Practice Problems

1. Use the Lagrange-multiplier method to find the stationary values of z:

(a) $z = xy$, subject to $x + 2y = 2$

(b) $z = x(y+4)$, subject to $x + y = 8$

(c) $z = x - 3y - xy$, subject to $x + y = 6$

(d) $z = 7 - y + x^2$, subject to $x + y = 0$

2. In the above problem, find whether a slight relaxation of the constraint will increase or decrease the optimal value of z. At what rate?

3. Write the Lagrangian function and the first-order condition for stationary values (without solving the equations) for each of the following:

(a) $z = x + 2y + 3w + xy - yw$, subject to $x + y + 2w = 10$

(b) $z = x^2 + 2xy + yw^2$, subject to $2x + y + w^2 = 24$ and $x + w = 8$

4. Consider the utility functions of the form $U = x_1^{\alpha_1} x_2^{\alpha_2}$. Given the budget constraint, $p_1 x_1 + p_2 x_2 = M$, show that the implied demand curves are

$$x_1^M = \frac{\alpha_1}{(\alpha_1 + \alpha_2)} \frac{M}{p_1}$$

$$x_2^M = \frac{\alpha_2}{(\alpha_1 + \alpha_2)} \frac{M}{p_2}$$

Find λ^M and $U^*(x_1^M, x_2^M)$ and verify that $\lambda^M = \frac{\partial U^*}{\partial M}$

5. Consider the profit-maximizing firm with the production function $y = f(x_1, x_2)$, facing output price *p* and factor prices w_1 and w_2. Suppose this firm is taxed according to the total cost of factor 2, i.e.,
$tax = tw_2x_2$

(a) Derive factor demand functions, i.e., show where they came from, etc. Are these choice functions homogeneous of any degree in any of the parameters?

(b) Show that if the tax rate rises, the firm will use less of factor 2.

(c) Show that $\frac{\partial x_1^*}{\partial t} = w_2 \frac{\partial x_2^*}{\partial w_1}$

(d) Suppose that factor 1 is held fixed at its profit-maximizing level. Show that the response of factor 2 to a change in the tax rate is less in absolute value than before.

6. A monopolist produces *y* at cost *C(y)*, and sells this output in two separated markets, producing total revenues $R(y) = R_1(y_1) + R_2(y_2)$, where $y = y_1 + y_2$

(a) Show that the profit-maximizing monopolist will equate the marginal cost of production to the marginal revenues in each market.

(b) Assuming that the SOSC (Second Order Sufficient Conditions) hold, what conditions on the slopes of the marginal-revenue and marginal-cost curves are implied?

(c) Using the equation $MR_i = p_i(1 + 1/\varepsilon_i)$, i = 1,2, where $\varepsilon_i < 0$ is the i^th^ price elasticity of demand, show that a discriminating monopolist will charge a higher price in the market whose demand is less elastic. Assume MR > 0.

7. Consider a monopolist whose total cost function is $c = kx^2$, and who faces the demand curve $x = a - bp$.

(a) What restrictions on the values of the parameters (a, b, k) would you be inclined to assert, a priori? Explain your restrictions in economic terms.

(b) Assume that this monopolist faces a tax per unit of output t > 0. Set up the monopolist's maximization problem and derive the explicit choice function $x = x^*(t)$.

(c) Confirm for the restrictions you have placed on (a, b, k) in part a, that $\frac{dx^*(t)}{dt} < 0$ Holds.

(d) What restriction does the SOSC of this model place on (a, b, k)? Are these weaker or stronger than your a priori restrictions?

(e) Substitute $x = x^*(t)$ in the FONC and confirm that an identity in *t* results.

(f) Confirm that, for this specification of the model, the second-order sufficient condition for profit maximization alone imply $\frac{dx^*(t)}{dt} < 0$.

8. Find the maximum or minimum values of the following functions $f(x_1, x_2)$ subject to the constraint $g(x_1, x_2) = 0$, by the method of Lagrange multipliers. Be sure to check the SOSC to see if a maximum or minimum (if either) is achieved. Assume that $x_1 > 0$ and $x_2 > 0$ at the solution.

a) $f(x_1, x_2) = x_1 x_2, g(x_1, x_2) = 2 - (x_1 + x_2)$.

b) $f(x_1, x_2) = x_1 + x_2, g(x_1, x_2) = 1 - x_1 x_2$.

c) $f(x_1, x_2) = x_1 x_2, g(x_1, x_2) = M - p_1 x_1 - p_2 x_2$, where (p_1, p_2, M) are parameters

d) $f(x_1, x_2) = p_1 x_1 + p_2 x_2, g(x_1, x_2) = U^0 - x_1 x_2$, where (p_1, p_2, U^0) are parameters

8´. Show that the SOSC for 8a) and 8b) are equivalent, and those for 8c) and 6d) are as well.

9. Use Cramer's rule to solve the following equation systems

(a) $\begin{aligned} 3x_1 - 2x_2 &= 11 \\ 2x_1 + x_2 &= 12 \end{aligned}$ (b) $\begin{aligned} -x_1 + 3x_2 &= -3 \\ 4x_1 - x_2 &= 12 \end{aligned}$

(c) $\begin{aligned} 8x_1 - 7x_2 &= -6 \\ x_1 + x_2 &= 3 \end{aligned}$ (d) $\begin{aligned} 6x_1 + 9x_2 &= 15 \\ 7x_1 - 3x_2 &= 4 \end{aligned}$

10. Use Cramer's rule to solve the following equation systems

(a) $\begin{aligned} 8x_1 - x_2 &= 15 \\ x_2 + 5x_3 &= 1 \\ 2x_1 + 3x_3 &= 4 \end{aligned}$ (b) $\begin{aligned} -x_1 + 3x_2 + 2x_3 &= 24 \\ x_1 + x_3 &= 6 \\ 5x_2 - x_3 &= 8 \end{aligned}$

(c) $\begin{aligned} 4x + 3y - 2z &= 7 \\ x + y &= 5 \\ 3x + z &= 4 \end{aligned}$ (d) $\begin{aligned} -x + y + z &= a \\ x - y + z &= b \\ x + y - z &= c \end{aligned}$

11. Find the solution of the equation system

$$\begin{aligned} 5x_1 + 3x_2 &= 30 \\ 6x_1 - 2x_2 &= 8 \end{aligned}$$

12. Find the solution of the equation system

$$\begin{aligned} 7x_1 - x_2 - x_3 &= 0 \\ 10x_1 - 2x_2 + x_3 &= 8 \\ 6x_1 + 3x_2 - 2x_3 &= 7 \end{aligned}$$