of Lemma 9:

Let X^{io} be the interior of box X^{i*}. By assumption $x^{i*} \in X^{io}$. Suppose $x^i_{p'}(\theta^*)$ is not a singleton, i.e., there exists $x^i \in x^i_{p'}(\theta^*)$ such that $x^i \neq x^{i*}$. Optimal choices are affordable and hence $x^{i*}, x^i \in B^i_{p'}(\theta^*) \cap X^i(\theta^*)$. Let $x^{i\prime} = \alpha x^{i*} + (1-\alpha)\, x^i$ with $\alpha \in (0,1)$ sufficiently large so that $x^{i\prime} \in X^{io}$. Such α exists since X^{io} is open. By quasi-convexity of $b^i_p(\cdot, \theta^*)$ set $B^i_{p'}(\theta^*)$ is convex and $x^{i\prime} \in B^i_{p'}(\theta^*)$. It follows that $x^{i\prime} \in B^i_{p'}(\theta^*) \cap X^i(\theta^*)$. By strict convexity of preferences one has $x^{i\prime} \succ^i_{\theta^*} x^{i*}$ contradicting optimality of x^{i*}.

Next suppose $z^i(\theta^*)$ is not a singleton, i.e., there exists $z^i \in z^i(\theta^*)$ such that $z^i \neq z^{i*} \in X^{io}$. Let $z^{i\prime} = \alpha z^{i*} + (1-\alpha)\, z^i$ with $\alpha \in (0,1)$ sufficiently large so that $z^{i\prime} \in X^{io}$. Let $\tau^* \equiv EV^i(\theta^*)$. Since $z^i - \tau^* d \in B^i_p(\theta^*)$, and $z^{i*} - \tau^* d \in B^i_p(\theta^*)$ and $B^i_p(\theta^*)$ is convex, therefore $z^{i\prime} - \tau^* d \in B^i_p(\theta^*)$. Since $z^{i*}, z^i \in \bar{\Psi}^i_{p'}(\theta^*)$, it follows that $z^i \succeq^i_{\theta^*} x^{i*}$ and $z^{i*} \succeq^i_{\theta^*} x^{i*}$. By strict convexity of preferences $z^{i\prime} \succ^i_{\theta^*} x^{i*}$. By continuity of Ψ^i there must exists neighborhood of $z^{i\prime}$ in X^{io} for which this strict preference is preserved. Otherwise there would exist sequence $z^{i,h} \to z^{i\prime}$ such that $x^{i*} \succeq^i_{\theta^*} z^{i,h}$ and hence for which $x^{i*} \in \Psi^i(z^{i,h}, \theta^*)$, which, along with continuity of Ψ^i implies $x^{i*} \in \Psi^i(z^{i\prime}, \theta^*)$ or $x^{i*} \succeq^i_{\theta^*} z^{i\prime}$, a contradiction. It then follows that for sufficiently small $\varepsilon > 0$ element $z^{i\prime\prime} \equiv z^{i\prime} - \varepsilon d \in \bar{\Psi}^i_{p'}(\theta)$ and $z^{i\prime\prime} - (\tau^* - \varepsilon)\, d = z^{i\prime} - \tau^* d \in B^i_p(\theta)$ and hence $(z^{i\prime\prime}, \tau^* - \varepsilon)$ satisfies the constraints of Program (5) and gives a lower value. This contradicts the assumption that (z^{i*}, τ^*) is a solution. $\square$

Lemma 10. *There exists neighborhood Θ^1 of θ^* such that the problem restricted to $X^{i*} \times \Theta^1$ satisfies the assumptions of Theorem 1.*

Proof of Lemma 10:

We need to verify (i) non-emptiness of correspondence $B^i_{p'}(\cdot) \cap X^{i*}$, (ii) continuity of this correspondence on Θ^1, and (iii) continuity of restriction $\Psi^i : X^{i^*} \times \Theta^1 \to X^{i^*}$.

(i) We first show that there exists neighborhood $\Theta^1 \subset \Theta^*$ of θ^* for which correspondence $B^i_{p'}(\cdot) \cap X^{i*}$ is non-empty. Since $x^{i*} \in B^i_{p'}(\theta^*)$ it follows that $b^i_{p'}(x^{i*}, \theta^*) \leq 0$. By construction $\underline{x} << x^{i*}$ and by Assumption 2 function $b^i_{p'}(\cdot, \theta^*)$ is strictly increasing, therefore $b^i_{p'}(\underline{x}, \theta^*) < 0$. Since function is differentiable, it is jointly continuous so there exists neighborhood of θ^*, denoted by $\Theta^1 \subset \Theta^*$, for which strict inequality is preserved, $b^i_{p'}(\underline{x}, \theta) < 0$. It follows that $\underline{x} \in B^i_{p'}(\theta)$ for all $\theta \in \Theta^1$. Since $\underline{x} \in X^{i*}$, correspondence $B^i_{p'}(\theta) \cap X^{i*}$ is non-empty in the neighborhood Θ^1.

(ii) We next demonstrate continuity of correspondence $B^i_{p'}(\cdot) \cap X^{i*}$ on Θ^1:

(upper hemicontinuity) Consider convergent sequence $x^{i,h}, \theta^h \to \bar{x}^i, \bar{\theta}$ in $X^{i*} \times \Theta^1$ such that $x^{i,h} \in B^i_{p'}(\theta^h) \cap X^{i^*}$ and $\theta^h \in \Theta^1$ for each h. By definition of budget correspondence, $b^i_{p'}(x^{i,h}, \theta^h) \leq 0$ and by differentiability this inequality is preserved in the limit $b^i_{p'}(\bar{x}^i, \bar{\theta}) \leq 0$. Hence $\bar{x}^i \in B^i_{p'}(\bar{\theta}) \cap X^{i*}$ and correspondence has a closed graph. Since the range of the correspondence is compact, correspondence is upper hemicontinuous.

(lower hemicontinuity) Consider sequence $\theta^h \to \bar{\theta} \in \Theta^1$ and let $\bar{y}^i \in B^i_{p'}(\bar{\theta}) \cap X^{i*}$. Note that $b^i_{p'}\left(\bar{y}^i, \bar{\theta}\right) \leq 0$. We consider two possibilities. Suppose first that $\bar{y}^i = \underline{x}$. Define constant sequence

$y^{i,h} = \underline{x} \in X^{i*}$ for all $h = 1, 2, ...$ For all $\theta^h \in \Theta^1$ and h one has $b^i_{p'}\left(y^{i,h}, \theta^h\right) = b^i_{p'}(\underline{x}, \theta^h) \leq 0$ since by previous step $B^i_{p'}(\theta^h) \cap X^{i*}$ is non-empty on Θ^1. Therefore $y^{i,h} \in B^i_{p'}(\theta^h) \cap X^{i*}$ and $y^{i,h} \to \underline{x} = \bar{y}^i$ which completes the argument. Suppose now $\bar{y}^i > \underline{x}$. For any $\alpha^k \in (0, 1)$ let $y^{i,k} \equiv \alpha^k \bar{y}^i + \left(1 - \alpha^k\right) \underline{x} < \bar{y}^i$. By strict monotonicity $b^i_{p'}\left(y^{i,k}, \bar{\theta}\right) < 0$. Moreover, by joint continuity of $b^i_{p'}(\cdot, \cdot)$ there exists $\delta^k > 0$ such that for all θ^k satisfying $\|\bar{\theta} - \theta^k\| \leq \delta^k$ inequality $b^i_{p'}\left(y^{i,k}, \theta^k\right) < 0$ is preserved and $y^{i,k} \in B^i_{p'}(\theta^k) \cap X^{i*}$. Define convergent subsequence $\theta^k, y^{i,k}$ as follows. Let $\varepsilon^k = 1/k$. For $k = 1$ pick α^k so that $\left\|\bar{y}^i - y^{i,k}\right\| \leq \varepsilon^k$. For the corresponding $\delta^{k=1}$ pick an element of sequence θ^h for which the distance from $\bar{\theta}$ is at most $\delta^{k=1}$. Since $\theta^h \to \bar{\theta}$, such element exists. Denote this element by by $\theta^{h_{(k=1)}}$. Repeat this step for $k = 2, 3, ...$, each time selecting an element from the sequence θ^h truncated to $h > h_{(k-1)}$ elements. By construction $y^{i,k} \to \bar{y}^i$ and $y^{i,k} \in B^i_{p'}(\theta^k) \cap X^{i*}$.

(iii) Continuity of restriction $\Psi^i : X^{i*} \times \Theta^1 \to X^{i*}$ follows straightforwardly from Assumption 3.
□

Lemma 11. *There exists neighborhood Θ^{**} of θ^* such that equivalent variation $EV^i(\cdot)$ and correspondences $z^i(\cdot)$ and $x^i_{p'}(\cdot)$ are continuous functions on Θ^{**}.*

Proof of Lemma 11 :

Step 1. By Lemma 10 in the problem restricted to $X^{i*} \times \Theta^1$ assumptions of Theorem 1 are satisfied and equivalent variation $EV^i_{X^i}(\cdot)$ is a continuous function on Θ_1 and correspondences $z^i_{X^i}(\cdot)$ and $x^i_{X^i}(\cdot)$ are non-empty and upper hemicontinuous. By Lemma 9 elements x^{i*} and z^{i*} are unique solutions of the optimization problems with unrestricted domain $X^i(\theta^*)$. Since $x^{i*}, z^{i*} \in X^{i*}$, they are necessarily unique solutions to the optimization programs on restricted domain X^{i*}. It follows that sets $x^i_{X^i}(\theta^*) = \{x^{i*}\}$ and $z^i_{X^i}(\theta^*) = \{z^{i*}\}$ are singletons. By upper hemicontinuity of $x^i_{X^i}(\cdot)$ there exists a neighborhood of θ^* denoted by $\Theta^2 \subset \Theta^1$, such that $x^i_{X^i}(\theta) \subset X^{io}$ for all $\theta \in \Theta_2$. Suppose not. There exists sequence $\theta^h \to \theta^*$ and $x^{i,h} \in x^i_{X^i}(\theta^h) \subset X^{i*}$ such that $x^{i,h} \notin X^{io}$. Since $x^{i,h} \in X^{i*}$, and X^{i*} is compact, there exists convergent subsequence $x^{i,k} \to x^{i\prime} \in X^{i*}$ such that $x^{i\prime} \neq x^{i*}$. Since $x^i_{X^i}(\cdot)$ is upper hemicontinuous, it has a closed graph, hence $x^{i\prime} \in x^i_{X^i}(\theta^*)$. This, however, contradicts the fact that the choice set is a singleton $x^i_{X^i}(\theta^*) = \{x^{i*}\}$. By analogous arguments there exists neighborhood $\Theta^3 \subset \Theta^2$ such that for all $\theta \in \Theta^3$ one has $z^i_{X^i}(\theta) \subset X^{io}$.

Step 2. We show that for all $\theta \in \Theta^3$, optimal choice $x^i \in x^i_{X^i}(\theta)$ is a unique maximizer of preferences on the unrestricted set $B^i_{p'}(\theta) \cap X^i(\theta)$ and hence $x^i_{X^i}(\theta) = x^i_{p'}(\theta)$ is a singleton. Suppose not. There exists $x^{i\prime} \in B^i_p(\theta) \cap X^i(\theta)$ such that $x^{i\prime} \neq x^i$ and $x^{i\prime} \succeq^i_\theta x^i$. By strictly convex preferences, for all $\alpha \in (0, 1)$ convex combination $x^{i\prime\prime} = \alpha x^i + (1 - \alpha)x^{i\prime}$ satisfies $x^{i\prime\prime} \succ^i_\theta x^i$ and for α large enough $x^{i\prime\prime} \in B^i_p(\theta) \cap X^i$, contradicting optimality of x^i in the restricted problem. It follows that correspondence $x^i_p(\theta) = x^i_{X^i}(\theta)$ is a function on Θ^3. By upper hemicontinuity of $x^i_{X^i}(\cdot)$ implied by Theorem 1 applied to the restricted problem, function $x^i_p(\cdot)$ is continuous on Θ^3.

Step 3. We show that for all $\theta \in \Theta^3$, element $z^i \in z^i_{X^i}(\theta)$ along with $\tau \equiv EV^i_{X^i}(\theta)$ is a unique solution to Program 5 on unrestricted domain. Suppose not. In the unrestricted problem there exists $z^{i\prime} \in z^i(\theta)$ such that $z^{i\prime} \neq z^i$, satisfying $z^{i\prime} \in \bar{\Psi}^i_{p'}(\theta)$ and $z^{i\prime} \in B^i_p(\theta) + \tau' d$ for some $\tau' \leq \tau$. Consider convex combination $z^{i\prime\prime} = \alpha z^i + (1-\alpha) z^{i\prime}$ where α is sufficiently large so that $z^{i\prime\prime} \in X^{io}$. Such α exists since $z^i \in X^{io}$. Let $x^i = x^i_{p'}(\theta) = x^i_{X^i}(\theta)$, which, by step 2, is uniquely defined. Since $z^i, z^{i\prime} \in \bar{\Psi}^i_{p'}(\theta)$ therefore $z^i \succeq^i_\theta x^i$ and $z^{i\prime} \succeq^i_\theta x^i$. By strict convexity of preferences $z^{i\prime\prime} \succ^i_\theta x^i$. By joint continuity of preferences there must exist a neighborhood of $z^{i\prime\prime}$ in X^{io} for which this strict preference is preserved.[17] Thus, for sufficiently small $\varepsilon > 0$ one has $z^{i\prime\prime\prime} \equiv z^{i\prime\prime} - \varepsilon d \in \bar{\Psi}^i_{p'}(\theta)$ and $z^{i\prime\prime\prime} - (\tau - \varepsilon) d = z^{i\prime\prime} - \tau d \in B^i_p(\theta)$ and hence $(z^{i\prime\prime\prime}, \tau - \varepsilon)$ satisfies constraints of Program (5) in the restricted problem and gives lower value than τ. This contradicts the assumption that (z^i, τ) solves the restricted problem. Therefore on set Θ^3 correspondence $z^i(\theta) = z^i_{X^i}(\theta)$ is a function and $EV^i(\theta) = EV^i_{X^i}(\theta)$. Finally by upper hemicontinuity of $z^i_{X^i}(\theta)$, implied by Theorem 1, function $z^i(\theta)$ is continuous. We complete the proof by defining $\Theta^{**} \equiv \Theta^3$. □

We finally demonstrate the cardinal test for joint continuity for strictly monotonic family of preferences.

Proof of Lemma 1:

The proof consists of two steps: showing upper and lower hemicontinuity.

Step 1. In this step we show upper hemicontinuity. Consider arbitrary convergent sequence in $X^{i*} \times \Theta^* \times X^{i*}$, denoted by $x^{i,h}, \theta^h, y^{i,h} \to \bar{x}^i, \bar{\theta}, \bar{y}^i$, such that $y^{i,h} \in \Psi^i(x^{i,h}, \theta^h)$ for each $h = 1, 2, ...$ Preferences admit utility representation, for which $U^i(y^{i,h}, \theta^h) \geq U^i(x^{i,h}, \theta^h)$. By joint continuity of U^i the inequality is preserved in the limit and $U^i(\bar{y}^i, \bar{\theta}) \geq U^i(\bar{x}^i, \bar{\theta})$, which implies $\bar{y}^i \in \Psi^i(\bar{x}^i, \bar{\theta})$.

Step 2. In this step we show lower hemicontinuity. Consider convergent sequence $x^{i,h}, \theta^h \to \bar{x}^i, \bar{\theta}$ in $X^i \times \Theta$ and $\bar{y}^i \in \Psi^i(\bar{x}^i, \bar{\theta})$. Utility function satisfies $U^i(\bar{y}^i, \bar{\theta}) \geq U^i(\bar{x}^i, \bar{\theta})$. We consider two cases. Suppose first $\bar{y}^i = \overline{x}$. Define convergent subsequence sequence as $y^{i,k} = \overline{x}$ for all $k = h$. Since by strict monotonicity $U^i(y^{i,k}, \theta) = U^i(\overline{x}, \theta) \geq U^i(x^i, \theta)$ for all $x^i, \theta \in X^{i*} \times \Theta^*$, so $y^{i,k} \in \Psi^i(x^{i,k}, \theta^k)$, and $y^{i,k} \to \overline{x} = \bar{y}^i$, which completes the argument. Suppose now that $\bar{y}^i < \overline{x}$. For $k = 1, 2, ...$, let $\varepsilon^k = 1/k$ and $y^{i,k} \equiv \alpha^k \bar{y}^i + \left(1 - \alpha^k\right) \overline{x} > \bar{y}$ where α^k is large enough so that $\|y^{k,i} - \bar{y}^i\| \leq \varepsilon^k$. By strict monotonicity $U^i(y^{i,k}, \bar{\theta}) > U^i(\bar{y}^i, \bar{\theta}) \geq U^i(\bar{x}^{i*}, \bar{\theta})$. By joint continuity of U^i, there must exist $\delta^k > 0$ such that for all (x^i, θ) satisfying $\|(x^i, \theta) - (\bar{x}^i, \bar{\theta})\| \leq \delta^k$ the strict inequality is preserved, i.e., $U^i(y^{i,k}, \theta^k) > U^i\left(x^i, \theta\right)$. For $k = 1$ let $y^{i,k}$ and $\delta^{k=1}$ be defined as follows. Choose an element of sequence $x^{i,h}, \theta^h$ for which distance from $(\bar{x}^i, \bar{\theta})$ no greater than $\delta^{k=1}$. Since $x^{i,h}, \theta^h \to \bar{x}^i, \bar{\theta}$, such element exists. Call it $x^{i,h_{(k=1)}}, \theta^{h_{(k=1)}}$. Repeat this step for $k = 2, 3, ...$ each time selecting an element from the sequence truncated to $h > h_{(k-1)}$ elements. By construction $y^{i,k} \to \bar{y}^i$ and $y^{i,k} \in \Psi^i(x^{i,k}, \theta^k)$. □

[17] Consider a function $g : \Theta \to X$ that is continuous at θ, for which $x = g^i(\theta)$. There exists neighborhood of θ such that for all θ' in the neighborhood one has $z \succ^i_{\theta'} g(\theta')$ and hence $z \in \Psi(g^i(\theta), \theta')$. Suppose not. There exists sequence $\theta_n \to \theta$ such that for the corresponding sequence $x_n \equiv g(\theta_n)$ one has $x_n \succeq^i_{\theta_n} z$ and hence $x_n \in \Psi(z, \theta_n)$. By continuity of $g(\cdot)$ one has $\lim_{n\to\infty} x_n = \lim_{n\to\infty} g(\theta_n) = g(\theta) = x$. Since Ψ has closed graph $x \in \Psi(z, \theta)$ and hence $x \succeq^i_\theta z$, a contradiction.

B Trader's problem

B.1 Definition of a reduced-form problem

We first introduce some definitions. For scalar $x_0^i \in \mathbb{R}$, consider the following program

$$v^i(x_0^i, \beta) \equiv \max_{\{x_t^i\}_{t=T+1}^{\infty}} E \sum_{t=T+1}^{\infty} \beta^t u^i(x_t^i), \tag{13}$$

subject to

$$E \sum_{t=T+1}^{\infty} \beta^t \zeta_t (x_t^i - e_t^i) \leq x_0^i \text{ and } x_t^i > \underline{x}^i \text{ for all } t > T.$$

The set of consumption flows that satisfies the constraints is empty whenever $x_0^i \leq \underline{x}_0^i(\beta) \equiv E \sum_{t=T+1}^{\infty} \beta^t \zeta_t (\underline{x}^i - e_t^i)$. The next lemma shows the converse: the domain is non-empty and the solution to the problem is uniquely defined whenever $x_0^i > \underline{x}_0^i(\beta)$.

Lemma 12. *Program (13) has a (unique) solution if and only if $x_0^i > \underline{x}_0^i(\beta)$.*

Proof of Lemma 12:

Derivative $u^{i\prime} : (\underline{x}^i, \infty) \to \mathbb{R}_{++}$ is a continuous and strictly decreasing bijection, therefore its inverse $\tilde{x}^i : \mathbb{R}_{++} \to (\underline{x}^i, \infty)$ is well-defined, is continuous and strictly decreasing. By Inada conditions constraint $x_{t,s}^i > \underline{x}^i$ is not binding for any date event (t, s) and the solution to the program can be determined by the first order necessary and sufficient conditions from the standard unconstrained Lagrangian problem (utility augmented by budget constraint). For any date-event (t, s) the first order condition with respect to consumption, $\beta^t \pi_{s,t} u^{i\prime}\left(x_{t,s}^i\right) = \pi_{s,t} \beta^t \zeta_{t,s} \lambda^i$, can be equivalently reformulated as $x_{t,s}^i = \tilde{x}^i(\lambda^i \zeta_t)$. Plugging the latter conditions in the budget constraint,

$$\eta\left(\beta, \lambda^i\right) \equiv (1-\beta) E \sum_{t=T+1}^{\infty} \beta^t \zeta_t \tilde{x}^i(\lambda^i \zeta_t) = (1-\beta) x_0^i + (1-\beta) E \sum_{t=T+1}^{\infty} \beta^t \zeta_t e_t^i, \tag{14}$$

give an equation that implicitly defines scalar λ^i. Fix $\beta \in (0,1)$. Function $\eta(\beta, \cdot)$ on the left hand side is a strictly decreasing bijection, mapping positive reals $\mathbb{R}_{++}$ onto an open interval $\{y^i \in \mathbb{R} | y^i > (1-\beta) E \sum_{t=T+1}^{\infty} \beta^t \zeta_t \underline{x}^i\}$. Right-hand side of the equation is some real number. Equation (14) has a (unique) solution if and only if the constant on the right hand side is in the range of the bijection. This holds if $x_0^i > \underline{x}_0^i(\beta) \equiv E \sum_{t=T+1}^{\infty} \beta^t \zeta_t (\underline{x}^i - e_t^i)$. Given strictly convex separable preferences solution $\lambda^i > 0$, along with consumption flow $\bar{x}^i$ where $\bar{x}_{t,s}^i = \tilde{x}^i(\lambda^i \zeta_{t,s})$ satisfy necessary and sufficient conditions for optimality. □

The reduced-form of the infinite horizon problem formally consists of three elements: consumption space, preferences and budget correspondence for each policy $p \in \mathbb{P}$. For any $\beta \in (0,1)$ reduced-form preferences, $\succeq_\beta^i$ over alternatives in $\tilde{X}^i(\beta) \equiv \{(x_0^i, \{x_t^i\}_{t=1}^T) | x_0^i > \underline{x}_0^i(\beta), x_t^i > \underline{x}^i$ for all $t\}$ are represented by utility function

$$\tilde{U}^i(x^i, \beta) \equiv v^i(x_0^i, \beta) + E \sum_{t=1}^{T} \beta^t u^i(x_t^i).$$

Finally, for policy $p = \{\zeta, e^i\} \in \mathbb{P}$ a budget correspondence in the reduced form problem $\tilde{B}^i_p(\beta)$ is derived from constraint $\tilde{b}^i_p\left(x^i, \beta\right) \equiv x^i_0 + E\sum_{t=1}^{T}\beta^t\zeta_t(x^i_t - e^i_t) \leq 0$. By $E\tilde{V}^i_{p,p',\tilde{d}}(\beta)$ we denote an equivalent variation in the reduced-form measured in terms of money x^i_0, that is, $\tilde{d} = (1, 0, \ldots, 0)$.

B.2 Equivalence of the two representations

For a stochastic process $x^i = \{x^i_t\}_{t=1}^{\infty}$ in the infinite horizon problem (henceforth referred to as IH), define a reduction $x^{i-} \equiv \left(x^{i-}_0, \{x^i_t\}_{t=1}^{T}\right)$ where $x^{i-}_0 \equiv E\sum_{t=T+1}^{\infty}\beta^t\zeta_t(x^i_t - e^i_t)$ is the value of consumption in future periods. For process $y^i = (y^i_0, \{y^i_t\}_{t=1}^{T})$ in the reduced form (RF) define extension $y^{i+} \equiv \{y^{i+}_t\}_{t=1}^{\infty}$ as $y^{i+}_t \equiv y^i_t$ for $t = 1, \ldots, T$ and $\{y^{i+}_t\}_{t=T+1}^{\infty}$ is a solution to Program (13) given y^i_0. In the next three lemmas we demonstrate equivalence of the alternative representations of the problem in terms of budget sets (Lemma 13), optimal choices (Lemma 14), and welfare indices (Lemma 15).

Fix arbitrary $\tau \in \mathbb{R}$ and numeraire vector $d \in \{d_t\}_{t=T+1}^{\infty} > 0$ for which market value satisfies $\kappa_d\left(\beta\right) \equiv E\sum_{t=T+1}^{\infty}\beta^t\zeta_t d_t < \infty$. We first demonstrate equivalence of two representations in terms of budget sets potentially shifted by vector τd. In this section we fix β and simplify notation $\kappa_d = \kappa_d\left(\beta\right)$.

Lemma 13. *Suppose consumption flow in IH satisfies $x^i \in X^i\left(\beta\right) \cap (B^i_p(\beta) + \tau d)$. Reduction x^{i-} is well-defined in RF and satisfies $x^{i-} \in \tilde{X}^i\left(\beta\right) \cap (\tilde{B}^i_p(\beta) + \kappa_d\tau\tilde{d})$. Conversely, for $x^i \in \tilde{X}^i\left(\beta\right) \cap (\tilde{B}^i_p(\beta) + \kappa_d\tau\tilde{d})$ in RF, its extension is well-defined and satisfies $x^{i+} \in X^i\left(\beta\right) \cap (B^i_p(\beta) + \tau d)$.*

Proof of Lemma 13*:*

Step 1. Fix $x^i \in X^i\left(\beta\right) \cap (B^i_p(\beta) + \tau d)$ in IH. Since $x^i \in X^i(\beta)$, for all $t = 1, \ldots, T$ one has $x^i_t > \underline{x}^i$ and $x^{i-}_0 \equiv E\sum_{t=T+1}^{\infty}\beta^t\zeta_t(x^i_t - e^i_t) > E\sum_{t=T+1}^{\infty}\beta^t\zeta_t(\underline{x}^i - e^i_t) = \underline{x}^i_0\left(\beta\right)$. Moreover, $x^i \in B^i_p(\beta) + \tau d$ and hence $E\sum_{t=1}^{\infty}\beta^t\zeta_t(x^i_t - \tau d_t - e^i_t) \leq 0$, where for the sake of notation we adopt convention $d_t = 0$ for all $t \leq T$. This implies

$$x^{i-}_0 \equiv E\sum_{t=T+1}^{\infty}\beta^t\zeta_t(x^i_t - e^i_t) < -E\sum_{t=1}^{T}\beta^t\zeta_t(\underline{x}^i - e^i_t) + \tau\kappa_d < \infty.$$

It follows that $\underline{x}^i_0\left(\beta\right) < x^{i-}_0 < \infty$, and reduction $x^{i-} \in \tilde{X}^i\left(\beta\right)$ is well-defined by Lemma 12. Moreover,

$$x^{i-}_0 - \tau\kappa_d + E\sum_{t=1}^{T}\beta^t\zeta_t(x^{i-}_t - e^i_t) = E\sum_{t=1}^{\infty}\beta^t\zeta_t(x^i_t - \tau d_t - e^i_t) \leq 0$$

where the last inequality holds since $x^i \in B^i_p(\beta) + \tau d$. It follows that $x^{i-} \in \tilde{B}^i_p(\beta) + \kappa_d\tau\tilde{d}$.

Step 2. Fix $x^i \in \tilde{X}^i\left(\beta\right) \cap (\tilde{B}^i_p(\beta) + \kappa_d\tau\tilde{d})$ in RF. Since $x^i \in \tilde{X}^i\left(\beta\right)$, $x^i_t > \underline{x}^i$ for $t = 1, \ldots, T$ and $x^i_0 > \underline{x}^i_0\left(\beta\right)$, by Lemma 12 solution to Program 13 exists $\left\{x^i_t\right\}_{t>T}$ that satisfies $x^i_t > \underline{x}^i$. It follows that extension $x^{i+} \in X^i(\beta)$ is well-defined. Moreover,

$$\begin{aligned} E\sum_{t=1}^{\infty}\beta^t\zeta_t(x_t^{i+}-\tau d_t-e_t^i) &= E\sum_{t=T+1}^{\infty}\beta^t\zeta_t(x_t^{i+}-e_t^i)-\tau E\sum_{t=1}^{\infty}\beta^t\zeta_t d_t+E\sum_{t=1}^{T}\beta^t\zeta_t(x_t^i-e_t^i) \\ &\leq x_0^i-\kappa_d\tau\tilde{d}+E\sum_{t=1}^{T}\beta^t\zeta_t(x_t^i-e_t^i)\leq 0 \end{aligned}$$

where the last inequality holds by the fact that $x^i \in \tilde{B}_p^i(\beta)+\kappa_d\tau\tilde{d}$. Therefore extension satisfies $x^{i+} \in B_p^i(\beta)+\tau d$. □

We next demonstrate equivalence in terms of optimal choices

Lemma 14. *Suppose x^i is optimal in IH on set $B_p^i(\beta)\cap X^i(\beta)$. Then x^{i-} is well-defined and optimal on $\tilde{B}_p^i(\beta)\cap\tilde{X}^i(\beta)$ in RF. Conversely, if in RF x^i is optimal on $\tilde{B}_p^i(\beta)\cap\tilde{X}^i(\beta)$ then x^{i+} is well-defined and optimal on $B_p^i(\beta)\cap X^i(\beta)$ in IH.*

Proof of Lemma 14:

Step 1. Fix optimal x^i on set $B_p^i(\beta)\cap X^i(\beta)$ in IH. Since $x^i \in B_p^i(\beta)\cap X^i(\beta)$, by Lemma 13 reduction is well-defined and $x^{i-} \in \tilde{B}_p^i(\beta)\cap\tilde{X}^i(\beta)$. Suppose x^{i-} is not optimal on this set. There exists $y^i \in \tilde{B}_p^i(\beta)\cap\tilde{X}^i(\beta)$ strictly preferred to x^{i-}. By Lemma 13 extension y^{i+} is well-defined and satisfies $y^{i+} \in B_p^i(\beta)\cap X^i(\beta)$. Finally

$$\begin{aligned} U^i(y^{i+},\beta) &= \sum_{t=1}^{\infty}\beta^t u_t^i(y_t^{i+}) = v^i(y_0^i,\beta)+E\sum_{t=1}^{T}\beta^t u_t^i(y_t^i) \\ &> v^i(x_0^{i-},\beta)+E\sum_{t=1}^{T}\beta^t u_t^i(x_t^{i-}) \geq E\sum_{t=1}^{\infty}\beta^t u_t^i(x_t^i) = U^i\left(x^i,\beta\right) \end{aligned}$$

where the strict inequality holds by the fact that y^i is strictly preferred to x^{i-} in RF. This contradicts optimality of x^i on $B_p^i(\beta)\cap X^i(\beta)$.

Step 2. Fix optimal x^i on set $\tilde{B}_p^i(\beta)\cap\tilde{X}^i(\beta)$ in RF. Since $x^i \in \tilde{B}_p^i(\beta)\cap\tilde{X}^i(\beta)$, by Lemma 13 extension is well-defined and $x^{i+} \in B_p^i(\beta)\cap X^i(\beta)$. Suppose x^{i+} is not optimal on this set. It follows that there exists $y^i \in B_p^i(\beta)\cap X^i(\beta)$ strictly preferred to x^{i+}. By Lemma 13 reduction y^{i-} is well-defined and satisfies $y^{i-} \in \tilde{B}_p^i(\beta)\cap\tilde{X}^i(\beta)$. Finally,

$$\begin{aligned} \tilde{U}^i\left(y^{i-},\beta\right) &= v^i(y_0^{i-},\beta)+E\sum_{t=1}^{T}\beta^t u_t^i(y_t^i) \geq \sum_{t=1}^{\infty}\beta^t u_t^i(y_t^i) \\ &> E\sum_{t=1}^{\infty}\beta^t u_t^i(x_t^{i+}) = v^i(x_0^i)+E\sum_{t=1}^{T}\beta^t u_t^i(x_t^i) = \tilde{U}^i\left(x^i,\beta\right) \end{aligned}$$

where the strict inequality holds by the fact that y^i is strictly preferred to x^{i+} in IH. This contradicts the optimality of x^i on $\tilde{B}_{p'}^i(\beta)\cap\tilde{X}^i(\beta)$. □

Finally, we demonstrate equivalence in terms of ordinal welfare indices.

Lemma 15. *Suppose equivalent variation $EV^i_{p,p',d}(\beta)$ in IH is attained on z^i. In RF equivalent variation is attained on z^{i-} and satisfies $\tilde{EV}^i_{p,p',\tilde{d}} = \kappa_d EV^i_{p,p',d}(\beta)$. Conversely, if in RF equivalent variation $\tilde{EV}^i_{p,p',\tilde{d}}(\beta)$ is attained on z^i, then in IH equivalent variation is attained on z^{i+} and satisfies $EV^i_{p,p',d}(\beta) = \tilde{EV}^i_{p,p',\tilde{d}}/\kappa_d$.*

Proof of Lemma 15:

Step 1. Suppose in IH equivalent variation $\tau \equiv EV^i_{p,p',d}(\beta)$ is attained on z^i. It follows that $z^i - \tau d \in B^i_p(\beta)$ and $z^i \in \bar{\Psi}^i_{p'}\left(\bar{\theta}\right) \subset X^i\left(\beta\right)$. By Lemma 13 reduction $z^{i-} \in \tilde{X}^i\left(\beta\right)$ is well-defined and it satisfies $z^{i-} \in \tilde{B}^i_p(\beta) + \kappa_d \tau \tilde{d}$. Let x^i be an optimal choice in IH. By Lemma 14, reduction x^{i-} is well-defined and optimal in RF. Then

$$\begin{aligned}
\tilde{U}^i\left(z^{i-}, \beta\right) &= v^i(z^{i-}_0, \beta) + E\sum_{t=1}^{T}\beta^t u^i_t(z^i_t) \geq E\sum_{t=1}^{\infty}\beta^t u^i_t(z^i_t) \\
&\geq E\sum_{t=1}^{\infty}\beta^t u^i_t(x^i_t) = v^i(x^{i-}_0) + E\sum_{t=1}^{T}\beta^t u^i_t(x^i_t) = \tilde{U}^i\left(x^{i-}, \beta\right)
\end{aligned}$$

which implies that $z^{i-} \in \bar{\Psi}^i_{p'}\left(\beta\right)$ in the reduced-form setting. It follows that $(z^{i-}, \kappa_d \tau)$ satisfy constraints of Program (5) within RF. Suppose that $z^{i-}, \kappa_d \tau$ does not solve this program. It follows that there exists $z^{i\prime} \in \bar{\Psi}^i_{p'}\left(\beta\right) \subset \tilde{X}^i\left(\beta\right)$ satisfying $z^{i\prime} \in \tilde{B}^i_{p'}(\beta) + \kappa_d \tau' \tilde{d}$ for some $\tau' < \tau$. By Lemma 13, extension of $z^{i\prime +} \in X^i\left(\beta\right)$ to IH is well-defined and satisfies $z^{i\prime +} \in B^i_{p'}(\beta) + \tau' d$. Moreover, one has

$$\begin{aligned}
U^i\left(z^{i\prime +}, \beta\right) &= \sum_{t=1}^{\infty}\beta^t u^i_t(z^{i\prime +}_t) = v^i(z^{i\prime}_0, \beta) + E\sum_{t=1}^{T}\beta^t u^i_t(z^{i\prime}_t) \\
&\geq v^i(x^{i-}_0, \beta) + E\sum_{t=1}^{T}\beta^t u^i_t(x^{i-}_t) \geq \sum_{t=1}^{\infty}\beta^t u^i_t(x^i_t) = U^i\left(x^i, \beta\right),
\end{aligned}$$

and hence $z^{i\prime +} \in \bar{\Psi}^i_{p'}\left(\beta\right)$ in the IH problem. Thus $z^{i\prime +}, \tau'$ satisfies constraints of Program (5) in IH and gives a smaller value, contradicting that z^i, τ is a solution. It follows that $\tilde{EV}^i_{p,p',\tilde{d}} = \kappa_d EV^i_{p,p',d}(\beta)$.

Step 2. Suppose equivalent variation $\tilde{EV}^i_{p,p',\tilde{d}}(\beta)$ is attained on z^i in RF and let $\tau \equiv \tilde{EV}^i_{p,p',\tilde{d}}(\beta)/\kappa_d$. Consequently $z^i - \tau \kappa_d \tilde{d} \in \tilde{B}^i_p(\beta)$ and $z^i \in \bar{\Psi}^i_{p'}\left(\bar{\theta}\right) \subset \tilde{X}^i\left(\beta\right)$. By Lemma 13 extension $z^{i+} \in X^i\left(\beta\right)$ is well-defined and satisfies $z^{i+} \in B^i_{p'}(\beta) + \tau d$. Let x^i be an optimal choice in RF. By Lemma 14, extension x^{i+} is well-defined and optimal in IH. Then

$$\begin{aligned}
U^i\left(z^{i+}, \beta\right) &= E\sum_{t=1}^{\infty}\beta^t u^i_t(z^{i+}_t) = v^i(z^i_0, \beta) + E\sum_{t=1}^{T}\beta^t u^i_t(z^i_t) \\
&\geq v^i(x^i_0, \beta) + E\sum_{t=1}^{T}\beta^t u^i_t(x^i_t) \geq E\sum_{t=1}^{\infty}\beta^t u^i_t(x^{i+}_t) = U^i\left(x^{i+}, \beta\right)
\end{aligned}$$

which implies that $z^{i+} \in \bar{\Psi}^i_{p'}\left(\beta\right)$ in IH. It follows that z^{i+}, τ satisfy constraints of Program (5) within IH. Suppose that z^{i+}, τ is not a solution. There must exist $z^{i\prime} \in \bar{\Psi}^i_{p'}\left(\beta\right) \subset X^i\left(\beta\right)$ satisfying

$z^{i\prime} \in B^i_{p'}(\beta) + \tau' d$ for some $\tau' < \tau$. By Lemma 13, reduction to RF $z^{i\prime -} \in \tilde{X}^i(\beta)$ is well-defined and satisfies $z^{i\prime -} \in \tilde{B}^i_{p'}(\beta) + \kappa_d \tau' \tilde{d}$. Moreover,

$$\begin{aligned} \tilde{U}^i\left(z^{i\prime -}, \beta\right) &= v^i(z_0^{i\prime -}, \beta) + E\sum_{t=1}^{T} \beta^t u_t^i(z_t^{i\prime}) = E\sum_{t=1}^{\infty} \beta^t u_t^i(z_t^{i\prime}) \\ &\geq E\sum_{t=1}^{\infty} \beta^t u_t^i(x_t^{i+}) = v^i(x_0^i, \beta) + E\sum_{t=1}^{T} \beta^t u_t^i(x_t^i) = \tilde{U}^i\left(x^i, \beta\right), \end{aligned}$$

hence $z^{i\prime -} \in \bar{\Psi}^i_{p'}(\beta)$ in RF. Thus $z^{i\prime -}, \kappa_d \tau'$ solves Program (5) in RF and attains a smaller value, contradicting that $z^i, \tilde{EV}^i_{p,p',\tilde{d}}(\beta) = \kappa_d \tau$ is a solution. It follows that $EV^i_{p,p',d}(\beta) = \frac{1}{\kappa_d} \tilde{EV}^i_{p,p',\tilde{d}}$. □

B.3 Ordinal convergence

In the next result, we characterize properties of the limit quasilinear preferences.

Proof of Lemma 2:

Fix policy $p' = \left\{\zeta', e^{i\prime}\right\}$. The first order necessary and sufficient conditions are derived from the unconstraint Lagrangian equation. Consumption in date-event (t, s) is determined by equality $\lambda^{i*}\zeta'_{t,s} = u^{i\prime}\left(x^{i*}_{t,s}\right)$ which by Inada conditions has a unique solution. It can be equivalently written as $x^{i*}_{t,s} = \tilde{x}^i\left(\lambda^{i*}\zeta'_{t,s}\right)$, where $\tilde{x}^i(\cdot)$ is the inverse of derivative $u^{i\prime}$ (a bijection). Consumption in period zero is then determined from budget constraint $x_0^{i*} = -E\sum_{t=1}^{T} \zeta'_t(\tilde{x}^i\left(\lambda^{i*}\zeta'_t\right) - e_t^{i\prime})$. Therefore, optimal choice x^{i*} is uniquely defined.

Consider policies $p = \left\{\zeta, e^i\right\}$ and $p' = \left\{\zeta', e^{i\prime}\right\}$ and let x^{i*} be optimal choice under policy p'. Program (5) specializes to $\min_{z^i \in X^{i*}(1), \tau \in \mathbb{R}} \tau$ subject to $\tilde{U}^i\left(z^i, 1\right) \geq \tilde{U}^i\left(x^{i*}, 1\right)$ and $z_0^i - \tau + E\sum_{t=1}^{T} \zeta_t(z_t^i - e_t^i) \leq 0$. In optimum with strictly monotone preferences, constraints hold with equality. Solving the second equation for τ and plugging it into the objective function gives

$$\tilde{EV}^i_{p,p',\tilde{d}}(1) = \min_{z^i \in \tilde{X}^i(1)} z_0^i + \sum_{t=1}^{T} E\zeta_t\left(z_t^i - e_t^i\right) : \tilde{U}^i\left(z^i, 1\right) = \tilde{U}^i\left(x^{i*}, 1\right). \tag{15}$$

The first order (necessary and sufficient) conditions for this program are $\lambda^{i*}\zeta_{t,s} = u^{i\prime}\left(z^{i*}_{t,s}\right)$ for all (t, s) or equivalently $z^{i*}_{t,s} = \tilde{x}^i\left(\lambda^{i*}\zeta_{t,s}\right)$ and $z_0^{i*} = x_0^{i*} + \frac{1}{\lambda^{i*}}E\sum_{t=1}^{T}(u^i(x_t^{i*}) - u^i(z_t^{i*}))$. With Inada conditions unique $z^{i*} \in X^{i*}(1)$ exists. Plugging values of $z_0^{i*}, x_0^{i*}, z_t^{i*}$ and x_t^{i*} back in the objective function of Program (15) gives $\tilde{EV}^i_{p,p',\tilde{d}} = S^i(p') - S^i(p)$ where

$$S^i(p) = S^i(\{\zeta, e^i\}) = -E\sum_{t=1}^{T} \zeta_t(\tilde{x}^i(\lambda^{i*}\zeta_t) - e_t^i) + \frac{1}{\lambda^{i*}}E\sum_{t=1}^{T} u^i(\tilde{x}^i(\lambda^{i*}\zeta_t)).$$

□

Proof of Proposition 2 :

Step 1. For the considered Markov chain, a regular symmetric transition matrix is diagonalizable with S independent eigenvectors and all real eigenvalues. The largest eigenvalue is equal to one, while other (possibly repeated) eigenvalues $m = 2, 3, ..., S$ satisfy $|r_m| < 1$. It follows that the probability distribution at t can be written as $\pi_t = \tilde{\pi} + \sum_{m=2}^{S} c_{0,m} (r_m)^t v_m$ where $\tilde{\pi}$ denotes the unique stationary distribution derived from the eigenvector with the largest eigenvalue, v_m is an eigenvector corresponding to r_m and c_0 are constants that express the initial distribution in terms of eigenvector basis. Let $\tilde{\zeta}$ and $\tilde{e}$ be stationary random variables of a Markov process and $\tilde{\zeta}_s, \tilde{e}_s$ be their realizations in event s. In terms of this notation, function $\eta(\beta, \lambda^i)$ from (14) can be written as

$$\begin{aligned}
\eta\left(\beta, \lambda^i\right) &\equiv (1-\beta) \sum_{t>T} \beta^t \sum_{s=1}^{S} \pi_{t,s} \tilde{\zeta}_s \tilde{x}^i(\tilde{\zeta}_s \lambda^i) \\
&= (1-\beta) \sum_{t>T} \beta^t \sum_{s=1}^{S} (\tilde{\pi}_s + \sum_{m=2}^{S} c_{0,m} (r_m)^t v_{m,s}) \tilde{\zeta}_s \tilde{x}^i(\tilde{\zeta}_s \lambda^i) \\
&= (1-\beta) \sum_{t>T} \beta^t E(\tilde{\zeta} \tilde{x}^i(\tilde{\zeta} \lambda^i)) + (1-\beta) \sum_{m=2}^{S} \sum_{s=1}^{S} c_{0,m} v_{m,s} \tilde{\zeta}_s \tilde{x}^i(\tilde{\zeta}_s \lambda^i) \sum_{t>T} (r_m \beta)^t \\
&= \beta^{T+1} E(\tilde{\zeta} \tilde{x}^i(\tilde{\zeta} \lambda^i)) + \sum_{m=2}^{S} \omega_m \sum_{s=1}^{S} v_{m,s} \tilde{\zeta}_s \tilde{x}^i(\tilde{\zeta}_s \lambda^i),
\end{aligned}$$

where corresponding weights ω_m are given by

$$\omega_m \equiv c_{0,m} (r_m \beta)^{T+1} \frac{1-\beta}{1 - r_m \beta}.$$

Since $|r_m| < 1$, for $m = 2, ..., S$ the weights are finite in some neighborhood of $\beta = 1$ and they are equal to zero at this value. Therefore, the weights and, hence, function $\eta(\beta, \lambda)$ itself, are well-defined and differentiable with respect to β near $\beta = 1$.

Fix arbitrary value x_0^i. In terms of parameters of the transition matrix, the constant on the right hand side of condition (14) is given by

$$(1-\beta)\, x_0 + \beta^{T+1} E(\tilde{\zeta} \tilde{e}^i) + \sum_{m=2}^{S} \omega_m \sum_{s=1}^{S} v_{m,s} \tilde{\zeta}_s \tilde{e}_s^i.$$

For $\beta = 1$ condition (14) reduces to $E(\tilde{\zeta} \tilde{x}^i(\tilde{\zeta} \lambda^i)) = E(\tilde{\zeta} \tilde{e}^i)$. By the arguments analogous to the ones in Lemma 12, this equation has unique solution denoted by λ^{i*}. Moreover, $\tilde{x}^i(\cdot)$ is strictly decreasing so that derivative $\partial \eta(1, \lambda^{i*}) / \partial \lambda^i = E(\tilde{\zeta}^2 \tilde{x}^{i\prime}(\tilde{\zeta} \lambda^{i*})) < 0$ is non-zero. By the implicit function theorem there exists threshold $\beta_{x_0} < 1$, a neighborhood of λ^{i*}, denoted by $V_{\lambda^{i*}}$ and a continuous bijection $\lambda_{x_0}^i : [\beta_{x_0}, 1] \to V_{\lambda^{i*}}$ such that $\lambda_{x_0}^i(\beta)$ is a unique solution to equation (14) for each $\beta \in [\beta_{x_0}, 1]$. Note that $\lim_{\beta \to 1} \lambda_{x_0}^i(\beta) = \lambda_{x_0}^i(1) = \lambda^{i*}$ for arbitrary value x_0 and hence the limit of the function does not depend on x_0. Let $\overline{x}_0$ and $\underline{x}_0$ be the bounds on consumption of commodity zero that define box X^{i*}. By $\bar{\beta}_0, \bar{\beta}_{\overline{x}}, \bar{\beta}_{\underline{x}}$ denote corresponding thresholds for x_0 equal to $0, \overline{x}_0$ and $\underline{x}_0$ respectively and let functions $\lambda_0^i(\cdot)$, $\lambda_{\overline{x}}^i(\cdot)$ and $\lambda_{\underline{x}}^i(\cdot)$ be the corresponding bijections.

Step 2. Consider the borrowing constraint

$$\begin{aligned}
\underline{x}_0^i(\beta) &\equiv \sum_{t>T}\beta^t\sum_{s=1}^{S}\pi_{t,s}\tilde{\zeta}_s\left(\underline{x}^i-\tilde{e}_{t,s}^i\right)\\
&= \sum_{t>T}\beta^t\sum_{s=1}^{S}(\tilde{\pi}_s+\sum_{m=2}^{S}c_{0,m}\left(r_m\right)^t v_{m,s})\tilde{\zeta}_s\left(\underline{x}^i-\tilde{e}_s^i\right)\\
&= \sum_{t>T}\beta^t E(\tilde{\zeta}(\underline{x}^i-\tilde{e}^i))+\sum_{m=2}^{S}\sum_{s=1}^{S}c_{0,m}v_{m,s}\tilde{\zeta}_s\left(\underline{x}^i-\tilde{e}_s^i\right)\sum_{t>T}\left(r_m\beta\right)^t\\
&= \frac{\beta^{T+1}}{(1-\beta)}E(\tilde{\zeta}(\underline{x}^i-\tilde{e}^i))+\sum_{m=2}^{S}r_m^{T+1}c_{0,m}\beta^{T+1}\frac{1}{1-r_m\beta}\sum_{s=1}^{S}v_{m,s}\tilde{\zeta}_s\left(\underline{x}^i-\tilde{e}_s^i\right).
\end{aligned}$$

By assumption $\tilde{e}_s^i > \underline{x}^i$ and $\tilde{\zeta}_s > 0$ for any s therefore $E(\tilde{\zeta}(\underline{x}^i-\tilde{e}^i)) < 0$ and the first term in the equation converges to $-\infty$ as $\beta \to 1$. Since other eigenvalues are strictly smaller than one, one has $1/\left(1-r_m\beta\right) \to 1/\left(1-r_m\right) > 0$ and therefore the second term converges to a finite limit. It follows that $\lim_{\beta\to1}\underline{x}_0^i(\beta) = -\infty$ and there must exist $\beta_{box}^i < 1$ such that for all $\beta \in [\beta_{box}^i, 1)$ one has $\underline{x}_0^i(\beta) < \underline{x}_0$ and the box satisfies $X^{i*} \subset X^i\left(\beta\right)$.

Step 3. Define $\beta^{i*} \equiv \max\{\bar{\beta}_0, \bar{\beta}_{\overline{x}}, \bar{\beta}_{\underline{x}}, \beta_{box}^i\} \in (0,1)$ where the first three elements are defined in Step 1 and the last one in Step 2. For $\beta \in [\beta^{i*}, 1)$ define function $\tilde{V}^i\left(x^i,\beta\right) \equiv \tilde{U}^i\left(x^i,\beta\right) - v^i(0,\beta)$ while for $\beta = 1$ let $\tilde{V}^i\left(x,1\right) \equiv \lambda^{i*}x_0^i + E\sum_{t=1}^{T}u^i(x_t^i)$. We next show that representation $\tilde{V}^i : X^{i*}\times[\beta^{i*},1] \to \mathbb{R}$ is jointly continuous. $\tilde{V}^i\left(x,\beta\right)$ is jointly continuous for all x^i, β for which $\beta < 1$ by the standard maximum theorem. Therefore it suffices to verify joint continuity for the points with $\beta = 1$. Consider an arbitrary sequence $x^{i,h}, \beta^h \to \bar{x}, 1 \in X^{i*}\times[\beta^{i*},1]$. By the envelope theorem, the derivative of the value function is given by the Lagrangian multiplier $\partial v^i(x_0^i,\beta)/\partial x_0^i = \lambda_{x_0}^i\left(\beta\right)$. Function $v^i(x_0^i,\beta) - v^i(0,\beta)$ is strictly concave and it attains zero at $x_0^i = 0$. Hence, for any element of the sequence $h = 1, 2, ...$ utility function is bounded from above by

$$\tilde{V}^i(x^{i,h},\beta^h) \le \lambda_0^i(\beta^h)x_0^{i,h} + E\sum_{t=1}^{T}\left(\beta^h\right)^t u^i(x_t^{i,h}). \tag{16}$$

For all $\beta^h \in [\beta^{i*},1]$ function $\lambda_0^i(\beta)$, defined in Step 1 is continuous and hence $\lim_{h\to\infty}\lambda_0^i(\beta^h) = \lambda_0^i(\lim_{h\to\infty}\beta^h) = \lambda^{i*}$. It follows that $\lim_{h\to\infty}\tilde{V}^i\left(x^{i,h},\beta^h\right) \le \lambda^{i*}\bar{x}_0^i + E\sum_{t=1}^{T}u_t^i(\bar{x}_t^i)$.

By strict concavity of $v^i(\cdot,\beta)$ for all x_0^i such that $x_0 \le x_0^i \le 0$ value function satisfies $v^i(x_0^i,\beta) - v^i(0,\beta) \ge \lambda_{\underline{x}}^i(\beta^h)x_0^{i,h}$ while for all $0 \le x_0^i \le \overline{x}_0$ one has $v^i(x_0^i,\beta) - v^i(0,\beta) \ge \lambda_{\overline{x}}^i(\beta^h)x_0^{i,h}$ and hence

$$\tilde{V}^i(x^{i,h},\beta^h) \ge \min[\lambda_{\underline{x}}^i(\beta^h)x_0^{i,h};\lambda_{\overline{x}}^i(\beta^h)x_0^{i,h}] + E\sum_{t=1}^{T}\left(\beta^h\right)^t u^i(x_t^{i,h})$$

Taking the limit gives $\lim_{h\to\infty}\tilde{V}^i(x^{i,h},\beta^h) \ge \lambda^{i*}\bar{x}_0^i + E\sum_{t=1}^{T}u^i(\bar{x}_t^i)$. The two inequalities imply $\lim_{h\to\infty}\tilde{V}^i\left(x^{i,h},\beta^h\right) = \tilde{V}^i\left(\bar{x},1\right)$ and utility representation $\tilde{V}^i$ is jointly continuous on $X^{i*}\times[\beta^{i*},1]$. Since for all $\beta \in [\beta^{i*},1]$ preferences are strictly monotone and they admit jointly continuous representation, by Lemma 1 correspondence $\Psi^i : X^{i*}\times[\beta^{i*},1] \to X^{i*}$ is continuous. □

Proof of Proposition 3 and Theorem 2:

We first verify Assumptions 2-3 in the reduced-form problem. Consider parametric space $[\beta^{i*}, 1]$ as defined in Lemma 2. For any date-event (t, s), one has $\partial\tilde{b}^i_p/\partial x^i_{t,s} = \pi_{t,s}\beta^t\tilde{\zeta}_s$ and for $t = 0$ the derivative is $\partial\tilde{b}^i_p/\partial x^i_0 = 1$. Therefore, $\overline{b} \geq \partial b^{i*}_p/\partial x^i_{t,s} \geq \underline{b}$ where $\overline{b} \equiv \max(1, \max_{t,s}\zeta_{t,s}\pi_{t,s}) > 0$ and $\underline{b} \equiv \min(1, (\beta^{i*})^T \min_{t,s}\zeta_{t,s}\pi_{t,s}) > 0$ are well-defined since $T < \infty$ and $S < \infty$, and $\pi_{t,s} > 0$ for all date events. Thus, Assumption 2 holds. By Lemma 2, optimal choice and equivalent variation at $\beta = 1$ are well-defined. In the reduced-form representation for all $\beta \in [\beta^{i*}, 1]$, preferences $\succeq^i_\beta$ are strictly convex on the respective domains $\tilde{X}^i(\beta)$. For any policy p and $\beta \in [\beta^{i*}, 1]$ function $\tilde{b}^i_p(\cdot, \beta)$ is linear in x^i, and hence it is quasi-convex. Finally, by Proposition 2 restriction $\Psi^i : X^{i*} \times [\beta^{i*}, 1] \to X^{i*}$ is continuous and hence Assumption 3 holds as well. By Proposition 1, there exists $\beta^{i**} \in (0, 1)$ such that equivalent variation $\tilde{EV}^i_{p,p',\tilde{d}}(\beta)$ as well as choice are continuous functions on $[\beta^{i**}, 1]$ and they converge to the corresponding quasilinear limits. By Lemma 14 optimal choice in the reduced form coincides with a truncation of the choice is in the infinite horizon problem. This completes the proof of Proposition 3.

For the welfare part (Theorem *2)* observe that function $\kappa_d(\beta) \equiv E\sum_{t=T+1}^{\infty}\beta^t\zeta_t d_t$ from Appendix B.2 is continuous, strictly increasing in β and has an upper bound. It follows that limit $\kappa_d(1) \equiv \lim_{\beta\to1}\kappa_d(\beta)$ is well-defined, bounded and strictly positive.

Since for any $\beta \in [\beta^{i**}, 1]$ index $\tilde{EV}^i_{p,p',\tilde{d}}(\beta)$ is well-defined and $\kappa_d(\beta) \leq \kappa_d(1) < \infty$, by Lemma 15 in the infinite horizon problem equivalent variation $EV^i_{p,p',d}(\beta)$ is well-defined and given by $EV^i_{p,p',d}(\beta) = \tilde{EV}^i_{p,p',\tilde{d}}(\beta)/\kappa_d(\beta)$. Consequently its limit satisfies

$$\lim_{\beta\to1} EV^i_{p,p',d}(\beta) = \frac{\lim_{\beta\to1}\tilde{EV}_{p,p',\tilde{d}}(\beta)}{\lim_{\beta\to1}\kappa_d(\beta)} = \frac{\tilde{EV}^i_{p,p',\tilde{d}}(1)}{\kappa_d(1)} = \frac{S^i(p') - S^i(p)}{\kappa_d(1)}$$

where the second equality follows from the continuity of equivalent variation in the reduced-form problem and market value of flow d at at $\beta = 1$ and the last equality from Lemma 2. We complete the proof by defining $\beta^{i***} \equiv \beta^{i**}$ □

C Gorman economy

We first prove basic results for an abstract Gorman economy. Consider an economy populated by $i = 1, ..., I$ consumers each with utility functions $U^i(x^i, \beta)$ over space $X^i = \{x^i \in \mathbb{R}^N | x^i_n \geq \underline{x}^i_n \text{ for all } n\}$ where N can be finite or infinite and with endowment given by $e^i = \{e^i_n\}_{n=1}^N$. By $e \equiv \sum_i e^i$ denote the aggregate endowment. Suppose for each i function $U^i(x^i, \beta)$ is twice continuously differentiable, strictly monotone, strictly concave, additively separable and satisfies Inada conditions. Finally, suppose that preferences are in Gorman polar form. Consider a utilitarian program that puts equal weights on each consumer,

$$U(e) \equiv \max_{x=\{x^i\}} \sum_{i=1}^{I} U^i(x^i, \beta) : \sum_i x^i \leq e. \tag{17}$$

Define $\bar{\xi} \equiv \nabla_e U(e)$ as the derivatives of the value function of the utilitarian program, and $\bar{x} \equiv x^i(\bar{\xi}, \bar{\xi} \cdot e^i)$ as the optimal choice of consumer i given prices $\bar{\xi}$ and endowment e^i and let $\bar{x} \equiv \{\bar{x}^i\}_{i=1}^I$.

Lemma 16. *Tuple $(\bar{\xi}, \bar{x})$ is a unique competitive equilibrium in this economy up to price normalization.*

Proof of Lemma 16:

Step 1. In this step we show that $(\bar{\xi}, \bar{x})$ is a competitive equilibrium. Since allocation $\bar{x}$ is defined as a tuple of optimal choices given prices $\bar{\xi}$ for all i, it suffices to demonstrate market clearing. Let $\hat{x}$ be a solution to the utilitarian program and let $\hat{\tau}^i \equiv \bar{\xi} \cdot \left(\hat{x}^i - e^i\right)$ be a monetary transfer to consumer i. By strict monotonicity of preferences, solution, $\bar{x}$, satisfies the utilitarian constraint with equality and $\sum_{i=1}^I \hat{\tau}^i = 0$. From the envelope theorem

$$\bar{\xi} = \nabla_x U^i \left(\hat{x}^i, \beta\right) \tag{18}$$

By the definition of transfer $\bar{\xi} \cdot \hat{x}^i = \bar{\xi}^i \cdot e^i + \hat{\tau}^i$ for each i and hence $\hat{x}^i$ satisfies the consumer's budget constraint with transfer with equality. These are necessary and sufficient conditions for optimality of interior $\hat{x}^i$ given prices $\bar{\xi}$, endowment e^i and transfer $\hat{\tau}^i$, and hence, $\hat{x}^i = x^i(\bar{\xi}, \bar{\xi} \cdot e^i + \hat{\tau}^i)$. But then

$$\sum_i \bar{x}^i = \sum_i x^i(\bar{\xi}, \bar{\xi} \cdot e^i) = \sum_i x^i(\bar{\xi}, \bar{\xi} \cdot e^i + \tau^i) = \sum_i \hat{x}^i = e$$

where the second inequality holds by aggregation property, resulting from the Gorman assumption and the last inequality by the fact that solution $\hat{x}$ satisfies the feasibility constraint with equality by strict monotonicity of preferences. It follows that, for the assumed allocation markets clear and the choices are optimal, and therefore $(\bar{\xi}, \bar{x})$ is a competitive equilibrium.

Step 2. In this step we show that $(\bar{\xi}, \bar{x})$ is the only equilibrium, up to price normalization. Consider arbitrary competitive equilibrium $(\bar{\xi}', \bar{x}')$. Let $\hat{\tau}^{i\prime} \equiv \bar{\xi}' \cdot \left(\hat{x}^i - e^i\right)$ and choice $\tilde{x}^i \equiv x^i(\bar{\xi}', \bar{\xi}'^i + \hat{\tau}^{i\prime})$. By definition of $\hat{\tau}^{i\prime}$ consumption profile $\hat{x}^i$ is just affordable with transfer given prices $\bar{\xi}'$, while $\tilde{x}^i$ is optimal. It follows that $U^i\left(\tilde{x}^i, \beta\right) \geq U^i\left(\hat{x}^i, \beta\right)$, and hence

$$\sum_{i=1}^I U^i(\tilde{x}^i, \beta) \geq \sum_{i=1}^I U^i(\hat{x}^i, \beta).$$

On the other hand

$$\sum_i \tilde{x}^i = \sum_i x^i(\bar{\xi}', \bar{\xi}'^i + \hat{\tau}^{i\prime}) = \sum_i x^i(\bar{\xi}', \bar{\xi}'^i) = \sum_i \bar{x}^{i\prime} = e,$$

where the second equality holds by Gorman aggregation and the last equality by the fact that $\bar{x}'$ is an equilibrium allocation and hence satisfied market clearing. It follows that $\tilde{x}$ satisfies the feasibility constraint in Program (17). It follows that $\tilde{x}$ necessarily solves the utilitarian problem. By strict convexity of preferences and convexity of allocation space $\times_i X^i$ solution to utilitarian problem is unique and hence $\tilde{x} = \hat{x}$. For interior solutions secured by the Inada assumptions, a necessary condition for optimality is proportionality of prices to the vector of marginal utilities i.e.,

$$const \times \bar{\xi}' = \nabla_x U^i\left(\tilde{x}^i, \beta\right) = \nabla_x U^i\left(\hat{x}^i, \beta\right) = \bar{\xi},$$

where *const* is some strictly positive constant, the second equality holds by the fact that $\hat{x}^i = \tilde{x}^i$ and the last equality by equation (18). It follows that $\bar{\xi}' = \frac{1}{const}\bar{\xi}$. Finally, by homogeneity of demand function of degree zero with respect to prices, one has $\bar{x}^{i\prime} \equiv x^i(\bar{\xi}', \bar{\xi}'^i) = x^i(\bar{\xi}, \bar{\xi} \cdot e^i) \equiv \bar{x}^i$ for all i. Therefore $(\bar{\xi}, \bar{x})$ is a unique equilibrium, up to price normalization. □

Proof of Proposition 4:

In the financial economy considered in this paper, for any i consumption space has structure $X^i \subset \mathbb{R}^\infty$ assumed in Lemma 16 where each commodity n corresponds to consumption in date-event $n = (t, s)$. Traders' preferences satisfy the assumptions of Lemma 16. It follows that the economy has a unique equilibrium up to price normalization. Note that in terms of prices $\bar{\xi}$, the pricing kernel process (in the paper called prices), is given by $\zeta_{t,s} = \bar{\xi}_{n=t,s}/\beta^t \pi_{t,s}$.

With additively separable utility functions $U^i(x, \beta) = E\sum_{t=1}^{\infty} \beta^t u^i(x_t)$ and separable constraints, utilitarian program (17) is an infinite sequence of independent optimization programs, one for each date event (t, s). Moreover, value function of the program is given by $U(e) = \sum_{t=1}^{\infty}\sum_{s=1}^{S} \pi_{t,s}\beta^t u(e_{t,s})$, where

$$u(\bar{y}) \equiv \max_{\{y^i\}_i} \sum_i u^i\left(y^i\right) : \sum_i y^i \leq \bar{y} \text{ and } y^i > \underline{x}^i.$$

Function $u(\cdot)$ is well-defined for all $\bar{y} > \sum_i \underline{x}^i$. By Lemma 16, for any β equilibrium prices admit normalization for which

$$\zeta_{t,s} = \frac{\bar{\xi}_{n=t,s}}{\beta^t \pi_{t,s}} = \frac{\partial U(e)}{\partial e_{t,s}} \frac{1}{\beta^t \pi_{t,s}} = u'(e_{t,s})\, \beta^t \pi_{t,s} \frac{1}{\beta^t \pi_{t,s}} = u'(e_{t,s}).$$

Since aggregate endowment $\{e_t\}_{t=T+1}^{\infty}$ satisfies stationarity assumptions, so do prices, that in each date event are given by a monotonic transformation of aggregate endowment. □

Proof of Lemma 3:

The argument for the uniqueness of equilibrium is standard and is omitted. (It follows straightforwardly from the Neghishi construction relying on Pareto efficiency of a competitive equilibrium). For the surplus part, by definition of aggregate equivalent variation and Lemma 2, one has $EV_{p,p',d}(\beta) \equiv \sum_i EV^i_{p,p^i,d}(\beta) = \sum_i S^i(p') - \sum_i S^i(p)$. Using the formula for individual surplus functions

$$\begin{aligned}\sum_i S^i(p) &= \sum_i [-E\sum_{t=1}^{T} \zeta_t(\tilde{x}^i(\lambda^{i*}\zeta_t) - e^i_t) + \frac{1}{\lambda^{i*}} E\sum_{t=1}^{T} u^i(\tilde{x}^i(\bar{\lambda}^i\zeta_t))] \\ &= \sum_i \frac{1}{\lambda^{i*}} E\sum_{t=1}^{T} u^i(\tilde{x}^i(\lambda^{i*}\zeta_t)) = \sum_i \frac{1}{\lambda^{i*}} E\sum_{t=1}^{T} u^i(\tilde{x}^i(\lambda^{i*}u'(e_t)))\end{aligned}$$

where the second equality holds by market clearing. Observe that the sum of individual surpluses can be expressed as a function of aggregate endowments and hence surplus is measurable with respect to aggregate endowment. Finally note that $\tilde{x}^i(\lambda^{i*}u'(e_t))$ for any i is a Pareto efficient allocation of e_t and hence aggregate surplus can be written as $\tilde{S}(e) = E\sum_{t=1}^{T} s(e_t)$ where

$$s(\bar{y}) \equiv \max_{\{y^i\}_i} \sum_i u^i(y)/\lambda^{i*} : \sum_i y^i \leq \bar{y}.$$

□

Proof of Proposition 5 and Theorem 3:

Endowments of consumers and equilibrium prices characterized in Proposition 4 satisfy all the assumptions made in Section 4. They are constant for all β and coincide with their limit values. By Theorem 2 for each trader i there exists $\beta^{i***} \in (0,1)$ such that equivalent variation in the infinite horizon setting is well-defined. Let $\beta^{***} \equiv \max_i \beta^{i***}$. For all $\beta \in (\beta^{***}, 1)$ the aggregate equivalent variation in the infinite horizon model is well-defined. Moreover, the limit satisfies

$$\lim_{\beta\to 1} EV_{p,p',d}(\beta) = \sum_i \lim_{\beta\to 1} EV^i_{p,p^i,d}(\beta) = \sum_i EV^{i*}_{p,p^i,d^*}(1) = EV^*_{p,p^i,d^*}(1) = \frac{\tilde{S}(e') - \tilde{S}(e)}{\kappa_d(1)}$$

where the second equality follows from Theorem 2, third from the definition of aggregate equivalent variation and the fourth from Lemma 3. □